LAGRANGIAN INTERSECTIONS AND THE SPECTRAL NORM IN CONVEX-AT-INFINITY SYMPLECTIC MANIFOLDS

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Abstract. Given a compact Lagrangian L in a semipositive convex-at-infinity symplectic manifold W, we establish a cup-length estimate for the action values of L associated to a Hamiltonian isotopy whose spectral norm is smaller than some $\hbar(L)$. When L is rational, this implies a cup-length estimate on the number of intersection points. This Chekanov-type result generalizes a theorem of Kislev and Shelukhin proving non-displaceability in the case when W is closed and monotone. The method of proof is to deform the pair-of-pants product on Hamiltonian Floer cohomology using the Lagrangian L.

1. *Introduction and main results*

1.1. *Introduction.* Let (W, ω) be a symplectic manifold and $L \subset W$ a closed Lagrangian submanifold. Understanding when a Hamiltonian diffeomorphism ϕ can displace L, and quantifying the intersections when it cannot, has been one of the driving forces of symplectic topology ever since Arnol'd's famous conjectures were formulated; see [Arn65, Arn13]. Let us denote by $\mathrm{Ham}_c(W, \omega)$ the group of compactly supported Hamiltonian diffeomorphisms, i.e., those diffeomorphisms ϕ which appear as the time-one map $\phi = \phi_1$ of a compactly supported Hamiltonian isotopy ϕ_t; recall that this means the non-autonomous vector field X_t generating ϕ_t is ω-dual to an exact one-form.

The Lagrangian version of one of the conjectures in the particular case of cotangent bundles states the following:

Conjecture 1 (Arnol'd)**.** *For every compactly supported Hamiltonian diffeomorphism ϕ of T^*L, the number of points in $\phi(L) \cap L$ is bounded from below by the minimal number of critical points of a smooth function on L.*

When the intersection is transverse, the conjectured lower-bound is replaced by the *Morse number* of L, i.e., the minimal number of critical points of a Morse function on L. In this direction, Gromov proves in his groundbreaking work [Gro85] the existence of at least one intersection point of any closed exact Lagrangian $L' \subset T^*L$ with the zero-section, and then sets $L' = \phi(L)$ to conclude $\phi(L) \cap L \neq \emptyset$.

As stated, Conjecture 1 remains open; however, in a classic result [Hof85], Hofer proves a slightly weaker version of the conjecture where the lower

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bound is replaced with one plus the cup-length $\mathrm{cl}_{\mathbb{K}}(L)$ of L with coefficients on a base field $\mathbb{K}$, where:

$$\mathrm{cl}_{\mathbb{K}}(L) = \max\{k \mid \exists\, a_1, \ldots, a_k \in H^{>0}(L; \mathbb{K}) \text{ such that } a_1 \cup \cdots \cup a_k \neq 0\}.$$

In another celebrated result [LS85], Laudenbach and Sikorav showed that, in the transverse case, the number of intersection points is bounded from below by the total Betti number of L.

A Lagrangian submanifold $L \subset W$ is called *weakly-exact* if ω vanishes on every smooth disk with boundary on L. More generally, L is called *rational* if $\omega(\pi_2(W, L)) \subset \mathbb{R}$ is a discrete subgroup, in which case we denote the positive generator by ρ_L. When (W, ω) is a tame symplectic manifold[1] Gromov shows in [Gro85, 2.3.B$_3'$] that a weakly-exact Lagrangian submanifold L is non-displaceable. Moreover, if W is closed, the foundational works [Flo88, Flo89a] of Floer imply that the number of intersection points $\phi(L) \cap L$ is at least $\mathrm{cl}_{\mathbb{K}}(L)+1$ in general, and $\dim H_*(L)$ when the intersection is transverse; see also [Hof88]. Recently, this cup-length estimate has been established by [HP22] for generalized cohomology theories.

Nonetheless, the existence of small displaceable Lagrangian tori in every symplectic manifold indicates that a generalization of Conjecture 1 beyond the weakly-exact setting requires additional hypothesis; one can, e.g., require that the Hamiltonian diffeomorphism is close to the identity in some sense.

In this direction, Polterovich [Pol93] proved that if L is rational and W is tame, then the *Hofer norm* of any ϕ displacing L is at least $\rho_L/2$, i.e.,

$$\|\phi\|_{\mathrm{Hof}} < \rho_L/2 \implies \phi(L) \cap L \neq \emptyset;$$

see [Hof90, LM95, Pol01] for discussion of the Hofer norm. This result was sharpened by Chekanov [Che98], as follows. For an admissible almost complex structure J on W, as defined in §2.3.2, define $\hbar(J, L) > 0$ to be the minimal symplectic area of a non-constant J-holomorphic disk with boundary on L or J-holomorphic sphere in W, and set:

$$\hbar(L) = \sup_{J \in \mathcal{J}} \hbar(J, L), \tag{1}$$

where $\mathcal{J}$ is the space of all admissible almost complex structures on W. Chekanov showed that if $\|\phi\|_{\mathrm{Hof}} < \hbar(L)$, then $\phi(L) \cap L \neq \emptyset$, and the number of intersection points is bounded from below by $\dim_{\mathbb{F}_2} H_*(L; \mathbb{F}_2)$ provided the intersection is transverse; see also [Liu05].

Spectral invariants provide a way of defining a *spectral norm* on $\mathrm{Ham}_c(W, \omega)$ which is bounded from above by the Hofer norm. They were introduced in symplectic topology by Viterbo [Vit92] via generating functions and, from a Floer theoretic perspective, by Schwarz [Sch00] and Oh [Oh05a, Oh05b] (in the closed setting) and by Frauenfelder and Schlenk [FS07] for convex-at-infinity symplectic manifolds (as defined in §2.1); see also [HZ94, §5.4] and [BP94, §1.5.B]. In short, for a Hamiltonian system ϕ_t, one associates real-valued measurements $c(\alpha, \phi_t)$ to classes α in the (quantum) cohomology of W; the definition is as a "min-max" action value of the Floer cohomology

[1] This means that $\omega(v, Jv) \geq g(v, v)$ holds for a metric g whose injectivity radius is bounded from below and whose sectional curvatures are bounded from above.

class representing the image of α under the map in [PSS96]; we review their construction in §2.4. The *spectral norm* of a compactly supported Hamiltonian diffeomorphism ϕ is defined by:

$$\gamma(\phi) = \inf_{\phi_1=\phi} -c(1,\phi_t) - c(1,\phi_t^{-1}); \tag{2}$$

see Proposition 2.19, and it satisfies $\gamma(\phi) \leq \|\phi\|_{\mathrm{Hof}}$.

When (W, ω) is a closed weakly-monotone symplectic manifold, Kislev and Shelukhin showed in [KS21, Theorem E] that if $\gamma(\phi) < \hbar(L)$ then ϕ does not displace L and, if $\phi(L) \cap L$ is transverse, then $\#(\phi(L)\cap L) \geq \dim_{\mathbb{F}_2} H_*(L; \mathbb{F}_2)$, sharpening Chekanov's result in this setting. The reason this improvement is possible boils down to the observation that the Floer continuation maps:

$$\mathfrak{c} : \mathrm{CF}(\phi_t) \to \mathrm{CF}(\psi_t)$$

are chain-homotopic to the multiplication operators:

$$\mu_2(x, -) : \mathrm{CF}(\phi_t) \to \mathrm{CF}(\psi_t)$$

given by taking the product with a cocycle $x \in \mathrm{CF}(\psi_t \circ \phi_t^{-1})$ representing the image of the unit under the PSS map.

In [KS21, Remark 50] it is suggested that the cup-length estimate for a suitable choice of coefficient field $\mathbb{K}$ should hold whenever $\gamma(\phi) < \hbar(L)$; see also [Gon21]. Proving such a cup-length estimate in the convex-at-infinity setting is the main goal of this paper.

1.2. *Main results.* Let (W, ω) be a semipositive convex-at-infinity symplectic manifold and L a compact Lagrangian submanifold of W. The class of convex-at-infinity symplectic manifolds contains all Liouville manifolds and compact symplectic manifolds; see §2.1.

Recall that, to a compactly supported Hamiltonian system ϕ_t, one can associate an action functional $\mathcal{A}_{\phi_t}$ on the covering space of "capped" paths $x(t)$ with endpoints of L; see §2.2 for the definitions. The critical points of $\mathcal{A}_{\phi_t}$ are in bijective correspondence with the capped Hamiltonian chords.

Theorem 1.1. *Let L be a compact Lagrangian submanifold of a convex-at-infinity symplectic manifold W. Suppose ϕ is a compactly supported Hamiltonian diffeomorphism such that $\gamma(\phi) < \hbar(L)$. Then,*

$$\phi(L) \cap L \neq \emptyset.$$

Moreover, if the intersection points are isolated, then for all Hamiltonian systems ϕ_t with $\phi_1 = \phi$ there exists an interval of length $\gamma(\phi)$ containing at least $\mathrm{cl}_{\mathbb{F}_2}(L) + 1$ critical values of $\mathcal{A}_{\phi_t}$.

When L is a rational Lagrangian submanifold with rationality constant ρ_L the action value of a path is well-defined modulo ρ_L. Since $\rho_L \leq \hbar(L)$, Theorem 1.1 yields:

Corollary 1.2. *Let L be a compact rational Lagrangian submanifold. The cup-length estimate $\#(\phi(L) \cap L) \geq \mathrm{cl}_{\mathbb{F}_2}(L) + 1$ holds for all compactly supported Hamiltonian diffeomorphisms ϕ satisfying $\gamma(\phi) < \rho_L$.*

When (W, ω) is a closed rational semipositive symplectic manifold our result sharpens that of [Sch98, Theorem 1.1] by replacing the Hofer norm with the spectral norm. More precisely, we obtain the following:

Corollary 1.3. *Let (W, ω) be a compact semipositive symplectic manifold and suppose that $\omega(\pi_2(W)) = \rho_W \mathbb{Z}$. If ϕ is a Hamiltonian diffeomorphism satisfying $\gamma(\phi) \le \rho_W$ then $\#\mathrm{Fix}(\phi) \ge \mathrm{cl}_{\mathbb{F}_2}(W) + 1$.*

Proof. Consider $(W \times W, \omega \oplus -\omega)$ with the diagonal Lagrangian Δ. We first show that Δ is rational with rationality constant $\rho_\Delta = \rho_W$. Let A be a relative class in $\pi_2(W \times W, \Delta)$ represented by:

$$V \colon (D, \partial D) \to (W \times W, \Delta) \quad \text{given by} \quad z \mapsto (v_1(z), v_2(z)),$$

where v_1 and v_2 are the projections of V onto the first and second factors. Note that if $z \in \partial D$ then $V(z) \in \Delta$; in particular, we have $v_1(z) = v_2(z)$. Consider the piecewise smooth sphere $u = v_1 \#(-v_2)$ obtained by gluing v_1 and v_2 along their common boundary (and reversing the orientation of v_2). The symplectic area of u in W equals the symplectic area of V in $W \times W$, hence $\rho_\Delta \mathbb{Z} \subset \rho_W \mathbb{Z}$. For the reverse inclusion, observe that every smooth sphere decomposes as $v_1 \#(-v_2)$ where $v_1|_{\partial D} = v_2|_{\partial D}$, and the previous argument can be run in reverse to conclude $\rho_W \mathbb{Z} = \rho_\Delta \mathbb{Z}$, as desired.

Next, set $\Phi = \mathrm{id} \times \phi$ and consider the Lagrangian submanifold $\Phi(\Delta)$ of $W \times W$. The intersection points $\Phi(\Delta) \cap \Delta$ correspond bijectively to the fixed points of ϕ. We appeal to the product formula for spectral invariants in [EP09, Theorem 5.1] to conclude that:

$$\begin{aligned}
\gamma(\Phi) &= \inf_{\Phi_1 = \Phi} -c(1, \Phi_t) - c(1, \Phi_t^{-1}) \\
&\le \inf_{\phi_1 = \phi} -c(1, \phi_t \times \mathrm{id}) - c(1, \phi_t^{-1} \times \mathrm{id}) \\
&= \inf_{\phi_1 = \phi} -c(1, \phi_t) - c(1, \phi_t^{-1}) = \gamma(\phi).
\end{aligned}$$

Hence, $\gamma(\Phi) \le \gamma(\phi) \le \rho_W = \rho_\Delta$, and we can therefore apply Corollary 1.2 to obtain:

$$\#\mathrm{Fix}(\phi) = \#(\Phi(\Delta) \cap \Delta) \ge \mathrm{cl}_{\mathbb{F}_2}(\Delta) + 1 = \mathrm{cl}_{\mathbb{F}_2}(W) + 1,$$

which concludes the proof of the corollary. □

1.3. *Proof overview.* Before delving into an overview of the proof of Theorem 1.1, let us first examine a simpler case to illustrate the underlying principles in our approach, while pointing to the difficulties that arise in the more general setting. Suppose that $L \subset W$ is a closed weakly-exact Lagrangian submanifold with cup-length $\mathrm{cl}_{\mathbb{F}_2}(L) = k$. To prove the cup-length estimate it is enough to show that there are at least $k+1$ critical values of the action functional. Indeed, the weakly-exact condition implies that the action value of a chord is independent of the choice of capping. In contrast, for rational L, the action $\mathcal{A}_{\phi_t}$ is defined modulo ρ_L. In the rational case one can still ensure the existence of at least $k+1$ Hamiltonian chords by showing that action values belong to an interval of length at most ρ_L; this excludes contributions of different cappings of the same chord.

To obtain a strictly decreasing sequence of $k+1$ action values $a_0 > \cdots > a_k$, it is sufficient to have a chain of k non-stationary Floer strips $u_1, \ldots, u_k$ with boundary on L; here *non-stationary* means the energy of the strip is non-zero, and *chain* means the positive asymptotics of u_j equals the negative asymptotic of u_{j+1}. See Figure 1 below for an illustration of such a chain.

Given such a chain, the sum of the energies of the u_j bounds the difference $a_0 - a_k$. Thus, if it is possible to construct chains of non-constant Floer strips satisfying a total energy bound less than ρ_L, one obtains the desired chain of action values.

One way to conclude a non-constant Floer strip u for the system ϕ_t relative the Lagrangian L is to require that $u(0,0)$ lies on a smooth cycle $f : P \to L$ which is disjoint from $\phi_1(L)$; here P is a compact manifold, without boundary, and f is a smooth map which we think of as representing a homology class. Such curves are used to define a *cap-action* of f on the Lagrangian Floer cohomology. In the weakly exact setting, well-known arguments using this cap-action explain how to construct chains of Floer strips whose length is the cup-length; we recall the arguments in §1.3.1.

The main difficulty in generalizing this argument is the bubbling of J-holomorphic disks. For one, the bubbling phenomenon impedes us from defining Lagrangian Floer cohomology and Lagrangian cap-action.

In §1.3.2 and §1.3.3 we explain how to construct chains of Floer strips of length k, with total action bound $a_0 - a_k \leq \gamma(\phi_t)$. The approach in §1.3.2 is based on the module action of Hamiltonian Floer cohomology on Lagrangian Floer cohomology considered in [KS21]; their Lagrangian Floer cohomology is only defined in action windows smaller than the disk bubbling threshold. In §1.3.3, we explain how to deform the pair-of-pants product on Hamiltonian Floer cohomology using a compact Lagrangian in such a way which circumvents the need to consider Lagrangian Floer cohomology entirely, while still concluding a configuration of strips as in Figure 1.

1.3.1. *The Lagrangian cap-action in the weakly exact case.* In the weakly-exact setting, Lagrangian Floer cohomology $\mathrm{HF}(L, \phi_t)$ is well-defined, since there is no disk bubbling; see, e.g., [KS21]. The PSS isomorphism provides an identification:

$$\mathrm{PSS}_{L,\phi_t} : H(L) \to \mathrm{HF}(L, \phi_t).$$

Moreover, every bordism class[2] Π of smooth maps $f : P \to L$, has a corresponding Lagrangian cap action map:

$$\mathrm{cap}_{\Pi} : \mathrm{HF}(L, \phi_t) \to \mathrm{HF}(L, \phi_t).$$

[2]Bordism classes of smooth maps are a useful way to represent homology classes and their intersection product. In this footnote we rapidly overview the relevant definitions. Two smooth maps f_0, f_1 valued in L, defined on compact manifolds P_0, P_1, are *cobordant* if there exists a cobordism Q from P_0 to P_1 and a smooth map $F : Q \to L$ which extends the maps f_0, f_1 on the boundary components P_0, P_1. The cobordism relation is an equivalence relation, and an equivalence class is called a *bordism class*. Two bordism classes Π_0 and Π_1 have a natural *intersection product* defined as the fiber product of any two transverse representatives. Standard results in transversality theory, as in [Mil65], imply that the bordism class of the resulting fiber product is independent of the choice of transverse representatives.

On the chain level, the map is defined by picking a representative f of Π and counting Floer strips $u : \mathbb{R}\times[0,1] \to W$ satisfying $u(\mathbb{R}\times\{0\}), u(\mathbb{R}\times\{1\}) \in L$ and $u(0,0) \in f(P)$. The cap action is associative:

$$\mathrm{cap}_{\Pi\cap\Pi'} = \mathrm{cap}_{\Pi} \circ \mathrm{cap}_{\Pi'},$$

and is compatible with the PSS isomorphism:

$$\mathrm{cap}_{\Pi}(\mathrm{PSS}_{L,\phi_t}(\Pi')) = \mathrm{PSS}_{L,\phi_t}(\Pi \cap \Pi');$$

see e.g., [LO96, §4] and [Flo89b] for associativity and, e.g., [PSS96, §3] and [Sch00, §2.3] for compatibility with PSS. The weakly-exact open-string case is handled analogously to the closed-string case.

It is a convenient fact that the cup-length in unoriented bordism is the same as cup-length in singular cohomology with $\mathbb{F}_2$ coefficients; see [Tho54, Theorem III.2], and, e.g., [Wil20, §3.4] and [BH81, Theorem B]. Therefore, there exist bordism classes $\Pi_1, \dots, \Pi_k$ of maps $f_i : P_i \to L$ of positive codimension, for $i = 1, \dots, k$, such that the intersection product $\Pi_1 \cap \dots \cap \Pi_k$ equals the point class [pt]. Because the codimension of Π_i is ≥ 1, we can make the images of f_i disjoint from $\phi_1(L) \cap L$ (assuming the intersections form an isolated set).

On the one hand, the cap action of the point class is non-trivial since:

$$\mathrm{cap}_{[\mathrm{pt}]}(\mathrm{PSS}_{L,\phi_t}([L])) = \mathrm{PSS}_{L,\phi_t}([\mathrm{pt}] \cap [L]) = \mathrm{PSS}_{L,\phi_t}([\mathrm{pt}]).$$

On the other hand, by associativity, we have:

$$\mathrm{cap}_{[\mathrm{pt}]} = \mathrm{cap}_{\Pi_1} \circ \dots \circ \mathrm{cap}_{\Pi_k}.$$

The non-triviality of the above chain of compositions implies, in particular, that there exists a sequence of k Floer strips with point constraints as illustrated in Figure 1.

The strict inequalities $\mathcal{A}_{\phi_t}(\gamma_{j,-}) < \mathcal{A}_{\phi_t}(\gamma_{j,+})$, for all $j \in \{1, \dots, k\}$, follow from the fact that each Floer strip is non-stationary because of the incidence constraint. Thus, there are at least $k+1$ action values, which concludes the argument.

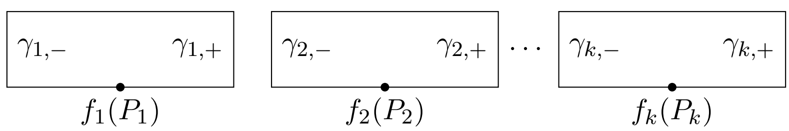

FIGURE 1. A sequence of Floer strips with point constraints and uniformly bounded energy converges-up-to-breaking to a sequence of Floer strips, satisfying action bounds $\mathcal{A}_{\phi_t}(\gamma_{j+1,-}) \leq \mathcal{A}_{\phi_t}(\gamma_{j,+}) < \mathcal{A}_{\phi_t}(\gamma_{j,-})$.

1.3.2. *An algebraic approach.* The approach in this section is heavily inspired by [KS21, Theorem E]. It relies on defining Lagrangian Floer cohomology, along with zero curvature operations, in an action window that is small enough to prevent bubbling yet sufficiently large to detect cohomological information of the Lagrangian. While this approach can likely be generalized to the convex-at-infinity setting, in this section we restrict ourselves to the case where L is a monotone Lagrangian of a closed symplectic manifold (W, ω) since the tools required have been carefully defined in [KS21].

Given a Hamiltonian system ϕ_t and an ω-compatible almost complex structure J, for each interval I of length $|I| < \hbar(J, L)$, let $\mathrm{CF}(\phi_t, L; \mathcal{D})^I$ be the Floer complex generated by capped (contractible) chords whose action values belong to the interval I; one should suppose that the endpoints of I are not action values of chords. For generic perturbation data $\mathcal{D}$, the differential is well defined since bubbling is prevented by the narrow action window. We denote by $\mathrm{HF}(\phi_t, L; \mathcal{D})^I$ the corresponding cohomology. As in §1.3.1, it is possible to define a Lagrangian cap-action associated to a bordism class Π. Depending on the action window, these maps could very well be trivial.

A new input compared to §1.3.1 is the following: to a cocycle $z \in \mathrm{CF}(\psi_t; \mathcal{D})$ in the Hamiltonian Floer cohomlogy of action $\mathcal{A}_{\psi_t}(z) = c$, there corresponds a multiplication operation:

$$[\mu(-, z)] : \mathrm{HF}(L, \phi_t; \mathcal{D})^I \to \mathrm{HF}(L, \psi_t \circ \phi_t; \mathcal{D})^{I+c+\epsilon},$$

where $\epsilon > 0$ is an error term related to the perturbation data $\mathcal{D}$. In short, $\mu(-, z)$ is defined by counting rigid one-punctured Floer strips with Lagrangian boundary conditions, whose ends are asymptotic to Hamiltonian chords of ϕ_t and $\psi_t \circ \phi_t$ and whose interior puncture is asymptotic to a one-periodic orbit of ψ_t belonging to the linear combination expressing z; see Figure 2.

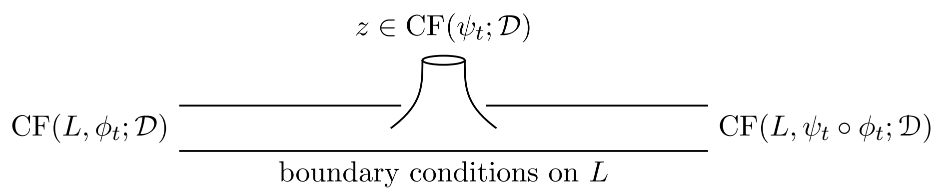

FIGURE 2. The module action of Hamiltonian Floer cohomology on Lagrangian Floer cohomology; see [KS21, §4.1].

The goal is to find an action interval I of length less than $\hbar(L)$ on which composing k times the restricted Lagrangian cap-action is a non-trivial operation, as this guarantees the existence of a chain of k Floer strips exactly as in §1.3.1.

Suppose ϕ_t is a Hamiltonian system with $\gamma(\phi_t) < \hbar(L)$. Essentially by the definition of the spectral norm, there are cocycles $x \in \mathrm{CF}(\phi_t^{-1}; \mathcal{D})$ and $y \in \mathrm{CF}(\phi_t; \mathcal{D})$, representing the unit elements, with actions:

$$u = \mathcal{A}_{\phi_t^{-1}}(x) = c(1, \phi_t^{-1}) \quad \text{and} \quad v = \mathcal{A}_{\phi_t}(y) = c(1, \phi_t),$$

so that $u+v = \gamma(\phi_t)$. The idea is to prove that, for some interval I of length shorter than $\hbar(L)$, the following composition:

$$(3) \quad [\mu(-, y)] \circ \mathrm{cap}_{[\mathrm{pt}]} \circ [\mu(-, x)] : \mathrm{HF}(L, \mathrm{id}; \mathcal{D})^I \to \mathrm{HF}(L, \mathrm{id}; \mathcal{D})^{I+\gamma(\phi)+2\epsilon}$$

is non-trivial, and hence $\mathrm{cap}_{[\mathrm{pt}]} : \mathrm{HF}(L, \phi_t; \mathcal{D})^{I+u+\epsilon} \to \mathrm{HF}(L, \phi_t; \mathcal{D})^{I+u+\epsilon}$ is non-trivial.

Note that $\mathrm{cap}_{[\mathrm{pt}]} = \mathrm{cap}_{\Pi_1} \circ \cdots \circ \mathrm{cap}_{\Pi_k}$ by the associativity of the Lagrangian cap-action, and once we know this k-fold composition is non-trivial, the argument proceeds exactly as in §1.3.1. Therefore, it is enough to show that (3) is non-trivial in the specified action windows. To this end, we recall the associativity relation:

$$[\mu(-, y)] \circ \mathrm{cap}_{[\mathrm{pt}]} \circ [\mu(-, x)] = \mathrm{cap}_{[\mathrm{pt}]} \circ [\mu(-, \mu_2(x, y))],$$

from [KS21, §5.1, (20)], where $\mu_2(x, y)$ represents the unit class in $\mathrm{CF}(\mathrm{id}; \mathcal{D})$. Furthermore, $\Phi = [\mu(-, \mu_2(x, y))]$ induces the interval shift map:

$$\Phi : \mathrm{HF}(L, \mathrm{id}; \mathcal{D})^I \to \mathrm{HF}(L, \mathrm{id}; \mathcal{D})^{I+\gamma(\phi_t)+2\epsilon}.$$

To conclude the argument, we observe that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{HF}(L, \mathrm{id}; \mathcal{D})^I & \longrightarrow & \mathrm{HF}(L, \mathrm{id}; \mathcal{D})^{I+\gamma(\phi_t)+2\epsilon} \\ \uparrow & & \uparrow \\ H(L) & \xrightarrow{\quad \cap[\mathrm{pt}] \quad} & H(L), \end{array}$$

where the top horizontal arrow is the composition $\mathrm{cap}_{[\mathrm{pt}]} \circ \Phi$ and the vertical arrows are the PSS morphisms. Moreover, *if the intervals I and $I + \gamma(\phi_t) + 2\epsilon$ each contain* 0*, then the vertical maps are injective*, and hence the top map is non-trivial, as desired. Thus the problem boils down to finding an interval I such that $|I| < \hbar(L)$ and such that $I, I + \gamma(\phi_t) + 2\epsilon$ both contain 0. It is clear that a necessary and sufficient condition for this to hold is that $\gamma(\phi_t) < \hbar(L)$. This concludes the sketch of the proof of Theorem 1.1 in the closed monotone setting using the framework introduced by [KS21].

1.3.3. *A moduli space approach.* The approach to proving Theorem 1.1 in this paper is to deform the pair-of-pants operation on Hamiltonian Floer cohomology using the compact Lagrangian L. The deformation is illustrated in Figures 3 and 4. The details of the deformation argument are in §2.6. Briefly, we compose the pair-of-pants product (see §2.5), and then apply an augmentation map defined using boundary conditions on L (see §2.6.1). This composition gives the configuration shown in Figure 3.

The crux of the matter is to construct curves with Lagrangian boundary condition which contain conformally embedded strips with a large modulus, as in Figure 4. As explained in Proposition 2.18 there always exist solutions in the deformed moduli space because the count of solutions represents a homologically non-trivial operation.

By an appropriate compactness argument, one concludes the existence of chains of Floer strips needed to prove Theorem 1.1; the argument proceeds as in §1.3.

There is one subtlety in the proof of Theorem 1.1 which we explain here. The arguments in §2.6 imply that for each Hamiltonian system ϕ_t with $\phi_1 = \phi$ there is an interval of length $\gamma(\phi_t)$ containing at least $\mathrm{cl}_{\mathbb{F}_2}(L) + 1$ action values. It is a standard fact that if ϕ_t' is another Hamiltonian system with

$\phi_1' = \phi$ then the action spectrums of ϕ_t and ϕ_t' coincide up to a shift by a constant depending only on the Hamiltonian loop $\phi_t' \circ \phi_t^{-1}$; see, for example, [KS21, Proposition 31].

Therefore, for all Hamiltonian systems ϕ_t generating ϕ, it is possible to find a closed interval $I(\phi_t')$ of length $\gamma(\phi_t')$ containing $\mathrm{cl}_{\mathbb{F}_2}(L) + 1$ action values of $\mathcal{A}_{\phi_t}$. The infimum of $\gamma(\phi_t')$ over ϕ_t' equals the spectral norm $\gamma(\phi)$. In the weakly-exact case the intervals $I(\phi_t')$ must remain in a fixed compact set (because $\phi_1(L) \cap L$ is finite). Otherwise, one can exploit the periodicity of the action spectrum and shift $I(\phi_t')$ to ensure the intervals do not drift off to infinity. In either case, a standard compactness argument for shrinking intervals contained in a compact set ensures the existence of an interval of length $\gamma(\phi)$ containing $\mathrm{cl}_{\mathbb{F}_2}(L) + 1$ action values.

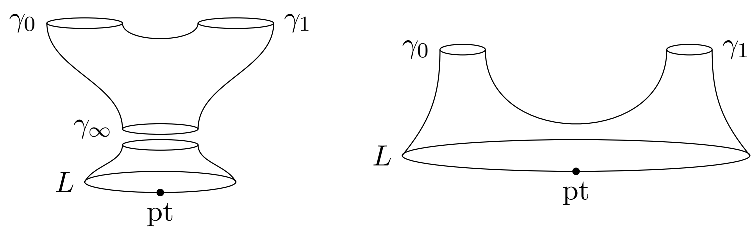

FIGURE 3. (left) Gluing the pair-of-pants onto a half-infinite cylinder; (right) deforming the conformal structure of the resulting Riemann surface.

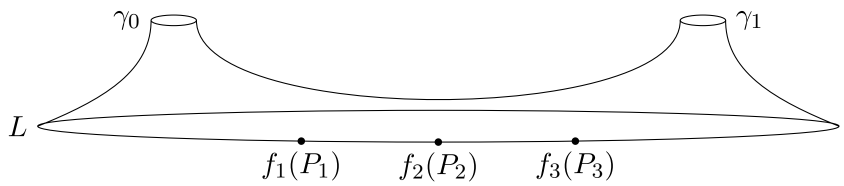

FIGURE 4. Deforming the conformal structure, and splitting the point constraints. The homological count of elements of such a deformed pair-of-pants will equal 1 if the intersection of the bordism classes equals the point class pt.

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2. *Floer cohomology in convex-at-infinity symplectic manifolds*

2.1. *Convex ends.* A convex end is a non-compact symplectic manifold modelled on the positive half of the symplectization of a contact manifold; see [EG91], [CE12, §11], and [Gin05, MS12, FS07, Lan13, Lan16]. Below, we define the class of symplectic manifolds called *convex-at-infinity* described in §1.2. In §2.1.2, it is shown that every convex-at-infinity manifold W can be expressed as the completion of a compact symplectic manifold Ω with contact type boundary $\partial\Omega$ (allowing $\partial\Omega = \emptyset$); here *contact type* is understood in the sense of [Wei79, McD91].

2.1.1. *Definition of a convex end.* Let (W, ω) be a symplectic manifold. Suppose there is a complete vector field Z whose time s flow ρ_s satisfies the following properties:

(i) for any sequence z_n there is a sequence $S_n < 0$ such that $\rho_{s_n}(z_n)$ has a convergent subsequence whenever $s_n < S_n$,
(ii) there is a compact set K_1 such that, if $\rho_{s_n}(z_n)$ and z_n converge, and z_n is not in K_1, then s_n is bounded from above,
(iii) $(\rho_s^*\omega)_z = e^s\omega_z$ holds for $s > 0$ and z outside a compact set K_2.

Note that if $Z_1 = Z_2$ holds outside a compact set, and Z_1 satisfies the above properties, then so does Z_2, and we say Z_1, Z_2 are equivalent. A *convex end* on W is an equivalence class of such vector fields Z. If W is a compact symplectic manifold, then we can take $K_1 = K_2 = W$, in which case the axioms are trivially satisfied for any vector field.

The flow by Z (outside K_2) will be referred to as the *Liouville flow*.

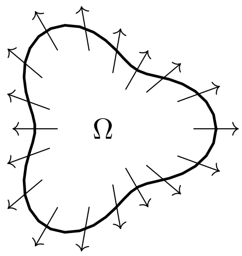

FIGURE 5. Every convex-at-infinity symplectic manifold W is the completion of a star-shaped domain Ω. In the case when W is closed, we have that $W = \Omega$ and $\partial\Omega = \emptyset$. In the case when W is open, the characteristic foliation of the boundary $\partial\Omega$ is the Reeb flow for some choice of contact form on the ideal boundary Y.

2.1.2. *Structural results about convex ends.* Let V be the open set of points in W lying on trajectories of Z which pass through K_1^c. Axioms (i) and (ii) ensure that ρ_s defines a free and proper $\mathbb{R}$-action on V; the details of the argument are left to the reader.

It follows that $V \to V/\mathbb{R}$ is a smooth submersion to a manifold whose fibers are the trajectories of Z. We claim that $V/\mathbb{R}$ is a compact manifold. Indeed, pick a sequence z_n in V and choose a compact neighborhood K_1' of K_1 which contains K_1 in its interior. There is s_n so $\rho_{s_n}(z_n) \in K_1'$, because the limit points of $\rho_s(z_n)$ as s converges to $-\infty$ are non-empty, by (i), and must be contained in K_1 by the properness established above. Then $\rho_{s_n}(z_n)$ has a convergent subsequence since K_1' is compact. Thus $V/\mathbb{R}$ is compact.

A variant of the Ehresmann fibration theorem implies that $V \to V/\mathbb{R}$ is a fiber bundle whose structure group is the group of translations on $\mathbb{R}$. Thus $V \to V/\mathbb{R}$ admits a section, denoted $\partial\Omega$. It follows that Z is transverse to $\partial\Omega$, and the flow map takes $\partial\Omega \times [0,\infty)$ onto a neighborhood of ∞ for W. Note that $\partial\Omega$ bounds a compact domain Ω in W, by axiom (i).

Replacing $\partial\Omega$ by $\rho_s(\partial\Omega)$, we may assume from axiom (iii) that $\rho_s^*\omega = e^s\omega$ holds on the image of $\partial\Omega \times [0,\infty)$. The region $\partial\Omega \times [0,\infty)$ is symplectomorphic to the positive half of a symplectization of $\partial\Omega$ with contact form $\alpha = \lambda|_{\partial\Omega}$, where $\lambda = \omega(Z,-)$. Let us refer to such a domain Ω as *star-shaped.*

The induced contact distribution on $V/\mathbb{R}$ is independent of Ω and the resulting contact manifold is called the *ideal contact boundary* of W.

2.1.3. *Reeb flow in the convex end.* A star-shaped domain Ω induces a function r by requiring that r is 1-homogeneous outside of Ω and $r|_{\partial\Omega} = 1$. The function r should be extended smoothly to all of W, in such a way that $r \le 1$ holds on Ω.

The restriction of λ to $\partial\Omega$ defines a contact form α for the ideal contact boundary. The Reeb flow for α is the ideal restriction of the Hamiltonian system generated by r.

For $r_0 \ge 1$, we introduce the domain $\Omega(r_0)$ where $r \le r_0$.

Let f be a smooth convex function such that $f(x) = x$ for $x \ge 1$, and $f(x) = 0$ for $x \le 0$. Consider $f(r - r_0) + r_0$ as generating an autonomous Hamiltonian system, and let R_t^α denote the flow induced by $X_{f(r-r_0)+r_0}$. This system equals the identity on $\Omega(r_0)$ and equals the Reeb flow for α on $\Omega(r_0 + 1)$.

2.1.4. *The group of Reeb-Hamiltonian isotopies.* Let $\mathcal{RH}(W; R^\alpha, r_0 + 1)$ be the space of smooth functions $H : W \to \mathbb{R}$ such that:

$$H = ar + c$$

where a, c are locally constant outside of $\Omega(r_0 + 1)$. When W is compact, the space consists of all smooth functions $W \to \mathbb{R}$.

Let $\mathrm{RHI}(W; R^\alpha, r_0 + 1)$ be the group of Hamiltonian isotopies $\varphi_t : W \to W$ whose generating functions H_t are in $\mathcal{RH}(W; R^\alpha, r_0 + 1)$, and such that:

$$\varphi_{t+1} = \varphi_t\varphi_1$$

defines a smooth extension to all $t \in \mathbb{R}$; the latter is equivalent to requiring that $(x, t) \mapsto H_t(x)$ is smooth on $W \times \mathbb{R}/\mathbb{Z}$.

Isotopies in $\mathrm{RHI}(W; R^\alpha, r_0 + 1)$ act as reparametrizations of the Reeb flow generated by R^α outside of $\Omega(r_0 + 1)$, and this property characterizes them. Consequently, $\mathrm{RHI}(W; R^\alpha, r_0 + 1)$ is closed under products and inverses.

Let ϕ_t be a compactly supported Hamiltonian isotopy, as in Theorem 1.1. Henceforth we fix r_0 large enough that ϕ_t is supported in $\Omega(r_0)$, and also that L is contained in $\Omega(r_0)$. A key role will be played by the isotopies $R^\alpha_{st} \circ \phi_t$, where R^α_{st} is as in §2.1.3. It is important to note that, because we have chosen r_0 large enough, it holds that $R^\alpha_{st} \circ \phi_t \in \mathrm{RHI}(W; R^\alpha, r_0 + 1)$.

Henceforth, we will write $\mathrm{RHI} = \mathrm{RHI}(W; R^\alpha, r_0 + 1)$, and similarly for $\mathcal{RH}$; the number r_0 and the contact form α will be implicit.

Finally, let us note that, if W is compact, then RHI is simply the usual group of Hamiltonian isotopies (and all of this discussion is unnecessary in this case).

2.2. *Cappings and the Hamiltonian action functional.* The Hamiltonian action functional is defined on a suitable covering space of the space of contractible loops in W or paths in W with endpoints on the Lagrangian L.

2.2.1. *Cappings of chords and orbits.* A contractible loop or path γ can be joined to a constant loop or path via a smooth map $u : [0, 1]\times S \to W$, where $S = \mathbb{R}/\mathbb{Z}$ or $S = [0, 1]$, such that (i) $u(0, t)$ is constant, (ii) $u(s, 0), u(s, 1) \in L$ in the case $S = [0, 1]$, and (iii) $u(1, t) = \gamma(t)$. Two such maps u_1, u_2 are equivalent provided their symplectic areas are the same and an equivalence class $[u]$ is called a *capping* of γ. The projection $(\gamma, [u]) \mapsto \gamma$ is a covering space.

2.2.2. *The Hamiltonian action functional.* Given an isotopy $\varphi_t \in \mathrm{RHI}$ one defines the Hamiltonian action functional on the covering space by:

$$\mathcal{A}_{\varphi_t}(\gamma, [u]) := \int_0^1 H_t(\gamma(t))\,\mathrm{d}t - \int u^*\omega, \tag{4}$$

where H_t is the unique normalized time-dependent family of smooth functions generating φ_t; see §2.2.3 for the normalization conditions. It is well-known, and easy to check, that the critical points of $\mathcal{A}_{\varphi_t}$ are lifts of contractible orbits or chords of the system φ_t.

2.2.3. *Normalization conditions.* Throughout this paper we assume that W is connected. We do not assume that the ideal boundary Y is connected, and for this reason we pick a connected component in Y to be the *distinguished* component. This choice is only used to define the normalization condition for Hamiltonian functions.

If (W, ω) is compact, say that $H \in \mathcal{RH}$ is *normalized* if H has mean zero with respect to the volume form ω^n; if (W, ω) is non-compact, say that $H \in \mathcal{RH}$ is *normalized* if $H = ar$ holds in the distinguished convex end. In other words, H is normalized if it is one-homogeneous in the distinguished non-compact end.

Crucially: *any constant normalized function is zero.*

In particular, given $\varphi_t \in \mathrm{RHI}$, there exists a *unique* normalized generating Hamiltonian H_t.

2.3. *Geometric preliminaries on Floer's equation.* In this paper we will consider moduli spaces of solutions to PDEs defined for maps $u : \Sigma \to W$ where $(\Sigma, \partial\Sigma, j)$ is a Riemann surface, potentially with boundary. The PDE depends on two additional inputs:

(i) an almost complex structure J on W,
(ii) a Hamiltonian connection $\mathfrak{H}$ on $W \times \Sigma \to \Sigma$,

The equation takes the following form: in local conformal coordinates $s+it$ on Σ, the Hamiltonian connection $\mathfrak{H}$ determines two Hamiltonian vector fields $V_{s,t}, X_{s,t}$ on W, varying smoothly on the domain of the chart, such that any solution u satisfies:

$$(\partial_s u - V_{s,t}(u)) + J(u)(\partial_t u - X_{s,t}(u)) = 0$$

in this coordinate chart.

In this section, we will define these moduli spaces and their inputs precisely, and explain the geometric aspects of this PDE relevant to our paper. In the later sections, we will explain how these moduli spaces are used to define the algebraic manipulations (Floer cohomology, continuation maps, pair-of-pants products, Lagrangian augmentation map, etc), culminating in the proof of Theorem 1.1.

2.3.1. *Metric structures.* A Riemannian metric g is said to be *translation-invariant in the end* provided the Liouville flow ρ_s acts by an isometry for g, outside of $\Omega(r_0+1)$ and for $s \geq 0$.

2.3.2. *Almost complex structures.* An almost complex structure J is said to be *admissible* provided that:

(i) $\omega(v, Jv)$ is positive for all non-zero tangent vectors v,
(ii) $J_{\rho_s(x)} \circ \mathrm{d}\rho_{s,x} = \mathrm{d}\rho_{s,x} \circ J_x$ holds for $x \notin \Omega(r_0+1)$ and $s \geq 0$.

Notice that in this case $\omega(v, Jv) \geq cg(v,v)$ holds some positive constant c, assuming that g is translation-invariant in the end.

There is another property of admissibility that we will require. To state it, we require introducing the moduli space of simple J-holomorphic spheres $\mathcal{M}^*(J)$ of solutions to:

$$\begin{cases} u : \mathbb{C}P^1 \to W, \\ \partial_s u + J(u)\partial_t u = 0, \\ u \text{ has an injective point.} \end{cases}$$

This moduli space is studied in detail in [MS12]. Each solution $u \in \mathcal{M}^*(J)$ determines a well-defined linearized operator, which is a linear elliptic differential operator:

$$D_u : u^*TW \to \Lambda^{0,1} \otimes u^*TW,$$

as explained in, e.g., [MS12]. We require that our J satisfies:

(iii) the cokernel of D_u is trivial for each $u \in \mathcal{M}^*(J)$.

As proven in [MS12], this property is generic and can be achieved by pertubing J in a C^∞-small way on $\Omega(1)$; here we note that every non-constant solution of $\mathcal{M}^*(J)$ must pass through $\Omega(1)$ because the symplectic form is exact outside of $\Omega(1)$.

For later use, we will also use the following property:

Lemma 2.1. *If J satisfies (i) and (ii), then for each number A, there is a compact set K such that every J-holomorphic plane $u : \mathbb{C} \to W$, or upper half-plane $u : \mathbb{H} \to W$ with $u(\mathbb{R}) \subset L$, satisfying:*

$$\int u^*\omega < A$$

is contained inside of K.

Proof. This is a consequence of Gromov's monotonicity theorem from [Gro85], using that J satisfies $\omega(v, Jv) \geq g(v, v)$ for some metric g which is translation invariant in the ends as in §2.3.1; see also the mean-value property approach of [RS01, MS12, CC23]. □

2.3.3. *Reeb-Hamiltonian connections.* A Reeb-Hamiltonian connection is a special type of Ehresmann connection $\mathfrak{H}$ on the trivial bundle $W \times \Sigma \to \Sigma$, i.e., $\mathfrak{H}$ is a smoothly varying linear subspace of $T(W \times \Sigma)$ which is everywhere complementary to the vertical distribution TW.

Each $\mathfrak{H}$ is generated by a *connection one-form*, which is any one-form $\mathfrak{a}$ on $W \times \Sigma$ which can locally be written in the form:

$$\mathfrak{a} = K_{s,t}\mathrm{d}s + H_{s,t}\mathrm{d}t, \tag{5}$$

above any local coordinate chart on Σ, where $K_{s,t}, H_{s,t}$ are smoothly varying elements of $\mathcal{RH}$. For use in achieving transversality for moduli spaces, we will also allow a perturbation term:

$$\mathfrak{p} = k_{s,t}\mathrm{d}s + h_{s,t}\mathrm{d}t$$

where $k_{s,t}, h_{s,t}$ are bounded smooth functions (not assumed to be in $\mathcal{RH}$). To simplify later equations, pick a compact coordinate disk $D \subset \Sigma$ disjoint from the boundary, and suppose $\mathfrak{p}$ is supported in $W \times D$.

To generate the connection $\mathfrak{H}$ from $\mathfrak{a}$ and $\mathfrak{p}$, one introduces the closed two-form $\Omega = \mathrm{pr}^*\omega - \mathrm{d}(\mathfrak{a} + \mathfrak{p})$ and defines:

$$\mathfrak{H} = TW^{\perp\Omega},$$

i.e., $\mathfrak{H}$ is a Ω-dual to the vertical distribution. We refer the reader to §A and [MS12, §8] for a review of the differential geometry of such connections.

Importantly, if $\mathfrak{H}$ is the Reeb-Hamiltonian connection satisfying (5) above a coordinate chart $s + it$ on Σ, and $V_{s,t}, X_{s,t}$ are the Hamiltonian vector fields associated to $K_{s,t} + k_{s,t}, H_{s,t} + h_{s,t}$, then $\partial_s + V_{s,t}$ and $\partial_t + X_{s,t}$ are the unique vector fields on $W \times \Sigma$ tangent to $\mathfrak{H}$ which project to ∂_s, ∂_t on Σ.

If $\mathfrak{H}$ is a Reeb-Hamiltonian connection, then its *curvature two-form* is defined to be the two-form on $W \times \Sigma$ given by the local expression:

$$\mathfrak{r} = (\partial_s(H_{s,t} + h_{s,t}) - \partial_t(K_{s,t} + k_{s,t}) + \omega(V_{s,t}, X_{s,t}))\mathrm{d}s \wedge \mathrm{d}t; \tag{6}$$

this quantity is independent of the local coordinates $s+it$ chosen on Σ, and defines a global smooth two-form. It is important to note that the coefficient of the curvature two-form of an unperturbed connection ($\mathfrak{p}=0$) is valued in the space of functions $\mathcal{RH}$.

Lemma 2.2. *The curvature function r for the perturbed connection differs from the curvature function for the unperturbed connection by an error bounded by the C^1 norms of $h_{s,t}$ and $k_{s,t}$ and the C^0 norms of $\partial_s h_{s,t}$ and $\partial_t k_{s,t}$, with respect to a metric which is translation invariant in the end.*

Proof. A straightforward computation shows that the new curvature function differs from the old curvature function by the amount:

$$\partial_s h - \partial_t k + \mathrm{d}h(X_K) - \mathrm{d}k(X_H + X_h),$$

suppressing the subscripts s,t from the notation.

Since $\omega(X_h,-)=\mathrm{d}h$, one can prove that $\|X_h\|$ is bounded by a constant times $\|\mathrm{d}h\|$. Because H,K are in $\mathcal{RH}$, X_H, X_K are equivariant with respect to translations, so $|\mathrm{d}h(X_K)|$ is bounded in terms of $\|\mathrm{d}h\|$, and similarly $|\mathrm{d}k(X_H+X_h)|$ is bounded in terms of $\|\mathrm{d}k\|$ and $\|\mathrm{d}k\|\,\|\mathrm{d}h\|$. Combining everything yields the desired result. □

As we will see below, the unperturbed connection having a non-positive curvature two-form is needed to establish important a priori estimates.

2.3.4. *Floer's equation for a Reeb-Hamiltonian connection.* Let $\mathfrak{H}$ be a Reeb-Hamiltonian connection and let J be an admissible almost complex structure. Associated to these choices, let $J^{\mathfrak{H}}$ be the unique almost complex structure on $W\times\Sigma$ such that:

(i) the fibers $W\times\{z\}\subset W\times\Sigma$ are almost complex submanifolds,
(ii) $\mathfrak{H}$ is a $J^{\mathfrak{H}}$-line, and,
(iii) the projection $(\mathfrak{H}, J^{\mathfrak{H}})\to(T\Sigma, j)$ is complex-linear.

A smooth map $u:\Sigma\to W$ is said to solve *Floer's equation with data* $(\Sigma,\mathfrak{H},J)$ provided the induced section $z\mapsto(z,u(z))\in\Sigma\times W$ is $J^{\mathfrak{H}}$-holomorphic. The moduli space of all solutions is denoted $\mathcal{M}(\Sigma,\mathfrak{H},J)$. See [Gro85, §1.4.C',§2.2] and [MS12, §8] for related discussion.

In local holomorphic coordinates $z=s+it$ we can decompose the derivatives of $z\mapsto(z,u(z))$ according to the decomposition of $T(W\times\Sigma)$ by:

$$\begin{cases}\partial_s+\partial_s u=(\partial_s u - V_{s,t}(u))+(\partial_s+V_{s,t}(u))\in TW\oplus\mathfrak{H},\\ \partial_t+\partial_t u=(\partial_t u - X_{s,t}(u))+(\partial_t+X_{s,t}(u))\in TW\oplus\mathfrak{H}.\end{cases}$$

Here we use that $\partial_s+V_{s,t}$ and $\partial_t+X_{s,t}$ are tangent to $\mathfrak{H}$, as mentioned above. It is then clear that $u\in\mathcal{M}(\Sigma,H,J)$ if and only if:

$$(\partial_s u - V_{s,t}(u)) + J(u)(\partial_t u - X_{s,t}(u)) = 0, \tag{7}$$

as we had written at the start of §2.3.

2.3.5. *Energy identity for solutions to Floer's equation.* Let $u \in \mathcal{M}(\Sigma, \mathfrak{H}, J)$. We define the *energy density two-form* to be the quantity:

$$\omega(\partial_s u - V_{s,t}(u), \partial_t u - X_{s,t}(u))\mathrm{d}s \wedge \mathrm{d}t.$$

This quantity is independent of the choice of local conformal coordinates. Indeed, if $\Pi_{\mathfrak{H}} : T(W \times \Sigma) \to TW$ is the projection whose kernel is $\mathfrak{H}$, then the energy density two-form is equal to $\omega(\Pi_{\mathfrak{H}}\mathrm{d}u, \Pi_{\mathfrak{H}}\mathrm{d}u)$, which is manifestly coordinate invariant.

Since $\partial_s u - V_{s,t}$ and $\partial_t u - X_{s,t}$ span a J-complex line, and J is admissible, the energy density is everywhere non-negative with respect to the complex orientation of Σ. We define the *energy* $E(u)$ to be the integral of the energy density two-form.

At this stage it is convenient to redefine $\mathcal{M}(\Sigma, \mathfrak{H}, J)$ to be the moduli space of *finite energy* solutions to Floer's equation. We will never consider solutions which have infinite energy.

The following computation is a cornerstone of Floer theory; see [MS12, Lemma 8.1.6] and [Sei08b, §8g] for similar identities.

Lemma 2.3. *Let $u \in \mathcal{M}(\Sigma, \mathfrak{H}, J)$ where Σ is a compact Riemann surface with boundary $\partial\Sigma$. Then:*

$$E(u) = \omega(u) + \int u^*\mathfrak{r} - \int_{\partial\Sigma} u^*\mathfrak{a},$$

where $\mathfrak{a}$ and $\mathfrak{p}$ are the connection one-forms for $\mathfrak{H}$, $\mathfrak{r}$ is the curvature two-form from (6)*, and $\omega(u)$ is the symplectic area of u.*

Proof. Fix a conformal chart $s + it$ on Σ. We prove it in the case $\mathfrak{p} = 0$ as the argument is exactly the same when $\mathfrak{p} \neq 0$ with just more notation. The local contribution to the energy is given by:

$$E(u) = \int \omega(\partial_s u - V_{s,t}(u), \partial_t u - X_{s,t}(u))dsdt.$$

Straightforward and standard computations give:

$$E(u) = \int \omega(\partial_s u, \partial_t u) - \partial_s(H_{s,t}(u)) + \partial_t(K_{s,t}(u)) + r(u)dsdt,$$

where $r(u) = \partial_s H_{s,t} - \partial_t K_{s,t} + \omega(V_{s,t}, X_{s,t})$ is the coefficient for the curvature two-form in (6). The integrand therefore simplifies to $u^*\omega - u^*\mathrm{d}\mathfrak{a} + u^*\mathfrak{r}$. Patching together these local contributions proves the full energy is given by the desired formula. □

In §2.3.7, we apply this to a punctured Riemann surface Σ by exhausting Σ by compact Riemann surfaces with boundary.

2.3.6. *Cylindrical ends.* Let Σ be a compact Riemann surface with boundary and a set Γ of interior punctures, which we decompose as $\Gamma = \Gamma_+ \sqcup \Gamma_-$ into positive and negative punctures.

A *cylindrical coordinate system* near a puncture $z \in \Gamma_\pm$ is a choice of biholomorphism:

$$\epsilon_z : \Sigma_\pm \to U_z \backslash \{z\} \subset \Sigma$$

where U_z is a neighborhood of z and $\Sigma_{\pm} = \mathbb{R}_{\pm} \times \mathbb{R}/\mathbb{Z}$. Suppose that Σ is equipped with a choice of cylindrical coordinate systems around each puncture.

Let us say that a Reeb-Hamiltonian connection $\mathfrak{H}$ is *adapted* to Σ (with its choice of cylindrical coordinates) provided that its normalized connection one-form $\mathfrak{a}$ satisfies:

$$\mathfrak{a} = H_t^z \mathrm{d}t \text{ and } \mathfrak{p} = 0$$

where $H_t^z \in \mathcal{RH}$, in the cylindrical end corresponding to $z \in \Gamma$. The functions H_t^z and the systems $\varphi_t^z \in \mathrm{RHI}$ generated by H_t^z are called the *asymptotics* at the puncture $z \in \Gamma$. See [Sei08b, §8d] for related discussion.

2.3.7. *A priori energy estimate.* Let $L \subset W$ be a Lagrangian submanifold contained in $\Omega(r_0)$, and let J be an admissible almost complex structure.

This section concerns a priori energy bounds for solutions of:

$$\begin{cases} \Sigma \text{ is a punctured Riemann surface with boundary,} \\ \mathfrak{H} \text{ is a Reeb-Hamiltonian connection adapted to } \Sigma, \\ u \in \mathcal{M}(\Sigma, \mathfrak{H}, J), \\ u(\partial\Sigma) \subset L, \end{cases} \tag{8}$$

This equation is the template equation that we will use in defining the Floer theory operations used in the later parts of the paper.

Since u is a finite energy solution, the asymptotic convergence result for solutions to Floer's equation on cylindrical ends (see [Sal97]) implies $u(s,t)$ converges to an *asymptotic orbit* $\gamma_z(t)$ of φ_t^z in the cylindrical end corresponding to $z \in \Gamma_{\pm}$, as $s \to \pm\infty$.

Lemma 2.4. *The a priori energy estimate holds for all solutions of* (8)*:*

$$E(u) \le \omega(u) + \int_0^1 \sum_{z\in\Gamma_-} H_t^z(\gamma_z(t)) - \sum_{z\in\Gamma_+} H_t^z(\gamma_z(t)) dt - \int_{\partial\Sigma} u^*\mathfrak{a} + \mathrm{const}(\mathfrak{r}),$$

where $\mathrm{const}(\mathfrak{r}) = \sup\left\{\int_\Sigma v^*\mathfrak{r} : v \textit{ is a smooth map } \Sigma \to W\right\}$.

The quantity $\mathrm{const}(\mathfrak{r})$ *is finite if the curvature of the unperturbed connection is non-positive outside of a compact set.*

If $\mathrm{const}(\mathfrak{r})$ is finite, then we say that $\mathfrak{H}$ has *curvature bounded from above.* The lemma gives a necessary and sufficient condition for this to hold in terms of the unperturbed connection.

Proof. Apply Lemma 2.3 to the subdomain $P_N \subset \Sigma$ obtained by removing the half-infinite cylinders $[N,\infty) \times \mathbb{R}/\mathbb{Z}$ from the positive cylindrical ends and $(-\infty,-N] \times \mathbb{R}/\mathbb{Z}$ from the negative cylindrical ends. Taking the limit $N \to \infty$ yields the desired upper bound.

It remains to prove that $\mathrm{const}(\mathfrak{r})$ is finite if and only if $\mathfrak{H}$ has non-positive curvature outside of a compact set. Note that $\mathfrak{r}$ vanishes over the cylindrical ends because $\mathfrak{H}$ is adapted to Σ. Cover the complement of the cylindrical ends in Σ by finitely many compact disks $D(1)$. Over each such disk, we can write $\mathfrak{r} = r\mathrm{d}s \wedge \mathrm{d}t$. It therefore suffices to bound the smooth functions

r. As stated above, the curvature of the perturbed connection differs from the curvature of the unperturbed connection by a bounded function. Thus if the unperturbed connection's curvature is non-positive outside of $\Omega(r_0)$, the function r will have a maximum value. Since there are finitely many disks we obtain the desired bound on $\mathrm{const}(\mathfrak{r})$. □

2.3.8. *Bubbling.* The bubbling analysis required is standard and our arguments follow [HS95, §A], [MS12, §4], and [CC23, §C].

An a priori energy bound for a sequence of solutions to (8) guarantees a gradient bound unless bubbling occurs, see, e.g., [HS95, §A]. The argument, with some necessary adjustments, works in the general setting of Floer's equation associated to a Hamiltonian connection with Lagrangian boundary conditions, as in [MS12, §8].

Because we will use the semipositivity framework to achieve gradient bounds, it is important to know that bubbling implies the existence of a holomorphic sphere incident to the limiting solution to Floer's equation; for this incidence result, we refer the reader to [HS95, Theorem A.1.(iii)].

2.3.9. *Maximum principle.* In this section we prove a maximum principle.

Lemma 2.5. *Let* $(\Sigma_n, \mathfrak{H}_n, u_n)$ *be a sequence of solutions to* (8)*. Suppose* Σ_n *converges to* Σ_∞*, as in* [Gro85, Don11]*,* $\mathfrak{H}_n$ *curvature bounded from above and converges to* $\mathfrak{H}_\infty$*, and* $\omega(u_n)$ *remains bounded. Suppose moreover that the asymptotics of* $\mathfrak{H}_n$ *are independent of* n*, and their asymptotics have a time-1 map with a compact fixed point locus. Then there exists a compact set which contains the image of* u_n *for all* n*.*

Proof. Lemma 2.4 and our assumptions on the curvature of $\mathfrak{H}_n$ and on the symplectic area $\omega(u_n)$ ensure that u_n has uniformly bounded energy.

By bubbling analysis, we can remove finitely a disjoint union of finitely many small open disks or half-disks $B_n \subset \Sigma_n$ so that the derivative of u_n is bounded on compact subsets in the complement of B_n. Moreover, $u_n(B_n)$ remains a finite distance from a union of J-holomorphic spheres or disks on L. We have proved in Lemma 2.1 that such J-holomorphic spheres or disks lie in a fixed compact set, and so $u_n(B_n)$ remains in a fixed compact set.

Therefore it suffices to consider the piece of u_n on the complement of the disks B_n, where its first derivative is bounded.

Define $P_n \subset \Sigma_n$ to be a compact complement of cylindrical ends, and suppose $B_n \subset P_n$ by shrinking the cylindrical ends if necessary. We first claim that there exists $\sigma > 0$ such that $u_n(\Sigma_n \setminus P_n) \subset \Omega(e^\sigma r_0)$ for all n, where Ω is a star-shaped domain in W. This follows from the maximum principle in [BC23, Proposition 2.2] for finite length Floer cylinders; this is where we use the assumption that the time-1 maps of the asymptotics have a compact fixed point locus.

Therefore, if the maximum principle fails for u_n, it must fail on the compact subset P_n. Since, the image of P_n under u_n intersects a fixed compact subset of W for every n and the sequence P_n satisfies the gradient bound away from

B_n, the images $u_n(P_n \setminus B_n)$ can not go off to infinity. Thus we have shown there is a fixed compact set K which contains $u_n(B_n)$, the cylindrical ends $u_n(\Sigma_n \setminus P_n)$, and $u_n(P_n \setminus B_n)$. Tne union of these sets is the entire image, as desired. See [Gro23] for related discussion. □

2.3.10. *Transversality and dimensions of moduli spaces.* Our approach to achieving transversality for moduli spaces of solutions to Floer's equation follows [MS12, §8]; see also [Sch95, §4.2], [HS95, FHS95, Wen20], and [BC23, §4.1].

Fix a Riemann surface Σ, suppose that $\mathfrak{H}$ is an adapted Reeb-Hamiltonian connection whose asymptotics have non-degenerate time-1 maps.

It is well-known that the linearization of the equation for $u \in \mathcal{M}(\Sigma, \mathfrak{H}, J)$ is a Fredholm operator whose Fredholm index is given by:

$$\text{Index}(u) = n\mathrm{X}(\Sigma) + \mathfrak{m}^{-1}(0) \cdot [u] + \sum_{z \in \Gamma_+} \mathrm{CZ}_{\mathfrak{m}}(\gamma_z) - \sum_{z \in \Gamma_-} \mathrm{CZ}_{\mathfrak{m}}(\gamma_z), \tag{9}$$

where $\mathrm{X}(\Sigma)$ is the Euler characteristic of the Riemann surface. The precise details are not too important in this paper, but we briefly recall what these objects are. Here $\mathfrak{m}$ is a generic section of $\det_{\mathbb{C}}(TW)^{\otimes 2} \to W$

(i) is non-vanishing along each orbit of the asymptotic systems, and,
(ii) lies in a certain distinguished homotopy class of non-vanishing sections when restricted to L, so that the zero set $\mathfrak{m}^{-1}(0)$ is Poincaré dual to the *Maslov class* of L.

These conditions imply the homological intersection number $\mathfrak{m}^{-1}(0) \cdot [u]$ is well-defined. The restriction $\mathfrak{m}|_{\gamma_z}$ is used to define the *Conley-Zehnder index* of the asymptotic orbit $\mathrm{CZ}_{\mathfrak{m}}(\gamma_z)$. Such an index formula appears already in [Sch95, Sal97]. We refer the reader to [Can22] for details on the index formula for Cauchy-Riemann operators with Lagrangian boundary conditions, and to [Can24, §5] for details on the Conley-Zehnder index $\mathrm{CZ}_{\mathfrak{m}}(\gamma_z)$.

An important case is when $L = \emptyset$, in which case we can take $\mathfrak{m} = \mathfrak{s} \otimes \mathfrak{s}$ where $\mathfrak{s}$ is a generic section of $\det(TW)$. Then:

$$\mathrm{PD}(\mathfrak{m}^{-1}(0)) = 2\mathrm{PD}(\mathfrak{s}^{-1}(0)) = 2c_1(TW).$$

We say that $\mathcal{M}(\Sigma, \mathfrak{H}, J)$ is *cut transversally* provided its linearized operator is surjective, in which case $\mathcal{M}(\Sigma, \mathfrak{H}, J)$ is a smooth manifold whose local dimension is given by the index (9).

The strategy to achieve transversality is to use the perturbation term $\mathfrak{p}$ built into the definition of a Reeb-Hamiltonian connection $\mathfrak{H}$. Recall that we fix a compact coordinate disk $D \subset \Sigma$, which we assume is disjoint from the boundary and the cylindrical ends, and let $\mathfrak{p} = h_{s,t}\mathrm{d}s + k_{s,t}\mathrm{d}t$ be a perturbation of the connection one-form supported in D as in §2.3.3.

Following the standard strategy, one considers $\mathcal{P}$ to be a sufficiently rich Banach space of perturbation terms $\mathfrak{p}$. Fix an unperturbed connection one-form $\mathfrak{a}$. One considers the universal moduli space $\mathcal{M}_{\text{uni}}(\Sigma, \mathfrak{a}, J)$ of solutions $(u, \mathfrak{p})$ where $u \in \mathcal{M}(\Sigma, \mathfrak{H}, J)$ where $\mathfrak{H}$ has connection one-form $\mathfrak{a}+\mathfrak{p}$. Similarly to the proof of [MS12, Theorem 8.3.1], one proves that $\mathcal{M}_{\text{uni}}(\Sigma, \mathfrak{a}, J)$ is cut transversally, and that any regular value $\mathfrak{p}$ of the projection $\mathcal{M}_{\text{uni}}(\Sigma, \mathfrak{a}, J) \to \mathcal{P}$ will make $\mathcal{M}(\Sigma, \mathfrak{H}, J)$ cut transversally.

The construction can also be done parametrically, i.e., given a one-parameter family of data $\mathfrak{a}_\tau, j_\tau, J_\tau$, for $\tau \in [0,1]$, one can find $\mathfrak{p}_\tau$ making the parametric moduli spaces transverse; this parametric transversality is necessary for the deformation arguments employed in §2.5, §2.6.

2.3.11. *Semipositivity.* The idea behind the semipositive condition is to preclude the bubbling phenomenon in §2.3.8 by controlling the dimension of moduli spaces of holomorphic spheres. This is achieved by constraining which Chern numbers can appear.

A symplectic manifold (W, ω) is called *semipositive* if, for every $A \in \pi_2(W)$, the following holds:

$$\omega(A) > 0 \text{ and } c_1(A) \geq 3 - n \implies c_1(A) \geq 0.$$

Floer theory was constructed in [HS95] for semipositive symplectic manifolds (*weakly monotone* in their terminology),

Lemma 2.6. *If J is admissible, $L = \emptyset$, and the perturbation term of $\mathfrak{H}$ is sufficiently generic, one can guarantee that every sequence $u_n \in \mathcal{M}(\Sigma, \mathfrak{H}, J)$ with uniformly bounded energy and whose index, given by* (9)*, is ≤ 1, also has bounded first derivatives.*

The argument can also be done 1-parametrically. Let us fix a smooth family of Riemann surfaces Σ_θ and Reeb-Hamiltonian connections $\mathfrak{H}_\theta$, $\theta \in [0,1]$. Here *smooth* means that Σ_θ has a fixed underlying smooth surface Σ, but we deform its complex structure in a one-parameter family.

One considers the parametric moduli space $\mathcal{M}_{\text{para}}$ whose solutions are pairs (θ, u) where $(\Sigma_\theta, \mathfrak{H}_\theta, u)$ solves (8).

Lemma 2.7. *If J is admissible, $L = \emptyset$, and the perturbation terms of $\mathfrak{H}_\theta$ is sufficiently generic, one can guarantee that every sequence $(\theta_n, u_n) \in \mathcal{M}_{\text{para}}$ such that u_n has uniformly bounded energy and has index ≤ 0, also has bounded first derivatives.*

We only prove Lemma 2.7, as the proof of Lemma 2.6 is analogous.

Proof. The argument is well-known, and we refer the reader to [HS95, §3] and [MS12, §3] for more details. We explain the salient points.

First of all, we pick the pertubation terms generically so that $\mathcal{M}_{\text{para}}$ is cut transversally. Its local dimension near a point (θ, u) is equal to $1 + \text{Index}(u)$. There is a natural evaluation map:

$$(\theta, u, z) \in \mathcal{M}_{\text{para}} \times \Sigma \mapsto u(z) \in W, \tag{10}$$

which defines a pseudochain whose local dimension at (θ, u, z) is $3 + \text{Index}(u)$.

Suppose we have a sequence (θ_n, u_n) with unbounded first derivative. Then a holomorphic bubble must form (we assume $L = \emptyset$, so $\partial\Sigma = \emptyset$ and no holomorphic disk can form).

Semipositivity implies that bubbles can only have Chern number either 0 or 1. Holomorphic spheres with zero first Chern number form a codimension 4 pseudocycle in W and hence they generically miss the 3-dimensional pseudocycle defined by the moduli space of solutions with $\mathrm{Index}(u) \le 0$.

A sphere of Chern number ≥ 1 cannot bubble off along any sequence where $\mathrm{Index}(u_n) \le 0$, because it would imply there is a non-zero component of $\mathcal{M}_{\mathrm{para}}$ with dimension ≤ -1. This completes the proof. □

It is well-known and well-explained in [HS95] that first derivative bounds are enough to prove the necessary compactness theorems used in Floer theory. This completes our overview of the differential geometric and analytical background.

2.4. *Hamiltonian Floer cohomology.* Let ϕ_t be a compactly supported Hamiltonian system supported in the domain $\Omega(r_0)$. To define Hamiltonian Floer cohomology for ϕ_t, it's necessary to deform ϕ_t on the non-compact end to make its orbits non-degenerate; we follow the approach of, for example, [Sei08a, FS07, Rit13, Mai22] and consider systems of the form $R^{\alpha}_{\epsilon t} \circ \phi_t$, as in §2.1.4.

Since $R^{\alpha}_{\epsilon t} = \mathrm{id}$ holds on the support of ϕ_t, it follows that $\phi_t \circ R^{\alpha}_{\epsilon t} = R^{\alpha}_{\epsilon t} \circ \phi_t$. If $\epsilon > 0$ is not a period of a closed Reeb orbit, then the system $R^{\alpha}_{\epsilon t} \circ \phi_t$ will have its fixed points contained in a compact subset of W. A compactly supported perturbation[3] δ_t will ensure the system $\delta_t \circ R^{\alpha}_{\epsilon t} \circ \phi_t$ has finitely many non-degenerate fixed points. This system lies in RHI; see §2.1.4.

Note that the class of systems of the form $\delta_t \circ R^{\alpha}_{\epsilon t} \circ \phi_t$ is unchanged if we permute the order of the three terms $\delta_t, R^{\alpha}_{\epsilon t}, \phi_t$ (the perturbation term δ_t will change, but the class of systems is preserved).

The Floer complex for the perturbed system is defined to be the $\mathbb{F}_2$-vector space:

$$\mathrm{CF}(\delta_t \circ R^{\alpha}_{\epsilon t} \circ \phi_t, J)$$

of semi-infinite sums of capped 1-periodic orbits of the perturbed system. The Floer differential depends on a choice of admissible almost complex structure J, and counts Floer cylinders going from right-to-left as shown in Figure 6.

The sums are semi-infinite in the following sense: for any action value a, there are only finitely many non-zero terms whose action is less than a. The cohomological differential increases action, and hence the subspace of semi-infinite sums of orbits whose action is at least a is a subcomplex.

The homology is denoted $\mathrm{HF}(\delta_t \circ R^{\alpha}_{\epsilon t} \circ \phi_t)$ and is called the *Floer cohomology* of the perturbed system. As the notation suggests, the homologies defined for different J's are canonically isomorphic following the argument in [HS95].

[3]Here δ_t is itself a Hamiltonian system, generated by a compactly supported time-dependent Hamiltonian function which we should assume is "C^2 small." We will frequently refer to limits $\delta_t \to 0$, which means that the Hamiltonian function generating δ_t should converge to zero in the C^∞ topology.

Counting continuation cylinders produces *continuation maps*:

$$\mathrm{HF}(\delta'_t \circ R^{\alpha}_{\epsilon' t} \circ \phi_t) \to \mathrm{HF}(\delta_t \circ R^{\alpha}_{\epsilon t} \circ \phi_t) \tag{11}$$

provided $\epsilon' \le \epsilon$; see §2.4.3. The Floer cohomology of the system ϕ_t is defined as a limit[4] over the continuation maps in (11):

$$\mathrm{HF}(\phi_t) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \mathrm{HF}(\delta_t \circ R^{\alpha}_{\epsilon t} \circ \phi_t);$$

By definition, the action of a cohomology class is given by the formula:

$$\mathcal{A}([x]) = \sup \{\mathcal{A}(x + \mathrm{d}\beta) : \beta \in \mathrm{CF}(\delta_t \circ R^{\alpha}_{\epsilon t} \circ \phi_t)\}, \tag{12}$$

where $\mathcal{A}(\sum a_i x_i) = \min \{\mathcal{A}(x_i) : a_i \neq 0\}$. For any element $\mathfrak{e} \in \mathrm{HF}(\phi_t)$, one can consider its image $\mathfrak{e}_{\delta,\epsilon} \in \mathrm{HF}(\delta_t \circ R^{\alpha}_{\epsilon t} \circ \phi_t)$ and take the action $\mathcal{A}(\mathfrak{e}_{\delta,\epsilon})$. In §2.7, it is shown that $\mathcal{A}(\mathfrak{e}_{\delta,\epsilon})$ converges as δ, ϵ converge to zero; we call this number the *min-max action value* $\mathcal{A}_{\phi_t}(\mathfrak{e})$ of the class $\mathfrak{e}$.

Standard continuation arguments, similar to those used in [HS95, Theorem 5.2], produce canonical isomorphisms $\mathrm{HF}(\mathrm{id}) \to \mathrm{HF}(\phi_t)$ which are coherent with respect to continuation maps. If $\mathfrak{e} \in \mathrm{HF}(\mathrm{id})$, then the min-max action value of the image of $\mathfrak{e}$ in $\mathrm{HF}(\phi_t)$ is called the *spectral invariant* of $\mathfrak{e}$ and is denoted $c(\mathfrak{e}, \phi_t)$.

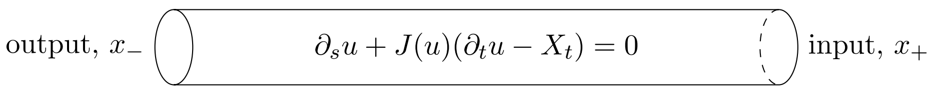

FIGURE 6. Cohomological convention for Floer cylinders.

2.4.1. *Spectral norm for a compactly supported system.* In §2.4.5 the distinguished unit element $1 \in \mathrm{HF}(\mathrm{id})$ is recalled. The *spectral norm* of the system is defined by the formula (2) in §1.1. It is proved in [Sch00, FS07] that this norm depends only on the time-one map ϕ_1, in the case when W is aspherical. See also [Oh05a, Oh05b] for the definition of the spectral norm in the presence of holomorphic spheres.

In any case, without appealing to these results, the spectral norm of a time-1 map ϕ_1 is defined to be the infimum of the spectral norms of all systems which generate ϕ_1.

2.4.2. *Floer differential.* If $\varphi_t \in \mathrm{RHI}$ is *non-degenerate*, i.e., has a time-1 map whose fixed points are non-degenerate, then one can form the vector space $\mathrm{CF}(\varphi_t, J)$ of semi-infinite sums of capped contractible orbits. Associated to this pair one considers the moduli space $\mathcal{M}(\varphi_t, J)$ of Floer cylinders, as in Figure 6.

Because we are using the semipositive framework for ensuring compactness, as in [HS95], we need to impose a few slightly technical conditions.

[4] This is a limit in the category theory sense as in, e.g, [Mac71, Alu09]. If one fixes a sequence of perturbations $\delta_{n,t}$ and slopes ϵ_n converging to zero, then the limit is canonically isomorphic to the inverse limit of the inverse system $\mathrm{HF}(\delta_{n,t} \circ R^{\alpha}_{\epsilon_n t} \circ \phi_t)$; here we assume the time-dependent Hamiltonian vector fields generating $\delta_{n,t}$ converge to zero in the C^∞ topology and the slopes ϵ_n converge to zero monotonically.

Recall from §2.3.2 that J being admissible requires the moduli space $\mathcal{M}^*(A, J)$ of simple parametrized holomorphic spheres is cut transversally. Then the evaluation map:

$$u \in \mathcal{M}^*(A, J)/\mathrm{Aut}(\mathbb{C}) \to u(\infty) \in W \tag{13}$$

defines a pseudocycle of dimension $2n + 2c_1(A) - 4$ for every homology class A; see [MS12, Chapter 6].

Consider now the evaluation map:

$$(\mathcal{M}(\varphi_t, J) \times \mathbb{R} \times \mathbb{R}/\mathbb{Z})/\mathbb{R} \to W \tag{14}$$

given by $(u, s, t) \mapsto u(s, t)$, and where $\mathbb{R}$ acts by translating s and reparametrizing u. We say that φ_t is *admissible* for J provided:

(i) φ_1 is non-degenerate,
(ii) $\mathcal{M}(\varphi_t, J)$ is cut transversally,
(iii) the evaluation map (14) is transverse to the pseudocycle (13).

A system φ_t can be made admissible by a generic perturbation of the form $\delta_t \circ \varphi_t$ where δ_t is compactly supported in $\Omega(r_0 + 1)$; see [FHS95].

Let $\mathcal{M}_d(\varphi_t, J)$ denote the d-dimensional component of the moduli space.

Define a differential on $\mathrm{CF}(\varphi_t, J)$ by the formula:

$$dx = \textstyle\sum_y n(y, x)y, \tag{15}$$

where y is required to have the induced capping, and:

$$n(y, x) = \#\{u \in \mathcal{M}_1(\varphi_t, J)/\mathbb{R} : y(t) = u(-\infty, t) \text{ and } u(+\infty, t) = x(t)\}$$

mod 2; the quotient by $\mathbb{R}$ is with respect to retranslations. The sum defining dx may be infinite; however, it is semi-infinite in the sense considered above, and hence is a well-defined element of the Floer complex.

As is usual in Floer theory, $d^2 = 0$ holds by considering the non-compact ends of the one-dimensional manifold $\mathcal{M}_2(\varphi_t, J)/\mathbb{R}$; see, e.g., [Flo89b, Theorem 4], [HS95, Theorem 5.1].

2.4.3. *Continuation maps.* Let $\varphi_{s,t}$ be a path in RHI, satisfying (i) $\varphi_{s,0} = \mathrm{id}$, and (ii) $\partial_s\varphi_{s,t} = 0$ for s outside a compact interval $[s_0, s_1]$. Let us denote the systems for $\pm s$ sufficiently large by $\varphi_{\pm,t}$. Write $X_{s,t} \circ \varphi_{s,t} = \partial_t\varphi_{s,t}$ for the generating Hamiltonian vector field.

Associated to this continuation data, define $\mathcal{M}(\varphi_{s,t}, J)$ to be the moduli space of continuation cylinder u solving:

$$\partial_s u + J(u)(\partial_t u - X_{s,t}(u)) = 0.$$

If $\varphi_{s,t}$ is *non-positive at infinity*, in the sense that its normalized generator $H_{s,t}$ satisfies $\partial_s H_{s,t} \le 0$ outside of $\Omega(r_0 + 1)$, then $\mathcal{M}(\varphi_{s,t}, J)$ satisfies an a priori energy bound. Indeed, this implies curvature of the connection whose connection one-form is $\mathfrak{a} = H_{s,t}dt$ is non-positive outside of $\Omega(r_0 + 1)$, and so we can appeal to the energy estimate in §2.3.7 for related discussion.

Pick a perturbation $\mathfrak{p}$ connection one-form supported in $[s_0, s_1] \times \mathbb{R}/\mathbb{Z}$, and let $\mathfrak{H}$ be the connection whose one-form is $\mathfrak{a} + \mathfrak{p}$, as in §A.2.6.

Let us say that $\varphi_{s,t}, \mathfrak{p}$ is *admissible continuation data* provided:

(i) $\varphi_{s,t}$ is non-positive at infinity, in the above sense,
(ii) $\varphi_{\pm,t}$ are admissible for defining the Floer complex with J,
(iii) the moduli space $\mathcal{M}(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, \mathfrak{H}, J)$ is cut transversally.
(iv) the evaluation map:

$$(u, s, t) \in \mathcal{M}(\mathbb{R} \times \mathbb{R}/Z, \mathfrak{H}, J) \mapsto u(s, t) \in W$$

is transverse to the simple J-spheres described in (13).

For any path $\varphi_{s,t}$ satisfying (i) and (ii), the other properties can be achieved by picking a generic pertubation term $\mathfrak{p}$.

Counting the points in this moduli spaces defines a *continuation map*:

$$\mathfrak{c} : \mathrm{CF}(\varphi_{+,t}, J) \to \mathrm{CF}(\varphi_{-,t}, J),$$

similarly to how the Floer differential was defined in (15).

Consideration of the 1-dimensional component of $\mathcal{M}(\varphi_{s,t}, J)$ proves $\mathfrak{c}$ is a chain map with respect to the Floer differentials $d_\pm$. The chain homotopy class of the map is unchanged under changes in the perturbation term $\mathfrak{p}$ or under homotopies of $\varphi_{s,t}$ with fixed endpoints, provided they remain non-positive during the entire homotopy.

For details, see [Flo89b, Theorem 4], [HS95, Theorem 5.2], [Sch95, §5.2], [Abo15, Lemma 6.13]. See [Rit09], [Can23, §2.2] for discussion in the context of convex-at-infinity manifolds.

2.4.4. *Independence of the choice of almost complex structure.* For most of our arguments we use a fixed admissible almost complex structure J; this simplifies notation while still enabling us to prove our main result. We note here that the spectral norm is independent of the choice of J. Indeed, for two choices of admissible complex structures J, J', continuation isomorphisms can be defined between $\mathrm{CF}(\varphi_t; J) \to \mathrm{CF}(\varphi_t; J')$, in a way that preserves the min-max action value of the unit element. This continuation argument is given in [HS95, Theorem 5.2] in the closed case. The convex-at-infinity setting does not complicate the argument; sharp energy estimates for the continuation cylinders are possible since the input and output systems coincide (the only difference is the choice of complex structure).

2.4.5. *PSS and the unit element.* The goal in this section is to construct the unit element in $\mathrm{HF}(\varphi_t)$ when the ideal restriction of $\varphi_t \in \mathrm{RHI}$ is a positive Reeb flow. Morally, the unit element is defined by considering continuation cylinders from the identity to φ_t, as in Figure 7. The construction is a special case of the PSS construction of [PSS96], see also [Sch95], and [Rit13] for discussion in the convex-at-infinity case.

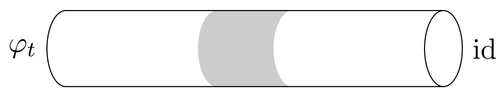

FIGURE 7. The unit is defined via a special kind of continuation cylinder; the shaded region interpolates between φ_t and id.

One picks a path $\varphi_{s,t}$ such that $\varphi_{s,t} = \varphi_t$ for $s \leq s_0$ and $\varphi_{s,t} = \mathrm{id}$ for $s \geq s_1$, and such that $\partial_s H_{s,t}(x) \leq 0$ holds for all s, for x outside of $\Omega(r_0 + 1)$.

As in the definition of the continuation map in §2.4.3, one lets $\mathfrak{H}$ be the Reeb-Hamiltonian connection whose connection one-form is $H_{s,t}\mathrm{d}t + \mathfrak{p}$, for some perturbation term $\mathfrak{p}$ supported in $[s_0, s_1] \times \mathbb{R}/\mathbb{Z}$.

Associated to this is the moduli space $\mathcal{M}(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, \mathfrak{H}, J)$ of continuation cylinders. Each element u in the moduli space has a removable singularity at the $s = +\infty$ end, and should be considered as map $u : \mathbb{C} \to W$ via the reparametrization $z = e^{-2\pi(s+it)}$; in other words, the domain we use for the PSS isomorphisms is $\mathbb{C}$ rather than $\mathbb{C}^\times$.

The resulting PDE has a Fredholm linearization (this is one reason we remove the singularity and pass from $\mathbb{C}^\times$ to $\mathbb{C}$). Under genericity conditions to avoid sphere bubbling similar to those in §2.4.2 and §2.4.3, the count of rigid elements in $\mathcal{M}(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, \mathfrak{H}, J)$ defines a closed element in $1_{\varphi_t} \in \mathrm{HF}(\varphi_t, J)$ called the *unit*. The necessary compactness results follow from the same considerations as those in §2.4.3. The homology class of the unit element does not depend on the particular choices for the same reason that the continuation map is well-defined up to chain homotopy; see, e.g., [Sch95, §5.2] for similar arguments on the invariance up to chain homotopy.

Standard gluing arguments show that the unit elements are natural with respect to continuation maps; see [Sch95, Rit13]. In particular, one can first define $1 \in \mathrm{HF}(\mathrm{id})$ and then propogate it to the other spaces $\mathrm{HF}(\varphi_t)$ using continuation maps.

2.5. *Pair-of-pants product.* The pair-of-pants product between Floer cohomology groups in compact symplectic manifolds is well-known; see for instance [PSS96, §3], [Sch95, §5.5.1.3], and [Sei97b, §6].

For any two systems $\varphi_{0,t}, \varphi_{1,t} \in \mathrm{RHI}$, there is a product:

$$\mu : \mathrm{HF}(\varphi_{0,t}) \otimes \mathrm{HF}(\varphi_{1,t}) \to \mathrm{HF}(\varphi_{\infty,t}),$$

where $\varphi_{\infty,t} = \varphi_{0,t}\varphi_{1,t}$; it is defined by counting elements of the moduli space $\mathcal{M}(\Sigma, \mathfrak{H}, J)$ of solutions to Floer's equation where $\mathfrak{H}$ is a certain Hamiltonian connection on the *pair-of-pants* surface $\Sigma = \mathrm{CP}^1 \setminus \{0, 1, \infty\}$.

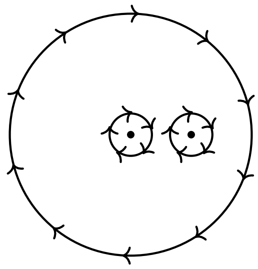

FIGURE 8. Pair-of-pants is CP^1 with three punctures $0, 1, \infty$. Circles around the punctures are oriented as the boundaries of cylindrical ends, i.e., $0, 1$ are positive punctures and ∞ is a negative puncture.

The product in convex-at-infinity manifolds is complicated by the need to ensure a maximum principle holds; see [Rit13, §6], [Rit14, Rit16], [Abo15, §10.3]. The results in §2.3.5 prove that the energy of $u \in \mathcal{M}(\Sigma, \mathfrak{H}, J)$ is given by a formula of the form:

$$\text{energy}(u) = \mathcal{A}_{\varphi_{\infty,t}}(\gamma_\infty) - \mathcal{A}_{\varphi_{1,t}}(\gamma_1) - \mathcal{A}_{\varphi_{0,t}}(\gamma_0) + \int u^*\mathfrak{r}, \tag{16}$$

where γ_i are the asymptotic orbits and $\mathfrak{r}$ is the curvature two-form of the connection $\mathfrak{H}$.

The strategy employed by this paper is to only consider connections $\mathfrak{H}$ which are *flat*, i.e., which satisfy $\mathfrak{r} = 0$. The upshot of this is that one obtains a priori energy bounds for $\mathcal{M}(\Sigma, \mathfrak{H}, J)$ in terms of the actions of the asymptotics; see §2.3.7 for more details. Technically speaking, there will be small error terms coming from the perturbation $\mathfrak{p}$, but these can be made arbitrarily small.

Similar use of flat connections on pairs-of-pants is employed in [Sch00, §4.1] to prove his spectral norm is sub-additive; see §2.5.3. The pair-of-pants constructed in [Rit13, §6], for autonomous $\varphi_{0,t} = \varphi_{1,t}$, also uses a flat Hamiltonian connection.

The paper [KS21] introduces a general construction of flat connections on Riemann surfaces Σ by conformally embedding strips $\mathbb{R} \times [0, 1]$ into Σ. As shown in Figure 9, [KS21] embeds strips and requires that Floer's equation appears in the standard form on each strip; outside the strips one requires solutions are holomorphic (i.e., no Hamiltonian term). In cylindrical ends near each puncture, the conformally embedded strips are supposed to converge to strips $\mathbb{R} \times [t_0, t_1]$ where $[t_0, t_1]$ is some sub-interval of $\mathbb{R}/\mathbb{Z}$. Their construction can be encoded into a Hamiltonian connection $\mathfrak{H}$ which is easily seen to be flat. Throughout our arguments, it is possible to use the construction of [KS21] to produce flat Hamiltonian connections with desired asymptotic systems; alternatively, one can use the methods in §A.3.

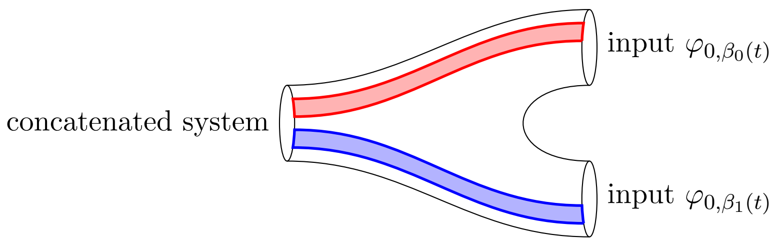

FIGURE 9. Zero curvature connections on the pair-of-pants via the strip technique of [KS21]. Here $\beta_i : [0, 1] \to [0, 1]$ is a non-decreasing surjective smooth function which is supported in a small sub-interval.

In the rest of this section we will review the definition of the pair-of-pants in §2.5.1, and prove in §2.5.2 that it respects the unit elements; this latter point plays a key role in the proof of Theorem 1.1.

2.5.1. *Definition of the pair-of-pants product.* Let $\varphi_{0,t}, \varphi_{1,t}, \varphi_{\infty,t} = \varphi_{0,t}\varphi_{1,t}$ be non-degenerate systems in RHI. The case relevant to our paper is when the ideal restriction of $\varphi^i_{i,t}$, $i = 0, 1$, is $R^\alpha_{\epsilon t}$ and $2\epsilon > 0$ is smaller than the minimal positive period of a Reeb orbit of α. The systems are assumed to be generic on the compact part of W so that the Floer complexes $\mathrm{CF}(\varphi_{i,t})$ are well-defined, as explained in §2.4.2.

Write $H_{i,t}$ for the normalized generator for $\varphi_{i,t}$.

Fix the pair-of-pants $\Sigma = \mathrm{CP}^1 \setminus \{0, 1, \infty\}$ with the choice of cylindrical ends:

$$\begin{cases} \epsilon_0(s,t) = 3^{-1}e^{-2\pi(s+it)}, \\ \epsilon_1(s,t) = 1 + 3^{-1}e^{-2\pi(s+it+i\pi)}, \\ \epsilon_\infty(s,t) = 2e^{-2\pi(s+it)}, \end{cases}$$

where ϵ_0, ϵ_1 are positive ends and ϵ_∞ is a negative end.

Let $\mathfrak{A}$ be the space of connection one-forms $\mathfrak{a}$ such that:

(i) $\mathfrak{a}(v) = a(v)r$ holds outside of $\Omega(r_0 + 1)$ for some $a(v) \in \mathbb{R}$, for each $v \in T\Sigma$; this is equivalent to requiring that the coefficient functions appearing $\mathfrak{a}$ are normalized functions in $\mathcal{RH}$, similarly to §2.3.3.
(ii) $\mathfrak{a} = H_{i,t}\mathrm{d}t$ in the cylindrical end around the ith puncture,
(iii) the connection generated by $\mathfrak{a}$ is flat outside of $\Omega(r_0 + 1)$.

Lemma 2.8. *The space $\mathfrak{A}$ is contractible.*

Proof. We claim that $\mathfrak{A}$ is closed under convex combinations. It is clear that (i) and (ii) are convex conditions. Condition (iii) is also closed under convex combinations; indeed, in any coordinate chart $z = s + it$ on Σ, if we write:

$$\mathfrak{a} = K_{s,t}\mathrm{d}s + H_{s,t}\mathrm{d}t,$$

then:

$$\mathfrak{r} = (\partial_s H_{s,t} - \partial_t K_{s,t} + \omega(V_{s,t}, X_{s,t}))\mathrm{d}s \wedge \mathrm{d}t$$

where $V_{s,t}, X_{s,t}$ are the Hamiltonian vector fields for $K_{s,t}, H_{s,t}$. By assumption (i), $\omega(V_{s,t}, X_{s,t}) = 0$ holds outside of $\Omega(r_0 + 1)$. Thus (iii) is also closed under convex combinations, as the non-linear term vanishes identically.

It remains only to prove that $\mathfrak{A}$ is non-empty. This follows from the construction in [Rit13] using closed one-forms on Σ; it also follows from the next result. □

Lemma 2.9. *The space $\mathfrak{A}$ contains connection one-forms $\mathfrak{a}$ whose curvature vanishes identically.*

Proof. This follows from the general construction of connection one-forms with zero curvature in §A.3. □

Let $\mathfrak{H}$ be any connection whose connection one-form is $\mathfrak{a} + \mathfrak{p}$ where $\mathfrak{a} \in \mathfrak{A}$ and where the perturbation term $\mathfrak{p}$ is as in §A.2.6. Associated to this data is the moduli space $\mathcal{M}(\Sigma, \mathfrak{H}, J)$ of finite-energy solutions to Floer's equation on Σ. We require that $\mathfrak{p}$ is chosen generically so that this the moduli space cut transversally and the natural evaluation map:

$$\mathrm{ev} : \mathcal{M}(\mathfrak{H} + \delta, J) \times \Sigma \to W$$

is transverse to the pseudocycle of simple J-holomorphic spheres; see §2.3.11.

Similarly to the definition of the continuation map, one counts the rigid elements in $\mathcal{M}(\Sigma, \mathfrak{H}, J)$ as defining a map:

$$\mu : \mathrm{HF}(\varphi_{0,t}) \otimes \mathrm{HF}(\varphi_{1,t}) \to \mathrm{HF}(\varphi_{\infty,t}).$$

Note that cappings of the asymptotic orbits at 0 and 1 induce a capping of the orbit at ∞. The a priori energy estimates of §2.3.7 and standard compactness results imply the sums converge (recalling that we allow semi-infinite sums). See, e.g., [Sch95, Rit13].

Because the space $\mathfrak{A}$ and the space of pertubations $\mathfrak{p}$ are contractible, the chain maps used to define μ form a single chain-homotopy class, and so the resulting operation on homology groups is independent of these choices.

Moreover, Lemma 2.9 implies a sharp estimate:

Proposition 2.10. *If $x, y \in \mathrm{HF}(\varphi_{0,t}), \mathrm{HF}(\varphi_{1,t})$ and $z = \mu(x, y) \in \mathrm{HF}(\varphi_{\infty,t})$, where $\varphi_{\infty,t} = \varphi_{0,t}\varphi_{1,t}$ then:*

$$\mathcal{A}_{\varphi_{\infty,t}}(z) \geq \mathcal{A}_{\varphi_{0,t}}(x) + \mathcal{A}_{\varphi_{1,t}}(y);$$

here we refer to the min-max action value of a Floer cohomology class as defined in (12)*.*

Proof. This follows immediately from the energy estimate in (16), Lemma 2.9, and the fact the contribution to the curvature from the pertubation term $\mathfrak{p}$ can be made arbitrarily small. □

It will be important for us to apply this proposition in the case when x, y, z are the unit elements introduced in §2.4.5.

2.5.2. *The product of the unit with itself is the unit.* The goal in this section is to prove that the unit elements constructed in §2.4.5 are compatible with the pair-of-pants product, in the following sense:

Proposition 2.11. *If $1_{\varphi_0}, 1_{\varphi_1} \in \mathrm{HF}(\varphi_{0,t}), \mathrm{HF}(\varphi_{1,t})$ are the unit elements, then $1_{\varphi_\infty} = \mu(1_{\varphi_0}, 1_{\varphi_1}) \in \mathrm{HF}(\varphi_{\infty,t})$, where $\varphi_{\infty,t} = \varphi_{0,t}\varphi_{1,t}$.*

Proof. This result is well-known, see, e.g., [Sch95, PSS96, Sei97b, Sei08a, Rit13, Sei15, Rit16], and we simply sketch the argument.

Let $\mathfrak{a} \in \mathfrak{A}$, and let $\mathfrak{b}$ be the flat connection given by:

$$\mathfrak{b} = H_{\infty,t}\mathrm{d}t \text{ in cylindrical coordinates } z = e^{-2\pi(s+it)}.$$

Let $\mathfrak{a}_R$ be obtained from $\mathfrak{a}$ by performing monotone cut-offs depending on a parameter R in the cylindrical ends of 0 and 1, but do not cut off at the ∞ puncture. In other words, $\mathfrak{a}_R$ is of the form:

$$\beta(R - s)H_{i,t}\mathrm{d}t \quad \text{in the ends } i = 0, 1,$$

where $\beta(x) = 1$ for $x \geq 1$, $\beta(x) = 0$ for $x \leq 0$, and $\beta'(x) \geq 0$. Similarly, let $\mathfrak{b}_R$ be obtained from $\mathfrak{b}$ by performing a monotone cut-off in the cylindrical end around 0 (but not 1 or ∞).

Crucially, because $\varphi_{0,t}, \varphi_{1,t}, \varphi_{\infty,t}$ are Reeb flows with positive speed outside of $\Omega(r_0+1)$, the connection one-forms $\mathfrak{a}_R$ and $\mathfrak{b}_R$ have non-positive curvature outside of $\Omega(r_0 + 1)$. Similar considerations were used in §2.4.3 and §2.4.5.

Consider the family of connection one-forms:

$$\mathfrak{c}_{\tau,R} = (1 - \tau)\mathfrak{a}_R + \tau\mathfrak{b}_R,$$

The same argument given in Lemma 2.8 shows that $\mathfrak{c}_\tau$ has non-positive curvature outside of $\Omega(r_0 + 1)$ for each τ.

Let $\mathfrak{H}_{\tau,R}$ be the connection generated by $\mathfrak{c}_{\tau,R} + \mathfrak{p}_{\tau,R}$, where $\mathfrak{p}_{\tau,R}$ is a perturbation term. The non-positivity of the curvature of $\mathfrak{c}_{\tau,R}$ outside of $\Omega(r_0+1)$ implies the solutions to $\mathcal{M}(\Sigma, \mathfrak{H}_{\tau,R}, J)$ satisfy a priori energy bounds. By picking the perturbation term generically, and appealing to the semipositivity assumption §2.3.11, we can ensure that no bubbling of holomorphic spheres appears over the one-dimensional region:

$$C = \{(\tau, R) \in ([0, 1] \times \{R_0\}) \cup (\{0, 1\} \times [R_0, \infty))\}\,.$$

Similarly to §2.4.5, each $u \in \mathcal{M}(\Sigma, \mathfrak{H}_{\tau,R}, J)$ is considered as defined on $\mathbb{C}$, since the cut-off operation implies these solutions have removable singularities at 0 and 1. Note that $\mathfrak{b}$ was already smooth at the puncture 1, and so we did not need to cut it off at 1.

For each generic point $(\tau, R) \in C$, the rigid elements of $\mathcal{M}(\Sigma, \mathfrak{H}_{\tau,R}, J)$ can be counted yielding a chain in $\mathrm{CF}(\varphi_{\infty,t})$. Standard arguments (similar to those in the construction of the unit element) show that this count defines a closed element. Deformation arguments show that the cycles depend on the choice of the generic point $(\tau, R) \in C$ only up to chain homotopy (i.e., addition of an exact element), and so the resulting cohomology class is well-defined.

The usual Floer theory gluing arguments as in [Sch95] applied with $\tau = 0$ and $R \to \infty$ imply that these counts equal $\mu(1_{\varphi_0}, 1_{\varphi_1})$. On the other hand, the counts when $\tau = 1$ equal 1_{φ_∞}. Thus we conclude the desired result. □

2.5.3. *Sub-additivity of spectral norm.* Propositions 2.10 and 2.11 imply:

Proposition 2.12. *Suppose that $\phi_{0,t}, \phi_{1,t}$ are compactly supported Hamiltonian systems on W; then $c(1, \phi_{0,t} \circ \phi_{1,t}) \geq c(1, \phi_{0,t}) + c(1, \phi_{1,t})$. Consequently the spectral norm defined in* (2) *satisfies $\gamma(\phi_{0,t} \circ \phi_{1,t}) \leq \gamma(\phi_{0,t}) + \gamma(\phi_{1,t})$.* □

The spectral invariant is super-additive because we are using cohomological conventions; see §2.4, but the spectral norm is sub-additive since (2) has a minus sign.

2.6. *Deforming the pair-of-pants product using a Lagrangian.* In this section we explain how to deform the pair-of-pants product using a compact Lagrangian. To begin, we consider the following operation defined by counting half-infinite cylinders.

If $\varphi_{\infty,t}$ is a small perturbation of the identity, which agrees with $R^{\alpha}_{2\epsilon t}$ outside of $\Omega(r_0+1)$, then one can count half-infinite cylinders $[0,\infty)\times\mathbb{R}/\mathbb{Z}$ with Lagrangian boundary conditions solving Floer's equation for $\varphi_{\infty,t}$. This defines an augmentation type map $\mathrm{HF}(\varphi_{\infty,t})\to\Lambda$; see §2.6.1. Here Λ is the $\mathbb{F}_2$-algebra consisting of all semi-infinite sums $\sum x_k\tau^{A_k}$ where A_k are areas of elements of $\pi_2(W,L)$; the sums are semi-infinite in the sense that only finitely many k have $\omega(A_k)\le C$ for any constant C. We are primarily interested in the one-dimensional subspaces $\Lambda_0=\mathbb{F}_2\cdot\tau^0$ corresponding to the disk of area 0.

We recall from (1) the constant $\hbar:=\hbar(L)$ associated to L.

Fix a parameter ν with $3\nu\in(0,\hbar)$; one should imagine ν is very close to 0.

By appropriately counting the solutions of the equation in Figure 10, in §2.6.1, we construct a map:

$$\mathfrak{v}:\mathrm{HF}_{>\nu-\hbar}(\varphi_{\infty,t})\to\Lambda_0. \tag{17}$$

We will show in §2.6.1 that:

Proposition 2.13. *Provided $\varphi_{\infty,t}$ is a sufficiently small perturbation of the identity then $1_{\varphi_{\infty,t}}$ is represented in $\mathrm{HF}_{>2\nu-\hbar}(\varphi_{\infty,t})$, and the map $\mathfrak{v}$ from* (17) *sends every representative of 1_{φ_∞} in $\mathrm{HF}_{>2\nu-\hbar}(\varphi_{\infty,t})$ to $1\in\Lambda_0$.*

Now the pair-of-pants product enters the discussion.

Let $\varphi_{0,t}=\delta_{0,t}\circ R_{\epsilon t}\circ\phi_t^{-1}$ and $\varphi_{1,t}=\phi_t\circ R_{\epsilon t}\circ\delta_{1,t}$ be small perturbations of a compactly supported Hamiltonian isotopy ϕ_t and its inverse, as in §2.4, and let $\varphi_{\infty,t}=\varphi_{0,t}\varphi_{1,t}$. Then $\varphi_{\infty,t}$ is a small enough perturbation of the identity so that the augmentation $\mathrm{HF}_{>\nu-\hbar}(\varphi_{\infty,t})\to\Lambda_0$ sends any unit element in $\mathrm{HF}_{>2\nu-\hbar}(\varphi_{\infty,t})$ to a non-zero element. The idea is to precompose this augmentation with the pair-of-pants product:

$$\mathrm{HF}_{>a}(\varphi_{0,t})\otimes\mathrm{HF}_{>b}(\varphi_{1,t})\to\mathrm{HF}_{>\nu-\hbar}(\varphi_{\infty,t})$$

for suitable negative constants a,b; see §2.6.2 for discussion of the precise choice of constants a,b and the relevance of the spectral norm.

Gluing the pair-of-pants to the half-infinite cylinder provides a chain level description for an operation:

$$\mathrm{HF}_{>a}(\varphi_{0,t})\otimes\mathrm{HF}_{>b}(\varphi_{1,t})\to\Lambda_0. \tag{18}$$

The operation (18) can be defined on chain-level in many ways, as explained in §2.6.3. The idea is to use the gluing trick and Proposition 2.13 to show:

Proposition 2.14. *Provided:*

(i) $\varphi_{0,t}$ and $\varphi_{1,t}$ are sufficiently small perturbations of ϕ_t^{-1},ϕ_t as above,
(ii) $\gamma(\phi)<\hbar$,

then the operation (18) *is non-zero for some ν and a,b with $a+b>2\nu-\hbar$.*

The proposition will be proved by considering (18) as a composition. Then we will deform the PDE in the manner described in Figure 3. As explained in Proposition 2.18 below this deformation does not change the induced map (18) on homology.

Via a sufficiently large deformation, one concludes a sequence of Floer strips with large modulus on L and point constraints as shown in Figure 4 and explained in §2.6.4. As in §1.3, one concludes multiple chords with endpoints on L, with varying actions, completing the proof of Theorem 1.1.

The rest of this section is concerned with the details of the argument.

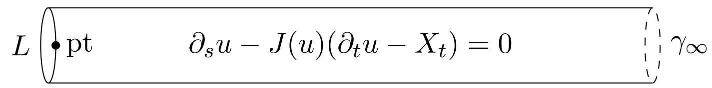

FIGURE 10. The augmentation $\mathrm{HF}_{>\nu-\hbar}(\varphi_{\infty,t}) \to \Lambda_0$ associated to a Lagrangian L with a point constraint $u(0,0) = \mathrm{pt}$.

2.6.1. *Augmentation associated to a Lagrangian with a point constraint.* In this section we are concerned with the proof of Proposition 2.13. We need to construct the augmentation described above in the case when:

$$\varphi_{\infty,t} := \delta_{0,t} R^{\alpha}_{2\epsilon t} \delta_{1,t},$$

where $\delta_{i,t}$ are C^∞ small perturbations and ϵ is a small positive number.

Let $\Omega(r_0) = \{r \leq r_0\}$ and R^{α}_{s}, be as in §2.1.3. Suppose that r_0 is large enough that L is contained in the interior of $\Omega(r_0)$. We also assume that the perturbations $\delta_{i,t}$ are such that the normalized Hamiltonians $\Delta_{i,t}$ which generate $\delta_{i,t}$ vanish outside of $\Omega(r_0+1)$; this is sufficient to ensure transversality of all non-constant Floer cylinders.

The normalized Hamiltonian generating $\varphi_{\infty,t}$ is given by:

$$H_{\infty,t} := \Delta_{0,t} + \Delta_{1,t} \circ R^{\alpha}_{-2\epsilon t} \delta_{0,t}^{-1} + 2\epsilon f(r \circ \delta_{0,t}^{-1} - r_0) + 2\epsilon r_0. \tag{19}$$

By picking ϵ and $\Delta_{0,t}, \Delta_{1,t}$ sufficiently small we may assume that:

$$\int_0^1 \max_{x \in \Omega(r_0+1)} H_{\infty,t}(x) \mathrm{d}t < \nu/2,$$

The augmentation map $\mathfrak{v} : \mathrm{HF}_{>\nu-\hbar}(\varphi_{\infty,t}) \to \Lambda_0$ is defined on capped orbits by the formula:

$$\mathfrak{v}(\gamma_\infty) := \sum n(\gamma_\infty, A)\tau^A,$$

where $n(\gamma_\infty, A)$ is the number of rigid half-infinite cylinders u as in Figure 10 such that $u \# \bar{\gamma}_\infty$ has symplectic area $A \leq 0$, where $\bar{\gamma}_\infty$ denotes the capping. The resulting Floer energy of the cylinder is given by:

$$E(u) = \int_0^1 H_t(u(0,t))dt - \mathcal{A}_{\varphi_{\infty,t}}(\gamma_\infty),$$

which is bounded from above by $\hbar - \nu/2$. In particular, by picking J so that $\hbar(L) - \nu/2 < \hbar(L, J)$, we can ensure that any solution contributing to $\mathfrak{v}$ has energy less than $\hbar(L, J)$, and hence no disk bubbling can occur. Henceforth we assume that we have such a choice of J.

Standard arguments, similar to those used to define the operator denoted $\bar{e}$ on [Sch95, pp. 195], then imply that:

$$\mathfrak{v} : \mathrm{HF}_{>\nu-\hbar}(\varphi_{\infty,t}) \to \Lambda_0$$

is a chain map, where Λ_0 is considered as a complex with zero differential, and hence induces the desired map on homology.

To complete the proof, we first need to show that the unit element $1_{\varphi_{\infty,t}}$ is represented in the $\mathrm{HF}_{>2\nu-\hbar}(\varphi_{\infty,t})$ for $\Delta_{i,t}$ and ϵ sufficiently small. One shows this by analyzing the action of the output of the PSS map.

The PSS construction gives one representative of $1_{\varphi_{\infty,t}}$ as a count of the solutions to the PSS equation described in §2.4.5; one can therefore bound the minimal action of the output from below by:

$$\min_{W\times\mathbb{R}/\mathbb{Z}} H_{\infty,t} + \text{symplectic area of the PSS cylinder in Figure 7.}$$

The first term converges to zero as $\Delta_{i,t}$ and ϵ go to zero; see (19). If the sum is not greater than $2\nu-\hbar < -\nu$, then the symplectic area must be uniformly negative. Then we eventually contradict the non-negativity of the energy of the PSS cylinder, because:

$$0 \leq \text{energy} \leq \int_{\gamma_-} H_{\infty,t}\mathrm{d}t + \int_{\mathbb{R}} \max_{W} \partial_s H_{s,t} + \text{symplectic area},$$

and the first two terms cannot be uniformly positive (as $\Delta_{i,t}$ and ϵ converge to zero), and so the last term cannot be uniformly negative.

Having established that $1_{\varphi_{\infty,t}}$ lives in the appropriate action window, we now show that $\mathfrak{v}$ sends $1_{\varphi_{\infty,t}}$ to $1 \in \Lambda_0$. To show this, one considers the moduli space $\mathcal{M}(\sigma, \varphi_{\infty,t})$ of cylinders u described in Figure 11, which have zero symplectic area.

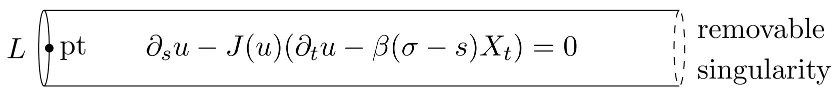

FIGURE 11. Here $\beta(x) = 1$ for $x \geq 1$ and $\beta(x) = 0$ for $x \leq 0$. The cylinders interpolate between the augmentation and the PSS equation from Figure 7. Taking a limit $\sigma \to \infty$ shows that the augmentation sends the unit to $1 \in \Lambda_0$.

Combining the energy estimate for the augmentation cylinders with the energy estimate for the PSS cylinders shows that, if $\Delta_{i,t}$ and ϵ are small enough compared to ν, then the Floer energy of the cylinders in Figure 11 is bounded above by $\hbar - \nu/2 < \hbar(L, J)$, and so disk bubbling does not happen.

If σ is suficiently negative, the only elements of $\mathcal{M}(\sigma, \varphi_{\infty,t})$ are the constant J-holomorphic disks. Arguing similarly to [MS12, §9.2], one thinks of the union of the index 0 components $\mathcal{M}_{\text{para}} = \bigcup_\sigma \mathcal{M}(\sigma, \varphi_{\infty,t})$ as a parametric moduli space; since there is a unique constant J-holomorphic disk satisfying the point constraint, the fibers over $\sigma < 0$ consists of a single point. It follows that, for generic compactly-supported perturbations of the parametric equation, $\mathcal{M}_{\text{para}} \to \mathbb{R}$ is smooth map between one-manifolds, and the fibers over generic $\sigma > 0$ represent the non-trivial class in the unoriented bordism group of 0-dimensional manifolds; in other words, the $\mathbb{F}_2$-valued counts of the zero symplectic area component of $\mathcal{M}(\sigma, \varphi_{\infty,t})$ is 1 for generic σ.

Taking the limit $\sigma \to \infty$, and applying standard compactness and gluing arguments, one concludes that $\mathfrak{v}$ takes the representative of the unit in $\mathrm{HF}_{>2\nu-\hbar}(\varphi_{\infty,t})$ constructed using PSS to 1.

It remains only to show that $\mathfrak{v}$ takes any other representative of the unit lying in $\mathrm{HF}_{>2\nu-\hbar}(\varphi_{\infty,t})$ to 1, provided that $\varphi_{\infty,t}$ is sufficiently small perturbation of the identity.

The idea is straightforward; briefly, because $\varphi_{\infty,t}$ is arbitrarily close to id, its *boundary depth* as in [Ush13] will be arbitrarily close to 0. In particular, we can assume the boundary depth is less than ν. Thus, any two representatives of the unit in $\mathrm{HF}_{>2\nu-\hbar}(\varphi_{\infty,t})$ will be equal in $\mathrm{HF}_{>\nu-\hbar}(\varphi_{\infty,t})$. Since $\mathfrak{v}$ is well-defined on the deeper filtered piece $\mathrm{HF}_{>\nu-\hbar}(\varphi_{\infty,t})$, we conclude that any two representatives of the unit in $\mathrm{HF}_{>2\nu-\hbar}(\varphi_{\infty,t})$ map to the same number in Λ_0, namely 1.

The fact that the boundary depth is small for systems which are close to the identity is well-known, but for completeness, we sketch the argument in the context of our paper:

Lemma 2.15. *If $\varphi_{\infty,t}$ is sufficiently small perturbation of the identity (of the form $\delta_{0,t}R^{\alpha}_{2\epsilon t}\delta_{1,t}$, as above), then any two cycles in $\mathrm{HF}_{>2\nu-\hbar}(\varphi_{\infty,t})$ which map to the same element in the full homology $\mathrm{HF}(\varphi_{\infty,t})$ map to the same element in $\mathrm{HF}_{>\nu-\hbar}(\varphi_{\infty,t})$; these maps are induced by inclusions of subcomplexes.*

Proof. The argument is well-known, although it is a bit technical. In the case when W is closed, it is proved in [Ush13, §5]. When W is open and convex-at-infinity, the same argument works using the PSS isomorphism in [FS07]. Their argument is given in the case when W is aspherical, but it generalizes in a straightforward way when W is semipositive (as proven in [Lan16]), and it shows there is an isomorphism:

$$\Phi : H^*(W) \otimes_{\mathbb{Z}/2} \Lambda \to \mathrm{HF}(\varphi_{\infty,t}), \tag{20}$$

where Λ is the Novikov field of power series $\sum \tau^w$ where w ranges over the symplectic areas of spheres in W. Moreover, Φ induces an interleaving of persistence modules, in the following sense. Introduce:

$$E_1 = -\int \min(H_{\infty,t})dt \quad \text{and} \quad E_2 = \int \max_{\Omega(r_0+1)}(H_{\infty,t})dt.$$

Then, for each A, there is a map:

$$\Phi_A : H^*(W) \otimes_{\mathbb{Z}/2} \Lambda_{>A} \to \mathrm{HF}_{>A-E_1}(\varphi_{\infty,t}),$$

where $\Lambda_{>A} \subset \Lambda$ is the subspace generated by terms τ^w with $w < -A$, and a map:

$$\Psi_A : \mathrm{HF}_{>A}(\varphi_{\infty,t}) \to H^*(W) \otimes_{\mathbb{Z}/2} \Lambda_{>A-E_2},$$

so that $\Psi_{A-E_1} \circ \Phi_A : \mathrm{HF}_{>A}(\varphi_{\infty,t}) \to \mathrm{HF}_{>A-E_1-E_2}(\varphi_{\infty,t})$ is the map on homology induced by the inclusion of a subcomplex, and $\Phi_{A-E_2} \circ \Psi_A$ is the inclusion map.

It then follows from the same argument in [Ush13] that the boundary depth of the persistence module $s \mapsto \mathrm{HF}_{>-s}(\varphi_{\infty,t})$ is at most $E_1 + E_2$. Since E_1

and E_2 can be made smaller than ν by making $\varphi_{\infty,t}$ arbitrarily close to the identity, the desired result follows. □

This completes the proof of Proposition 2.13. □

2.6.2. *Relevance of the spectral norm condition.* This section is concerned with the proof of Proposition 2.14.

The pair-of-pants operation defined in §2.5 respects the action filtrations and induces a morphism:

$$\mathrm{HF}_{>a}(\varphi_{0,t}) \otimes \mathrm{HF}_{>b}(\varphi_{1,t}) \to \mathrm{HF}_{>a+b-\rho_1}(\varphi_{\infty,t}); \tag{21}$$

here we use the energy estimate $E \leq \mathcal{A}(\gamma_\infty) - \mathcal{A}(\gamma_0) - \mathcal{A}(\gamma_1) + \rho_1$, where $\rho_1 > 0$ is a small error term due to perturbing a flat connection.

Thus, if $1_{\varphi_{0,t}} \in \mathrm{HF}_{>a}(\varphi_{0,t})$ and $1_{\varphi_{1,t}} \in \mathrm{HF}_{>b}(\varphi_{1,t})$, and $a + b - \rho_1 > 2\nu - \hbar$, composing (21) with the augmentation $\mathfrak{v} : \mathrm{HF}_{>2\nu-\hbar}(\varphi_{\infty,t}) \to \Lambda_0$ yields a *non-trivial* product by Proposition 2.13:

$$\mathrm{HF}_{>a}(\varphi_{0,t}) \otimes \mathrm{HF}_{>b}(\varphi_{1,t}) \to \Lambda_0 \tag{22}$$

sending $1_{\varphi_{0,t}} \otimes 1_{\varphi_{1,t}}$ to $1 \in \Lambda_0$. See Figure 3 for an illustration of this composition.

If $\varphi_{0,t} = \delta_{0,t} \circ R_{\epsilon t} \circ \phi_t^{-1}$ and $\varphi_{1,t} = \phi_t \circ R_{\epsilon t} \circ \delta_{1,t}$, we can pick the perturbation terms small enough that:

$$\mathcal{A}(1_{\varphi_{0,t}}) + \mathcal{A}(1_{\varphi_{1,t}}) \geq c(1, \phi_t^{-1}) + c(1, \phi_t) - \rho_2 = -\gamma(\phi_t) - \rho_2,$$

for an arbitrarily small $\rho_2 > 0$; this approximation follows from the results in §2.7.

Since $\gamma(\phi_t) < \hbar$, we can pick the parameter ν and the perturbations so that $2\nu + \rho_1 + \rho_2 < \hbar - \gamma(\phi_t)$. Then $\mathcal{A}(1_{\varphi_{0,t}}) + \mathcal{A}(1_{\varphi_{1,t}}) - \rho_1 > 2\nu - \hbar$ and so (22) is non-trivial. Finally pick:

$$\mathcal{A}(1_{\varphi_{0,t}}) > a \text{ and } \mathcal{A}(1_{\varphi_{1,t}}) > b,$$

maintaining $a + b > 2\nu - \hbar$, to complete the argument. □

2.6.3. *Product operation with multiple point constraints.* We continue with the set-up of §2.6.2, and interpret the non-trivial operation (22) as a count of solutions to Floer's equation associated to a Hamiltonian connection over a twice-punctured disk; see Figure 3 and 4.

Let $\Sigma_R = D(\frac{1}{2}, R) \setminus \{0, 1\}$, where R ranges in $(\frac{1}{2}, \infty)$. As $R \to +\infty$, the Riemann surface Σ_R breaks into a configuration of a pair-of-pants glued at its negative end to a half-infinite cylinder. As $R \to \frac{1}{2}$, the punctures converge to the boundary of the surface.

Let $\mathfrak{a} \in \mathfrak{A}$ be a flat connection-one form on the pair-of-pants, as in §2.5.1, on the pair-of-pants surface. Restrict $\mathfrak{a}$ to Σ_R and let $\mathfrak{H}_R$ be the resulting flat connection.

We will transform $\mathfrak{H}_R$ in such a way which various properties are satisfied. The argument is based on the coordinate transformations $g_{R,*}\mathfrak{H}_R$ analyzed in §A.3. Here $g_R : \Sigma_R \to G$ is a smooth family valued in $G = \mathrm{RHI}/\mathrm{RHI}_0$ where RHI_0 is the group of contractible loops in RHI (so G is a universal cover). As explained in §A, such coordinate transformations preserve the class of flat Reeb-Hamiltonian connections, and can be used to make the connection appear in a standard form in certain regions.

We claim that:

Lemma 2.16. *There exists a family* g_R *so that* $g_R = \mathrm{id}$ *for* R *sufficiently positive, and so that:*

(i) In travelling cylindrical ends around the punctures the connection appears in the standard form for Floer's equation for $\varphi_{0,t}$*, and* $\varphi_{1,t}$*.*
(ii) As $R \to \frac{1}{2}$*, there is a conformally embedded neck of large modulus* $N(R) \subset \Sigma_R$*, as proved in §2.6.5, and the connection appears in the standard form for Floer's equation on the strip for the system* $\varphi_{1,t}$*.*
(iii) The monodromy around the boundary circle is always conjugate to a time reparametrization of $\varphi_{\infty,t}$*.*

Proof. It is straightforward to achive (i). Indeed, the original connection $\mathfrak{H}_R$ already satisfies this, and so all we need to do is make sure that $g_{R,z} = \mathrm{id}$ holds for z near the punctures.

To achieve (ii), we argue as follows. First pick small geodesic disks around the punctures which are disjoint from a neighborhood of a holomorphic collar $C_{\infty,R}$ around the boundary circle. Via a coordinate tranformation as in §A.3.4 supported in the collar, one ensures the monodromy around the boundary circle is $\varphi_{\infty,t}$. This is possible because the monodromy around the boundary is conjugate to $\varphi_{\infty,t}$ (because the boundary circle is freely homotopic to a very large circle where the monodromy is known to $\varphi_{\infty,t}$); see Proposition A.5. The relevant parts of the domain are illustrated in Figure 12. This step is only necessary when R approaches $1/2$, because the monodromies around large circles are already known to be $\varphi_{\infty,t}$

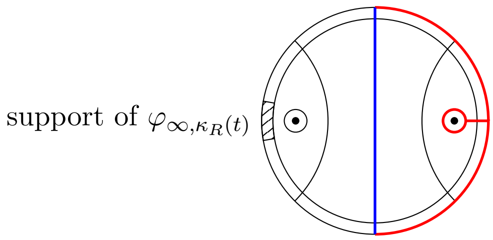

FIGURE 12. Construction of a flat connection with special properties; shown are the two cylindrical ends, the collar around the boundary, and the holomorphic neck $N(R)$ of large modulus. The coloured paths are used to compare monodromies.

Pick a time-reparametrization $\kappa_R : [0,1] \to [0,1]$ depending smoothly on R, so that $\kappa_R(t) = t$ for R sufficiently positive, and so that $\kappa_R'(t)$ is supported in a very small interval as shown in Figure 12 as R approaches $1/2$. Again by a coordinate transformation supported in the collar $C_{\infty,R}$, we can suppose that the monodromy along the boundary is $\varphi_{\infty,\kappa_R(t)}$, $\varphi_{\infty,\kappa_R(t)}$ and $\varphi_{\infty,t}$ have the same time-1 map in G. Thus we achieve that the monodromy along arcs in the boundary which are disjoint from the support shown in Figure 12 are the identity.

This construction ensures that monodromy along the transverse arcs of the neck $N(R)$ equals $\psi^{-1}\varphi_{1,t}\psi$ in G. We can correct the connection by pushing forward by ψ to ensure the monodromy along the transverse arcs equal $\varphi_{1,t}$. Indeed, we can push forward by a family g so that $g_z = \psi$ holds for $z \in \partial\Sigma$, and $g_z = \mathrm{id}$ near the punctures. This achieves (ii).

It follows that the monodromy along the boundary is $\psi\varphi_{\infty,\kappa_R(t)}\psi^{-1}$, as required by (iii), completing the proof. □

Let $\mathfrak{a}_R$ denote the normalized connection one-form for $g_{R,*}\mathfrak{H}_R$ obtained from this lemma. The upshot of these slightly technical requirements is that, as $R \to 1/2$, the connection one-form $\mathfrak{a}_R$ appears in the form $H_{1,t}\mathrm{d}t$ in the strip coordinates on the neck $N(R)$.[5]

Relabel $\mathfrak{H}_R$ to be the connection generated by the one-form $\mathfrak{a}_R + \mathfrak{p}_R$, where the perturbation term $\mathfrak{p}_R$ is as in §2.3.3.

Let $z_1, \ldots, z_k$ be smooth maps $(1/2, \infty) \to \partial\Sigma(R)$ such that

$$z_1(R) = \cdots = z_k(R) = 1/2 + R$$

for R sufficiently positive. Fix also smooth maps $f_i : P_i \to L$ representing bordism classes of maps in L.

Introduce $\mathcal{M}_{\text{para}}$ as the parametric moduli space of pairs $(p_1, \ldots, p_k, R, u)$ where u is a finite energy solution to Floer's equation for $(\Sigma_R, \mathfrak{H}_R, J)$ and which satisfy the incidence condition:

$$u(z_i(\sigma)) = f_i(p_i). \tag{23}$$

We pick the perturbation term so that $\mathcal{M}_{\text{para}}$ is cut transversally.

Each curve appearing in $\mathcal{M}_{\text{para}}$ is asymptotic at the ith-labeled puncture, $i = 0, 1$, to a 1-periodic orbit γ_i of $\varphi_{i,t}$.

Restrict $\mathcal{M}_{\text{para}}$ to only those u whose asymptotic orbits have cappings such that the disk formed by gluing u to the two cappings has zero symplectic area, and the sum of the actions of the capped orbits is greater than $2\nu - \hbar$.

Lemma 2.17. *With the above construction, the energy of u appearing in $\mathcal{M}_{\text{para}}$ is bounded from above by $\hbar - \nu/2$ provided that $\varphi_{\infty,t}$ is a sufficiently small perturbation of the identity, as in §2.6.1. In particular, no disk (or sphere) bubbling can occur in $\mathcal{M}_{\text{para}}$, since $\hbar - \nu/2 < \hbar(L, J)$ as in §2.6.1.*

Proof. The energy is given by the integral formula:

$$E(u) = \int_{\partial\Sigma(R)} u^*\mathfrak{a}_R - \mathcal{A}_{\varphi_{0,t}}(\gamma_0) - \mathcal{A}_{\varphi_{1,t}}(\gamma_1) + \int_{\Sigma(R)} u^*\mathfrak{r}.$$

[5]One should also note that one can use the construction of [KS21] described in §2.5 to construct the desired family of flat connections.

Because the curvature of $\mathfrak{a}_R$ vanishes, the integral of $\mathfrak{r}$ can be made arbitrarily small by choosing the perturbation $\mathfrak{p}_R$ very small. In particular, we can make it at most ν.

The integral of $u^*\mathfrak{a}_R$ over the boundary $\partial\Sigma(R)$ is bounded from above by:

$$\int_0^1 \max_{\Omega(r_0+1)} \tilde{H}_t dt$$

where $\tilde{H}_t$ is the generator of $\psi\varphi_{\infty,\kappa_R(t)}\psi^{-1}$. Since ψ preserves $\Omega(r_0+1)$ and the time-average of the Hamiltonian generator is invariant under time-reparametrization, this quantity is equal to:

$$\int_0^1 \max_{\Omega(r_0+1)} H_{\infty,t} dt,$$

which we assume is smaller than $\nu/2$ as in §2.6.1. This leads to the estimate:

$$E(u) < 3\nu/2 - \mathcal{A}_{\varphi_{0,t}}(\gamma_0) - \mathcal{A}_{\varphi_{1,t}}(\gamma_1) < \hbar - \nu/2,$$

as desired. □

Let $\mathcal{M}_{\text{para},1}$ be the one-dimensional component, and consider the projection map:

$$(p_1, \dots, p_k, R, u) \in \mathcal{M}_{\text{para},1} \mapsto R \in (1/2, \infty), \tag{24}$$

whose fiber over R is denoted by $\mathcal{M}(R)$. For generic parameter R, $\mathcal{M}(R)$ is a compact 0-dimensional manifold.

Suppose that $\varphi_{0,t}, \varphi_{1,t}$ are as in §2.6.2, so that the actions of the unit elements $1_{\varphi_{0,t}}$ and $1_{\varphi_{1,t}}$ are greater than a, b, respectively, and $a + b > 2\nu - \hbar$. In this setting, we can interpret the count of elements in $\mathcal{M}(R)$ (for generic R) as defining a map:

$$\mathfrak{w}_R : \mathrm{CF}(\varphi_{0,t})_{>a} \otimes \mathrm{CF}(\varphi_{1,t})_{>b} \to \Lambda_0. \tag{25}$$

One sends a generator $\gamma_0 \otimes \gamma_1$ to the number of elements in $\mathcal{M}(R)$ whose asymptotics equal γ_0, γ_1 and which form a contractible disk when glued to the cappings.

Proposition 2.18. *The map $\mathfrak{w}_R$ is a chain map for all generic R. If the bordism classes represented by $f_1, \dots, f_k$ have a total intersection product equal to the point class in L, then $\mathfrak{w}_R$ is non-trivial on homology.*

Proof. That $\mathfrak{w}_R$ is a chain map follows from the usual Floer theoretic arguments, similar to those in [Flo89b, §1.c] and [Sch95, Rit13]

When R is very positive, the two punctures become arbitrarily close to each other, and Σ_R is a large disk $D(\frac{1}{2}, R)$ with 0 and 1 as the punctures. Gluing arguments are then used to show that $\mathfrak{w}_R$ equals the composition of $\mathfrak{v}$ with the pair-of-pants product as in §2.6.2, and is therefore non-trivial on homology.

Finally, standard deformation arguments show that $\mathfrak{w}_R$ and $\mathfrak{w}_{R'}$ are chain homotopic for generic R, R'. In particular, the non-triviality for very negative values of R implies the non-triviality for all generic values of R. □

In the next subsection we exploit the non-triviality of $\mathfrak{w}_R$ for values of R close to $\frac{1}{2}$, for specific choice of the punctures $z_i(R)$, so as to conclude chains of non-stationary Floer strips, as described in §1.3.

2.6.4. *Floer strips with multiple point constraints.* The deformation argument in §2.6.3 produces Floer strips of arbitrarily large modulus for the system $\varphi_{1,t}$; moreover, by picking the punctures $z_i(R)$ to be equally spaced along the lower boundary of the Floer strip, one concludes k sub-strips of large modulus, each with an incidence constraint at the midpoint of its lower boundary; see Figure 13, and compare with Figure 4.

Moreover, since the total energy of a solution u to the moduli space considered in §2.6.3 is less than $\hbar - \nu/2$ (by construction), the Floer strip in Figure 13 also has total energy less than $\hbar - \nu/2$.

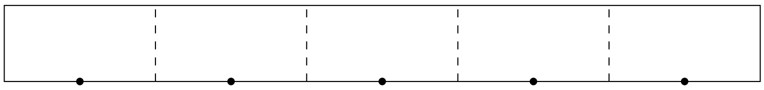

FIGURE 13. The Floer strip with k incidence constraints obtained by restricting the curves in §2.6.3 to the neck; this can be decomposed into a sequence of k sub-strips.

By a standard compactness argument (using Arzelà-Ascoli), one concludes k infinite-length Floer strips $v_1, \dots, v_k$, each of which has an incidence constraint at $(0, 0)$, i.e., $v_i(0, 0)$ lies in the image of $f_i : P_i \to L$. By a second limiting process, one can turn off the pertubations so that $\varphi_{1,t} = \phi_t \circ R_{\epsilon t} \circ \delta_{1,t}$ converges to ϕ_t; in this fashion, one concludes a sequence $u_1, \dots, u_k$ of infinite-length Floer strips for the original system ϕ_t so that u_i incident to the images of $f_i : P_i \to L$; see Figure 1.

Note that we do not claim that the positive asymptotic chord $\gamma_{i,+}$ of u_i matches the negative asymptotic chord $\gamma_{i+1,-}$ of u_{i+1}. However, because the total energy prior to breaking was less than $\hbar$, we conclude that:

$$\mathcal{A}_{\phi_t}(\gamma_{k,+}) \le \mathcal{A}_{\phi_t}(\gamma_{i+1,-}) \le \mathcal{A}_{\phi_t}(\gamma_{i,+}) \le \mathcal{A}_{\phi_t}(\gamma_{i,-}) < \mathcal{A}_{\phi_t}(\gamma_{k,+}) + \hbar;$$

in particular, the actions of the asymptotics lie in an interval of length $\hbar$.

We can take $f_1, \dots, f_k$ to have positive codimension and be disjoint from the intersection $L \cap \phi_1(L)$ (which is presumed to be an isolated set) while still satisfying the condition that the total intersection product is the point class (as required by Proposition 2.18). Then it is clear that each u_i is non-constant (otherwise the image of f_i would intersect $L \cap \phi_1(L)$). Thus we conclude strict inequalities $\mathcal{A}_{\phi_t}(\gamma_{i,+}) < \mathcal{A}_{\phi_t}(\gamma_{i,-})$.

It then follows that:

$$\mathcal{A}_{\phi_t}(\gamma_{1,-}) > \mathcal{A}_{\phi_t}(\gamma_{1,+}) > \mathcal{A}_{\phi_t}(\gamma_{2,+}) > \cdots > \mathcal{A}_{\phi_t}(\gamma_{k,+})$$

are $k + 1$ action values contained in an interval of length $\hbar$. Since $\hbar$ can be taken arbitrarily close to $\gamma(\phi_t)$, we conclude $k + 1$ action values in an interval of length $\gamma(\phi_t)$. The argument given in §1.3.3 upgrades this to an interval of length $\gamma(\phi)$, finishing the proof of Theorem 1.1.

2.6.5. *A conformal embedding of an infinite strip into a disk.* In this section we give a formula for a biholomorphism w between the infinite strip $\mathbb{R}\times[0,1]$ and the closed unit disk $D(1)$ with punctures $\{-1,1\}$.

Consider $w_1 : \mathbb{R}\times[0,1] \to \mathbb{H}^\times$ defined by $s+it \mapsto e^{\pi(s+it)}$, where $\mathbb{H}^\times$ is the punctured closed upper half plane. Note that w_1 maps the vertical line $\{s\}\times[0,1]$ onto the half circle $C_s := \{e^{\pi(s+it)} : t \in [0,1]\}$. Consider also the Möbius transformation:

$$w_2 : \mathbb{H} \to D(1), \quad z \mapsto \frac{z-i}{z+i}.$$

This is a biholomorphism between $\mathbb{H}^\times$ and $D(1)\setminus\{-1,1\}$ (note that 0 is sent to -1 and ∞ is sent to $+1$). The image $w_2(C_s)$ is a circular arc which is orthogonal to the boundary of the disk. The desired biholomorphism is $w = w_2 \circ w_1$; see Figure 14 for an illustration.

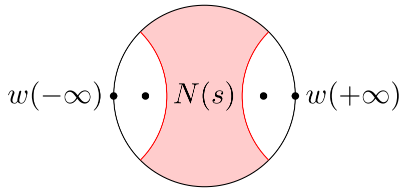

FIGURE 14. The image $N(s)$ of the rectangle $[-s,s]\times[0,1]$ under w.

2.7. *Convergence of action values.* In this section we prove that the action values $\mathcal{A}_{\phi_t}(\mathfrak{e}) \in \mathbb{R}$ for classes $\mathfrak{e} \in \mathrm{HF}(\phi_t)$, as defined by the limiting procedure in §2.4, are well-defined and finite. The crux of the argument is to establish the continuity of action values with respect to changing the slope ϵ of the approximation $\delta_t \circ R^\alpha_{\epsilon t} \circ \phi_t$.

Consider a Hamiltonian system ϕ_t supported in $\Omega(r_0)$. Similarly to §2.4, consider perturbations $\delta_t \circ R^\alpha_{\epsilon t} \circ \phi_t$ so that δ_t is supported in $\Omega(r_0+1)$, and which are admissible for defining the Floer complex $\mathrm{CF}(\delta_t \circ R^\alpha_{\epsilon t} \circ \phi_t, J)$, as in §2.4.2.

Between any two such perturbation data (δ_-, ϵ_-) and (δ_+, ϵ_+) with $\epsilon_- \geq \epsilon_+$, we take the continuation map associated to the linear interpolation:

$$H_{s,t} = (1-\beta(s))K^-_t + \beta(s)K^+_t,$$

where $\beta : \mathbb{R} \to [0,1]$ is a non-decreasing cut-off function satisfying $\beta(s) = 0$, for $s \leq 0$ and $\beta(s) = 1$, for $s \geq 1$.

Recall that $\mathrm{HF}(\phi_t)$ is defined as the inverse limit of the groups $\mathrm{HF}(\delta_t \circ R^\alpha_{\epsilon t} \circ \phi_t)$ with respect to the above continuation maps. Write $\mathfrak{e}_{\delta,\epsilon}$ for the image of $\mathfrak{e}$ in $\mathrm{HF}(\delta_t \circ R^\alpha_{\epsilon t} \circ \phi_t)$. In this section we prove:

Proposition 2.19. *The limit:*

$$\mathcal{A}_{\phi_t}(\mathfrak{e}) = \lim_{\epsilon\to 0}\lim_{\delta\to 0} \mathcal{A}(\mathfrak{e}_{\delta,\epsilon})$$

exists, where $\mathcal{A}(e_{\delta,\epsilon})$ *is defined in* (12).

This is necessary to define the spectral invariants and the spectral norm (2), and is also used in §2.6.2.

2.7.1. *Hofer norm estimate.* First we establish an inequality in one direction:

Lemma 2.20. *Let (δ_-, ϵ_-) and (δ_+, ϵ_+) be two perturbation data with $\epsilon_- \geq \epsilon_+$, both smaller than the minimal Reeb period. Let K_t^-, K_t^+ be the normalized Hamiltonian functions generating the systems $\delta_{\pm,t} \circ R^\alpha_{\epsilon_\pm t} \circ \phi_t$, and introduce the error term:*

$$e(K_t^+, K_t^-) = \int_0^1 \max_x (K_t^+(x) - K_t^-(x), 0)\mathrm{d}t < \infty;$$

the integrand is the maximum of the non-negative part of $K_t^+(x) - K_t^-(x)$. Then, for any element $\mathfrak{e} \in \mathrm{HF}(\phi_t)$, we have the following estimate for the action values:

$$\mathcal{A}(\mathfrak{e}_{\delta_-,\epsilon_-}) \geq \mathcal{A}(\mathfrak{e}_{\delta_+,\epsilon_+}) - e(K_t^+, K_t^-).$$

Moreover, if $\epsilon_+ = \epsilon_-$ then

$$|\mathcal{A}(\mathfrak{e}_{\delta_-,\epsilon_-}) - \mathcal{A}(\mathfrak{e}_{\delta_+,\epsilon_+})| \leq e(K_t^+, K_t^-).$$

Proof. Fix some small constant $\rho > 0$. One picks a chain level representative $\sum x_i$ for $\mathfrak{e}_{\delta_+,\epsilon_+}$ so that $\min \mathcal{A}(x_i) \geq \mathcal{A}(\mathfrak{e}_{\delta_+,\epsilon_+}) - \rho$. The actions of the chain level sum output by the continuation map $\mathfrak{c}(\sum x_i) = \sum y_j$ can be estimated to yield:

$$\mathcal{A}(\mathfrak{e}_{\delta_-,\epsilon_-}) \geq \min \mathcal{A}(y_j) \geq \min \mathcal{A}(x_i) - e(K_t^+, K_t^-).$$

The proof of the rightmost inequality is standard and follows from the energy estimate; see, e.g., [HS95, Oh05b, Gin05] or §2.3.5. Taking $\rho \to 0$ yields the desired result.

To see why $e(K_t^+, K_t^-)$ is finite, observe that $K_t^\pm = \epsilon_\pm r + c_\pm$ where $c_\pm$ is locally constant outside of $\Omega(r_0 + 1)$, and hence:

$$K_t^+ - K_t^- = (c_+ - c_-) - (\epsilon_- - \epsilon_+)t;$$

this is bounded from above because we assume $\epsilon_- \geq \epsilon_+$. □

2.7.2. *Estimating the error term.* Let ϕ_t be a Hamiltonian system generated by a normalized Hamiltonian H_t. Let (δ, ϵ) be an admissible perturbation data; the system $\delta_t \circ R^\alpha_{\epsilon,t} \circ \phi_t$ is generated by:

$$K_{\delta,\epsilon,t} = \Delta_t + \epsilon f(r \circ \delta_t^{-1} - r_0) + H_t \circ (\delta_t R^\alpha_{\epsilon t})^{-1} + \epsilon r_0, \tag{26}$$

where Δ_t is the compactly supported Hamiltonian function generating δ_t and the constant term ϵr_0 is used to make $K_{\delta,t}$ normalized. It is convenient to compare with the reference Hamiltonian:

$$G_{\epsilon,t} = \epsilon f(r - r_0) + H_t + \epsilon r_0.$$

Then the following estimate holds:

$$|K_{\delta,\epsilon,t} - G_{\epsilon,t}| \leq |\Delta_t| + |H|_{C^1} |\delta_t \circ R^\alpha_{\epsilon t}|_{C^0} + |f|_{C^1} |\delta_t|_{C^0}, \tag{27}$$

where $|f| := \max_x |f(x)|$ for a map $f : W \to \mathbb{R}$ and the C^0-norm is with respect to a metric that is translation invariant at infinity. Fixing $\rho, \epsilon > 0$, define:

$$B(\epsilon, \rho) := \{\delta : |K_{\delta,\epsilon} - G_\epsilon| < \rho \text{ and } \delta \circ R^\alpha_{\epsilon t} \circ \phi_t \text{ is admissible for } J\}.$$

Similarly let $C(\epsilon_0, \rho)$ be the set of pairs (ϵ, δ) so that $\epsilon < \epsilon_0$ and $\delta \in B(\epsilon, \rho)$.

From the estimate (27), one sees that to define Floer cohomology as in §2.4 it is enough to take the limit of $\mathrm{HF}(\delta_t \circ R^\alpha_{\epsilon t} \circ \phi_t)$ over the subcategory where $(\epsilon, \delta) \in C(\epsilon_0, \rho)$.

Associated to the set of perturbations $B(\epsilon, \rho)$, let:

$$A(\mathfrak{e}; \epsilon, \rho) = \{\mathcal{A}(\mathfrak{e}_{\delta,\epsilon}) : \delta \in B(\epsilon, \rho)\};$$

by the estimate §2.7.1 the diameter of $A(\mathfrak{e}; \epsilon, \rho)$ is less than 2ρ. Associated to the larger set $C(\epsilon_0, \rho)$, let:

$$\bar{A}(\mathfrak{e}; \epsilon_0, \rho) = \{\mathcal{A}(\mathfrak{e}_{\delta,\epsilon}) : (\epsilon, \delta) \in C(\epsilon_0, \rho)\} = \textstyle\bigcup_{\epsilon < \epsilon_0} A(\mathfrak{e}; \epsilon, \rho).$$

The key to convergence of the action values is:

Proposition 2.21. *If $\epsilon_0(1 + r_0) < \rho/3$, then the diameter of $\bar{A}(\mathfrak{e}; \epsilon_0, \rho)$ is bounded from above by 4ρ.*

The proof is given in §2.7.3, and Proposition 2.19 will follow as an easy corollary.

2.7.3. *Comparing perturbations with different slopes.* We consider specific pairs:

$$\varphi_{\pm,t} = \delta_{\pm,t} \circ R^\alpha_{\epsilon_\pm t} \circ \phi_t.$$

The construction is delicate, as we want to have precise control over the continuation map $\mathrm{HF}(\varphi_{+,t}) \to \mathrm{HF}(\varphi_{-,t})$. As in the statement of Proposition 2.21, we fix $\rho, \epsilon_0 > 0$ so that $\epsilon_0(1 + r_0) < \rho/3$.

Assume that $\epsilon_+ < \epsilon_- < \epsilon_0$ and pick a perturbation data $\delta_+ \in B(\epsilon_+, \rho')$ where $\rho' < \rho/3$. Consider the system $\varphi_{+,t} = \delta_{+,t} \circ R^\alpha_{\epsilon_+ t} \circ \phi_t$ which is generated by:

$$K_t^+ = \Delta_{+,t} + \epsilon_+ f(r \circ \delta_t^{-1} - r_0) + H_t \circ (\delta_{+,t} R^\alpha_{\epsilon_+ t})^{-1} + \epsilon_+ r_0,$$

as in (26). Assume that $\Delta_{+,t}$ is supported in $\Omega(r_0 + 0.5)$; this is sufficient to make $\varphi_{+,t}$ admissible. It follows that:

$$K_t^+ = \begin{cases} K_t^+ & r < r_0 + 0.5, \\ h_+(r) & r_0 + 0.5 \le r < r_0 + 1, \\ \epsilon_+ r + H_t & r_0 + 1 \le r. \end{cases}$$

It is important to bear in mind that H_t is locally constant on the region $r > r_0$. Note that K_t^+ is normalized as H_t is normalized.

Define:

$$K_t^- = \begin{cases} K_t^+ & r < r_0 + 0.5, \\ h_-(r) & r_0 + 0.5 \le r < r_0 + 1, \\ \epsilon_- r + H_t & r_0 + 1 \le r, \end{cases}$$

where $h_-(r) = h_+(r)$ for r near $r_0 + 0.5$, and $\partial_r h_-(r) \geq \partial_r h_+(r)$; such a function exists since $\epsilon_- > \epsilon_+$. We also suppose that $\partial_r h_-, \partial_r h_+$ are non-negative and everywhere smaller than the minimal period of a Reeb orbit for α; this ensures that there are no orbits of $K_t^{\pm}$ in the region $r_0 + 0.5 \leq r$. We claim that:

Lemma 2.22. *If $\delta_{-,t} \circ R^{\alpha}_{\epsilon_-,t} \circ \phi_t$ is the system generated by K_t^-, then it holds that $\delta_- \in B(\epsilon_-, \rho)$.*

Proof. We compute:

$$\begin{aligned}|K^- - G_{\epsilon_-}| &= |K^- - G_{\epsilon_-}|_{\Omega(r_0+1)} \\ &\leq |K^- - K^+|_{\Omega(r_0+1)} + |K^+ - G_{\epsilon_+}|_{\Omega(r_0+1)} + |G_{\epsilon_+} - G_{\epsilon_-}|_{\Omega(r_0+1)} \\ &\leq |h_- - h_+|_{\Omega(r_0+1)} + \rho' + (\epsilon_- - \epsilon_+)(|f(r - r_0)|_{\Omega(r_0+1)} + r_0) \\ &\leq \epsilon_0(r_0 + 1) + \rho/3 + \epsilon_0(1 + r_0) < \rho,\end{aligned}$$

which is equivalent to $K^- \in B(\epsilon_-, \rho_0)$. □

Next, observe that the periodic orbits of the two systems $\varphi_{+,t}$ and $\varphi_{-,t}$ are identical and they all lie in $\Omega(r_0)$. By a strong maximum principle argument similar to [FS07, §2], all of the Floer continuation cylinders associated to the linear interpolation between $K_t^{\pm}$ remain in $\Omega(r_0)$; see also [Vit99, §1.3], [Rit13, §D.3].

Moreover, since the two Hamiltonians K_t^+ and K_t^- coincide on $\Omega(r_0)$, the continuation cylinders solve the translation invariant Floer equation in $\Omega(r_0)$. By index reasons, the only rigid continuation cylinders are the stationary solutions (i.e., the continuation cylinders are s-independent). Moreover, every such cylinder contributes to the continuation map $\mathfrak{c}$, and hence $\mathfrak{c}$ is the identity map, bearing in mind that the (capped) orbits of K_t^+ and K_t^- are the same.

This observation implies that the continuation map from K_t^+ to K_t^- preserves action and is an isomorphism on chain level. Thus it holds that $\mathcal{A}(\mathfrak{e}_{\delta_+,\epsilon_+}) = \mathcal{A}(\mathfrak{e}_{\delta_-,\epsilon_-})$, as desired.

In particular, this implies that $A(\mathfrak{e}; \epsilon_-, \rho)$ and $A(\mathfrak{e}; \epsilon_+, \rho)$ intersect; see §2.7.2. Since both sets have diameter bounded by 2ρ, we conclude Proposition 2.21.

To conclude, we observe that the above argument can be performed for any choice of $\epsilon_+ < \epsilon_- < \epsilon_0$. This implies that the action values $\mathcal{A}(\mathfrak{e}_{\delta,\epsilon})$ converge, proving Proposition 2.19. A straightforward interleaving argument[6] shows that the limit is independent of the choice of contact form, the choice of r_0, and the choice of complex structure; the details are left to the reader.

[6] Two sequences $x_1, x_2, \dots$ and $y_1, y_2, \dots$ of real numbers are said to be *interleaved* if for all n there exists $m > n$ so that $x_m \leq y_n$ and $y_m \leq x_n$. If x_n, y_n both converge, and they are interleaved, then they must converge to the same limit.

Appendix **A.** *Flat Hamiltonian connections*

Our approach to Floer's equation on general Riemann surfaces is via bundles with a Hamiltonian connection. See [MS12, §8], [Sei97a, §7], [Pol01, §9.3] for a similar approach.

A.1. *Ehresmann connections.* To every smooth fiber bundle $\pi : E \to B$ one associates the *vertical sub-bundle* $V = \ker \mathrm{d}\pi \subset TE$. An *Ehresmann connection* is a smoothly varying choice of linear complement $\mathfrak{H} \subset TE$.

A.1.1. *Complete connections.* Let $E \to B$ be a fiber bundle with an Ehresmann connection $\mathfrak{H}$. Every vector field Y on B has a unique lift to a horizontal vector field $Y_{\mathfrak{H}} \in \mathfrak{H}$. An Ehresmann connection is called *complete* provided that every compactly supported vector field Y on B lifts to a complete vector field on E.

A.1.2. *Monodromy diffeomorphisms.* Let $b(t)$ be a path in B. If $\mathfrak{H}$ is a complete Ehresmann connection on $E \to B$, then for every $e_0 \in E_{b(0)}$ there is a unique *horizontal lift* $e(t)$ so $e(0) = e_0$, $\pi(e(t)) = b(t)$, and $e'(t) \in \mathfrak{H}$. The map which associates e_0 to $e(1)$ is a diffeomorphism $E_{x(0)} \to E_{x(1)}$ and is called the *monodromy* of $b(t)$.

As an example, solutions of the ODE $y'(x) = F(x, y(x))$, $y(0) = y_0$, are horizontal lifts of $x(t) = t$ for the Ehresmann connection:

$$\mathfrak{H} = \{\mathrm{d}y - F(x, y(x))\mathrm{d}x = 0\}$$

on $\mathbb{R}^2$. The map which associates an initial condition y_0 to $y(1)$ is the prototypical example of a monodromy diffeomorphism.

A.1.3. *Flat connections.* An Ehresmann connection $\mathfrak{H}$ is called *flat* provided that, for every choice of $e \in E_b$, there exists a germ of a section $\mathfrak{s}$ of $E \to B$ at b satisfying $\mathfrak{s}(b) = e$ and $\mathrm{im}(\mathrm{d}\mathfrak{s}) = \mathfrak{H}$. If $\mathfrak{H}$ is a complete flat connection, then deformations of a path $b(t)$ relative its endpoints do not change the monodromy.

A.1.4. *Curvature of an Ehresmann connection.* Let Y_1, Y_2 be two vector fields on B defined in a neighborhood of b. Let $Y_{i,\mathfrak{H}}$ denote their horizontal lifts to E. For $e \in E_b$, define:

$$R_{\mathfrak{H},e}(Y_1, Y_2) := [Y_{1,\mathfrak{H}}, Y_{2,\mathfrak{H}}] - [Y_1, Y_2]_{\mathfrak{H}}.$$

It is a standard exercise in manipulating the Lie bracket to show that $R_{\mathfrak{H},e}(Y_1, Y_2)$ is valued in the vertical sub-bundle $V \subset TE$. Moreover, $R_{\mathfrak{H},e}$ commutes with multiplication by smooth functions; in particular, $R_{\mathfrak{H},e}$ is induced by a tensor $TB_b \wedge TB_b \to V_e$. The resulting tensor:

$$R_{\mathfrak{H}} : \pi^*(TB \wedge TB) \to V$$

is called the *curvature tensor* of $\mathfrak{H}$.

It is clear that connection is flat if and only if its curvature tensor is everywhere zero.

A.2. *Hamiltonian connections on trivial bundles.* Let $E = W \times B \to B$ be a trivial bundle and suppose the fiber (W, ω) is a symplectic manifold. A *Hamiltonian connection* is an Ehresmann connection $TE = TW \oplus \mathfrak{H}$ (where TW is identified with the vertical sub-bundle V of the fibration $E \to B$) so that:

(i) $\mathfrak{H}$ is the Ω-complement to TW,
(ii) $\Omega := \mathrm{pr}^*\omega - \mathrm{d}\mathfrak{a}$, and,
(iii) $\mathfrak{a}$ is a one-form on E which vanishes on TW.

It follows that $\mathfrak{H}$ is a linear complement to TW and hence defines an Ehresmann connection.

A.2.1. *One-forms vanishing on the vertical bundle.* If $\mathfrak{a}$ is a one-form on E which vanishes on TW, and $x_1, \ldots, x_k$ are coordinates on B pulled back to coordinates on E, then $\mathfrak{a}$ can be written as $\mathfrak{a} = H_1 \mathrm{d}x_1 + \cdots + H_k \mathrm{d}x_k$. Here H_i is considered a function on $W \times B$, i.e., it is domain dependent. In this sense, $\mathfrak{a}$ can be considered as a one-form on B taking values in $C^\infty(W, \mathbb{R})$; see, e.g., [Sei08b, §8e], [KS21, pp. 3293].

A.2.2. *Monodromy of a Hamiltonian connection is Hamiltonian.* Fix local coordinates $x_1, \ldots, x_k$ on the base B of the trivial bundle $W \times B \to B$. Consider the Hamiltonian connection $\mathfrak{H}$ induced by $\mathfrak{a} = H_1 \mathrm{d}x_1 + \cdots + H_k \mathrm{d}x_k$ as in §A.2.1. Let $x : [0,1] \to B$ be a path in B remaining in the local coordinate chart. Then the induced monodromy diffeomorphism $W \to W$ is given by the time-one map of the Hamiltonian system:

$$\gamma'(t) = \textstyle\sum_{i=1}^k x_i'(t) X_{H_i}(\gamma(t)).$$

Indeed, the velocity of $(x(t), \gamma(t))$ is Ω-orthogonal to $V = TW$ when:

$$\Omega = \mathrm{pr}^*\omega - \mathrm{d}\mathfrak{a},$$

as can be checked directly. As a consequence, the monodromy of any Hamiltonian connection along any path is a Hamiltonian diffeomorphism.

A.2.3. *Curvature of a Hamiltonian connection is Hamiltonian.* Let $\mathfrak{H}$ be the Hamiltonian connection induced by a one-form $\mathfrak{a}$ as above. Then:

$$R_{\mathfrak{H}}(\partial_i, \partial_j) \lrcorner\, \omega = -\mathrm{d}\Big(\frac{\partial H_i}{\partial x_j} - \frac{\partial H_j}{\partial x_i} + \{H_i, H_j\}\Big) = -\mathrm{d}\mathfrak{r}(\partial_i, \partial_j). \tag{28}$$

In particular, the vertical curvature vectors define a Hamiltonian vector field on W, and the generating Hamiltonians can be encoded as a curvature two-form $\mathfrak{r}$. See [KS21, pp. 3293] for similar formula.

Note that if $\mathfrak{H}$ is flat, then $\mathfrak{r}$ is not necessarily zero, but is the pullback of a two-form from B (assuming W is connected). Typically the way to ensure that $\mathfrak{r} = 0$ for flat connections is to enforce some normalization conditions on the Hamiltonian functions appearing in $\mathfrak{a}$; see §A.2.8.

A.2.4. *Coordinate changes for Hamiltonian connections.* The next lemma is used for constructing and manipulating Hamiltonian connections; it ensures that the class of Hamiltonian connections is closed under a large family of coordinate changes.

Lemma A.1. *Let $E = W \times B$. A contractible family g_x of Hamiltonian diffeomorphisms, where $x \in B$, has a total map $g(w,x) = (g_x(w), x)$ which satisfies:*

$$g^*\mathrm{pr}^*\omega = \mathrm{pr}^*\omega + \mathrm{d}\mathfrak{b},$$

where $\mathfrak{b}$ is a one-form which vanishes on the vertical bundle.

Proof. Indeed, if $g_{x,t}$ is a path of systems $g_{x,1} = g_x$ and $g_{x,0} = \mathrm{id}$, then differentiating $g_t(w,x) = (g_{x,t}(w), x)$ with respect to t yields:

$$\frac{\partial}{\partial t} g_t^*\mathrm{pr}^*\omega = \mathrm{d}(g_t^*\mathrm{pr}^*[\mathrm{d}H_{x,t}]) = -\mathrm{d}(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(H_{x,t} \circ g_{x,t})\mathrm{d}x_i) = \mathrm{d}\beta_t, \tag{29}$$

where $H_{x,t}$ are the generators of the systems $t \mapsto g_{x,t}$. The existence of the path $g_{x,t}$ is what we mean when we say $g_{x,t}$ is a contractible family.

Integrating β_t over $t \in [0,1]$ constructs the primitive one-form $\mathfrak{b}$ which vanishes on the vertical bundle. In particular, if $\mathfrak{H}_2$ is a Hamiltonian connection, then $\mathfrak{H}_1 = g_*^{-1}\mathfrak{H}_2$ is also Hamiltonian. □

A.2.5. *Conjugation of monodromy and coordinate changes.* Let $\mathfrak{H}$ be a flat Hamiltonian connection on $W \times B$. Let g_x represent a coordinate change as in §A.2.4. Given a path $x(t)$ in the base, we have:

$$g_{x(1)}^{-1} \circ (\text{monod. of } g_*\mathfrak{H} \text{ along } x(t)) \circ g_{x(0)} = (\text{monod. of } \mathfrak{H} \text{ along } x(t)),$$

where “monod.” is short for “monodromy”. This is used in §2.6.3 to obtain a connection with a desired monodromy.

A.2.6. *Reeb-Hamiltonian connections.* Here we recall $\Omega(r_0 + 1)$ is a star-shaped domain determined by $r \geq r_0 + 1$ is the radial function associated to the Reeb flow for α, as explained in §2.1.3.

As in §2.3.3, a Hamiltonian connection $\mathfrak{H}$ is called α-Reeb outside of $\Omega(r_0+1)$ provided each function $\mathfrak{a}(X)$ equals $ar + c$, for some constant a and where c is locally constant outside of $\Omega(r_0 + 1)$, for all tangent vectors $X \in TB_x$.

Henceforth, we fix the parameter r_0 and contact form α, and say $\mathfrak{H}$ is a *Reeb-Hamiltonian connection* when it is α-Reeb outside of $\Omega(r_0 + 1)$.

A.2.7. *Group of Hamiltonian isotopies.* Using the notation from §2.1.4, introduce the group $G = \mathrm{RHI}/\mathrm{RHI}_0$ of Reeb-Hamiltonian isotopies modulo contractible loops. Thus G is the universal cover of a certain group of Hamiltonian diffeomorphisms.

In particular, a contractible family $g_x \in G$, $x \in B$, acts on the space of connections on $W \times B \to B$ which are α-Reeb outside of $\Omega(r_0 + 1)$, by the formula in §A.2.4.

The results of §A.2.2 specialize and imply that the monodromy of an α-Reeb, connection $\mathfrak{H}$ is valued in G, and moreover that all such connections are complete.

A.2.8. *Normalization conditions.* We say that a one-form $\mathfrak{a}$ on $W\times B$ (which vanishes on vertical vectors) is *normalized* if $\mathfrak{a}(X)$ is normalized according to §2.2.3 on the fiber $W \times \{x\}$ for every vector $X \in TB_x$. Every Reeb-Hamiltonian connection can be generated by such a normalized one-form.

An important property of normalized connections $\mathfrak{H}$ is that the corresponding curvature two-form $\mathfrak{r}$ appearing in (28) is normalized, and if the connection is flat the curvature is identically zero. Moreover, if normalized one-forms $\mathfrak{a}$ and $\mathfrak{b}$ generate the same connection $\mathfrak{H}$ then $\mathfrak{a} = \mathfrak{b}$.

A.3. *Locally trivial flat Hamiltonian connections.* In this section we fix the group $G = \mathrm{RHI}/\mathrm{RHI}_0$ from §A.2.7.

A.3.1. *Locally trivial flat connection.* Let $E \to \Sigma$ be a locally trivial bundle with fiber W and structure group G. An Ehresmann connection $\mathfrak{H}$ on E is called *flat* if around each point of the base there is a chart around the point so that the connection appears flat as in §A.1.3. Such a bundle $(\pi : E \to \Sigma, \mathfrak{H})$ will be called a *locally trivial Hamiltonian bundle with a flat connection.*

For example, if one can pass to a subatlas where the changes of trivialization are locally constant on each intersection $U_0 \cap U_1$, then the locally defined flat connections glue together to form a flat connection $\mathfrak{H}$.

A.3.2. *Pairs of pants.* The pair-of-pants Σ is a two-sphere with three punctures; we can fix this as CP^1 with the punctures at $0, 1$ and ∞.

Proposition A.2. *There is a simply-connected Riemann surface Σ' with a free-and-proper action of the free-group F_2 by biholomorphisms, and a biholomorphism $\Sigma'/\mathrm{F}_2 \to \Sigma$. The monodromy of the covering space around 0 equals the action of the first generator of F_2, and the monodromy around the 1 equals the second generator of F_2.*

Proof. This is a consequence of the theory of covering spaces in the context of Riemann surfaces; see, e.g., [Don11]. □

This gives a construction of flat Reeb-Hamiltonian connections with any desired monodromy around 0 and around 1:

Proposition A.3. *Fix any choice of monodromy $\mathrm{F}_2 \to G$. The diagonal quotient:*

$$E : (W \times \Sigma')/\mathrm{F}_2 \to \Sigma'/\mathrm{F}_2 \simeq \Sigma$$

is a locally trivial G-bundle with a flat connection. Moreover, this bundle is trivial as a bundle with structure group G.

Proof. Take contractible open sets U in Σ which admit smooth lifts to Σ'; each choice of lift gives an identification of $E|_U$ with $W\times U$. It is straightforward to check that transition functions are locally constant, which completes the first part of the proof (by the definition of locally flat in §A.3.1).

For the second part, note that $\mathrm{F}_2 \to G$ can be continuously homotoped to the constant map through group homomorphisms. This process produces a bundle over $\Sigma \times [0,1]$. The bundle restricted to $\Sigma \times \{1\}$ is isomorphic to the restriction over $\Sigma \times \{0\}$, which is clearly the trivial bundle $\Sigma \times W$. □

A.3.3. *Locally trivial G-bundles on the pair-of-pants are trivial.* Fix a Hamiltonian bundle with flat connection over the pair-of-pants $E \to \Sigma$. The goal in this section is to prove the following result allowing us to realize every locally trivial G-bundle with flat connection as a trivial bundle with a Reeb-Hamiltonian connection in the sense of §A.2.

Proposition A.4. *Every locally trivial G-bundle with flat connection $\mathfrak{H}$ on the pair-of-pants Σ, as in §A.3.1, can be trivialized in such a way that $\mathfrak{H}$ is Hamiltonian with connection form $\Omega = \mathrm{pr}^*\omega - \mathrm{d}\mathfrak{a}$, as in §A.2 and, in particular, §A.2.6.*

Proof. Using the covering space furnished by Proposition A.2, one can show that E is isomorphic to one of the bundles constructed in Proposition A.3, and hence is trivial in the category of fiber bundles with structure group G.

Standard ideas in Cech-cohomology imply that the cocycle of transition functions derived from any atlas is trivial. One can pick an atlas $\{(U_\alpha, \eta_\alpha)\}$ where the transitions are constant on each intersection. The triviality of the cocycle implies the existence of maps $g_\alpha : U_\alpha \to G$ so that:

$$g_\beta g_\alpha^{-1} = g_{\alpha\beta} = \eta_\beta \eta_\alpha^{-1}.$$

Note that g_α are not required to be constant. Consider g_α as inducing maps $g_\alpha : U_\alpha \times W \to U_\alpha \times W$.

In the following, suppose that the atlas uses a good open cover as in [BT82, §5], i.e., suppose that every finite intersection of open sets in the cover is empty or contractible.

Consider the map $E \to \Sigma \times W$ which on $\pi^{-1}(U_\alpha)$ equals $g_\alpha^{-1}\eta_\alpha$. This is well-defined, since on the intersection $U_\alpha \cap U_\beta$ we have $g_\beta^{-1}\eta_\beta = g_\alpha^{-1}\eta_\alpha$. Since the charts η_α take $\mathfrak{H}$ to the standard flat connection, the induced connection on $U_\alpha \times \Sigma$ appears in the form:

$$(g_\alpha^{-1})_*(\text{standard flat connection}).$$

As in §A.2.4, such a connection is Hamiltonian for:

$$\Omega_\alpha = g_\alpha^*(\mathrm{pr}^*\omega) = \mathrm{pr}^*\omega - \mathrm{d}\mathfrak{a}_\alpha.$$

Following §2.2.3, the Hamiltonian functions $\mathfrak{a}_\alpha(\partial_s), \mathfrak{a}_\alpha(\partial_t)$ are chosen to be normalized in each fiber $\{z\} \times W$. Moreover, because g_α is valued in G, these Hamiltonian functions satisfy the Reeb condition outside $\Omega(r_0 + 1)$ as in §A.2.6.

The connection two-forms $\Omega_\alpha, \Omega_\beta$ necessarily agree on their overlap because:

$$(g_\alpha^{-1})^*(\Omega_\alpha - \Omega_\beta) = \mathrm{pr}^*\omega - (g_\beta g_\alpha^{-1})^*\mathrm{pr}^*\omega = \mathrm{pr}^*\omega - g_{\alpha\beta}^*\mathrm{pr}^*\omega = 0,$$

using that $g_{\alpha\beta}$ is constant.

Writing $\Omega_\alpha = \mathrm{pr}^*\omega - \mathrm{d}\mathfrak{a}_\alpha$, we have that $\lambda_{\alpha\beta} = \mathfrak{a}_\alpha - \mathfrak{a}_\beta$ is closed on each intersection. Since $\lambda_{\alpha\beta}$ vanishes on vertical direction, it is exact, and can be written as $\lambda_{\alpha\beta} = \mathrm{d}f_{\alpha\beta}$ where $f_{\alpha\beta}$ is an $\mathbb{R}$-valued function pulled back from the base $U_\alpha \cap U_\beta$ (again, using that W is connected).

The normalization conditions imply that $f_{\alpha\beta}$ is constant, as follows. Compute

$$\frac{\partial f_{\alpha\beta}}{\partial x_i} = \lambda_{\alpha\beta}(\partial_i) = \mathfrak{a}_\alpha(\partial_i) - \mathfrak{a}_\beta(\partial_i).$$

If W is compact, integrate this over the fiber to conclude that $f_{\alpha\beta}$ is constant. If W is non-compact, use that the right-hand side is one-homogenous (in the ends of the fibers) while the left hand side is constant on each fiber; so that the left hand side must be zero. In particular, $\mathrm{d}f_{\alpha\beta}$ vanishes identically, and $\mathfrak{a}_\alpha = \mathfrak{a}_\beta$ holds on the overlap. Hence there is a globally defined $\mathfrak{a}$ so that $\mathfrak{H}$ is the Reeb-Hamiltonian connection for $\Omega = \mathrm{pr}^*\omega - \mathrm{d}\alpha$. □

A.3.4. *Cylindrical ends.* Let $\Sigma = [a, b] \times \mathbb{R}/\mathbb{Z}$ be a cylinder and suppose that $\mathfrak{H}$ is a flat Hamiltonian connection on $\Sigma \times W$. Introduce the notation $\mathfrak{H}(\varphi_t)$ for the standard flat connection with $\mathfrak{a} = H_t\mathrm{d}t$ where H_t is the normalized generator of φ_t.

The results of this section explain how to use the coordinate changes described in §A.2.4 to make $\mathfrak{H}$ appear in the standard form $\mathfrak{H}(\varphi_t)$ in a given cylindrical end.

Proposition A.5. *If the monodromy of $\mathfrak{H}$ around any circle $\{s_0\} \times \mathbb{R}/\mathbb{Z}$, in the universal cover, is represented by a conjugate of the system φ_t, then one can find a contractible family $g : \Sigma \to G$ so that $g_*^{-1}\mathfrak{H} = \mathfrak{H}(\varphi_t)$.*

Proof. Suppose $a = 0$, $b = 1$ and $s_0 = 0$ for simplicity. Let $\psi_{s,t}$ be the monodromy of $\mathfrak{H}$ along the path $\{s\} \times [0, t]$, and let $\kappa_{s,t}$ be the monodromy along the path $[0, s] \times \{t\}$.

By assumption, there is a homotopy φ_t^τ starting at $\varphi_{0,t} = \rho\varphi_t\rho^{-1}$ so that $\varphi_{1,t} = \psi_{0,t}$; the homotopy has fixed endpoints at $t = 0, 1$.

Let $g_\tau(s, t) = \kappa_{\tau s,t} \circ \varphi_t^\tau \circ \rho \circ \varphi_t^{-1}$. Then $g_1(s, t) = f_{s,t} \circ \varphi_t^{-1}$ where f satisfies:

$$\mathfrak{H} = f_*\mathfrak{H}(\mathrm{id}).$$

On the other hand, since:

$$(\varphi_t^{-1})^*\mathrm{pr}^*\omega = \mathrm{pr}^*\omega - \mathrm{pr}^*\omega(-, X_t) \wedge \mathrm{d}t = \mathrm{pr}^*\omega - \mathrm{d}(H_t\mathrm{d}t),$$

it follows that $(\varphi)_*\mathfrak{H}(\mathrm{id}) = \mathfrak{H}(\varphi)$ and thus $g_*\mathfrak{H}(\varphi) = \mathfrak{H}$. Since $g_0 = \rho$ is constant and G is connected, $g = g_1$ is a contractible family, as desired. □

A similar argument works with strips $[a, b] \times [0, 1]$.

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