

# EQUIVARIANT FLOER COHOMOLOGY FOR CONTACTOMORPHISMS OF QUOTIENT SPACES

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ABSTRACT. This paper establishes the orderability of contact manifolds which are quotients of fillable contact manifolds under finite group actions compatible with the filling, the prototypical example being  $\mathbb{R}P^{2n-1}$  as the quotient of  $S^{2n-1}$ . Our approach employs an equivariant formulation of the so-called contact Floer cohomology theory. This leads us to develop an analogue of Givental’s nonlinear Maslov index using the  $\mathbf{k}[[x]]$ -module structure on an equivariant version of contact Floer cohomology. A key idea is that mapping cones of continuation maps detect crossings with the discriminant (recall that Givental’s index is a continuous integer valued function on the complement of the discriminant). To properly handle the inherent non-canonicity in defining such mapping cones, we lift the structure of contact Floer cohomology to chain level by defining it as an  $\infty$ -functor on a suitable  $\infty$ -categorification of the Eliashberg–Polterovich orderability relation on the universal cover of the contactomorphism group.

## 1. Introduction

Let  $G$  denote a finite group; our main results concern rigidity phenomena for contact manifolds  $Y$  which admit the following additional structure:

**Definition 1.1.** *A  $G$ -filling of a compact contact manifold  $Y$  is an open<sup>1</sup> convex-at-infinity symplectic manifold  $(W, \omega)$  with a contact-at-infinity symplectic  $G$ -action<sup>2</sup> whose ideal restriction acts freely on the ideal contact boundary  $\partial W$ , and a contactomorphism<sup>3</sup>  $Y \rightarrow \partial W/G$ . We say that  $Y$  admits an aspherical  $G$ -filling if the filling can be chosen so that  $\omega$  vanishes on spherical homology classes.*

In a sentence: we prove  $Y$  is orderable in the sense of [EP00] if it admits an aspherical  $G$ -filling  $W$  and the  $G$ -action on  $W$  is not free.

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<sup>1</sup>Here *open* means that each connected component of  $W$  is non-compact.

<sup>2</sup>Each  $g \in G$  acts by a symplectomorphism of  $W$  which commutes with the Liouville flow in the convex end of  $W$ ; see [AAC25, §2.1] for our conventions on convex ends.

<sup>3</sup>Note that  $Y$  inherits a preferred coorientation from any  $G$ -filling.

**1.1. Background.** To start, let us recall some basic concepts in contact geometry. Let  $(Y^{2n-1}, \xi)$  be a contact manifold. The group of diffeomorphisms preserving its contact structure is denoted by  $\text{Cont}(Y)$ ; elements of this group are called *contactomorphisms*. A *contact isotopy*<sup>4</sup> is a path  $\varphi_t \in \text{Cont}(Y)$ , defined for  $t \in [0, 1]$ , based at  $\varphi_0 = id$ , and we denote by  $\text{CI}(Y)$  the group of all such contact isotopies. The quotient of  $\text{CI}(Y)$  by the subgroup of contractible loops is the universal cover of  $\text{Cont}(Y)$ .

Attempts to study  $\text{Cont}(Y)$  and its universal cover have long been a major theme of contact geometry. In [EP00], Eliashberg and Polterovich propose to study these groups via the following relation: let us say that  $\varphi_{0,t} \leq \varphi_{1,t}$  if there is a path in  $\text{CI}(Y)$  from  $\varphi_{0,t}$  to  $\varphi_{1,t}$ , say  $\varphi_{s,t}$ , such that:

$$(1) \quad s \mapsto \varphi_{s,1}(y) \text{ is never negatively transverse to } \xi \text{ for each } y \in Y.$$

Note that, in order for this to make sense, we require that  $\xi$  is cooriented.

If  $\varphi_{0,t}$  and  $\varphi_{1,t}$  differ by a contractible loop, then they are mutually related by  $\leq$ , and so the relation descends to the universal cover of  $\text{Cont}(Y)$ . If the induced relation on the universal cover of  $\text{Cont}(Y)$  is a partial order, then  $Y$  is called *orderable*; otherwise,  $Y$  is called *non-orderable*.

It was shown in [EP00] that certain contact manifolds are orderable:

- real projective spaces  $\mathbb{R}P^{2n-1}$  (with standard contact structure),
- prequantization bundles over certain symplectic manifolds,
- spherical cotangent bundles  $ST^*M$ , for certain  $M$ .<sup>5</sup>

Since [EP00], much research has been done concerning orderability of contact manifolds. A particularly noteworthy milestone is [EKP06] which establishes the non-orderability of the ideal boundary of stabilizations  $W \times \mathbb{C}^d$  of Liouville manifolds  $W$  when  $d \geq 2$ ; in particular,  $S^3, S^5, \dots$ , with their standard contact structure, are all non-orderable.<sup>6</sup>

The tool used to prove orderability of  $\mathbb{R}P^{2n-1}$  is Givental's "non-linear generalization of the Maslov index" from [Giv90, Giv91], defined on the universal cover of the contactomorphism group of projective spaces using equivariant cohomology of generating functions. Let us recall here that Givental's index is a map  $\mu : \text{CI}(\mathbb{R}P^{2n-1}) \rightarrow \mathbb{Z}$  satisfying:

$$(G1) \text{ (monotonicity) } \mu(\varphi_{0,t}) \leq \mu(\varphi_{1,t}) \text{ if } \varphi_{0,t} \leq \varphi_{1,t};$$

<sup>4</sup>To lighten the notation, we use the symbol  $\varphi_t$  rather than, say  $\{\varphi_t : t \in [0, 1]\}$ , to denote contact isotopies. In the same way, a path of contact isotopies is denoted  $\varphi_{s,t}$ . This notational convention is the same as the one used in, e.g., [Can23, CHK23, Can24b].

<sup>5</sup>For the state of the art result that all spherical cotangent bundles are orderable we refer the reader to [CN16, Theorem 1.1].

<sup>6</sup>See also [HS24] for the case of  $W \times \mathbb{C}$  if  $W$  is  $T^*S^1$  or is a Weinstein manifold with dimension at least 4.

- (G2) (*continuity from above*)  $\mu(\varphi_t) = \lim_{s \rightarrow 0^+} \mu(R_{st}\varphi_t)$  for any positive contact isotopy  $R_s$ ;<sup>7</sup>
- (G3) (*normalization*)  $\mu(id) = 0$ ;
- (G4) (*non-triviality*)  $\mu$  attains arbitrarily large values;
- (G5) (*discriminant*)  $\mu$  is constant on the path-connected components of the complement of the discriminant.

Recall the *discriminant* in  $\text{CI}(Y)$  is the subset of those  $\varphi_t$  such that  $\varphi_1$  has a fixed point  $x$  satisfying  $\varphi_1^* \alpha_x = \alpha_x$ , for some/any contact form  $\alpha$  near  $x$ .

One of our main results (Theorem 1.11) involves the construction of invariants satisfying (G1) through (G5) for contact manifolds  $Y$  other than real projective spaces, and using Floer theory (settling Givental's claim in [Giv91, pp. 43] that a Floer theoretic approach to his index should be possible).

**Remark 1.2.** Givental's index famously generalizes the Maslov index: under the inclusion  $\text{Sp}(2n) \rightarrow \text{Cont}(\mathbb{R}P^{2n-1})$  his index pulls-back to the Maslov index; in particular, this yields (G3) and (G4). The index we define in Theorem 1.11 also has this property, see Proposition 1.12.

Property (G5) suggests that “sufficiently positive paths” must intersect the discriminant. This is related to the dynamics of Reeb vector fields. Recall that a *Reeb vector field* is any vector field  $R$  whose flow preserves  $\xi$  and which is positively transverse<sup>8</sup> to the contact distribution  $\xi$ . Let us denote the time  $s$  flow of  $R$  by the symbol  $R_s$ . It then follows that:

$$\psi_{s,t} = \varphi_t^{-1} \circ R_{st} \text{ (defined for } s \in \mathbb{R}\text{)}$$

satisfies  $\lim_{s \rightarrow \pm\infty} \mu(\psi_{s,t}) = \pm\infty$ . In particular, since Givental's index is locally constant on the complement of the discriminant,  $\psi_{s,t}$  must intersect the discriminant for infinitely many values of  $s$ .

In Sandon's work [San11a, San11b, San12, San13], the values of  $s$  when  $\psi_{s,t}$  intersects the discriminant were discovered to be the critical values of suitable generating functions and were called *lengths of translated points*.

**Definition 1.3.** *The spectrum  $\text{Spec}_R(\varphi_t)$  of  $\varphi_t$  relative to a Reeb flow  $R$ , is the set of numbers  $s \in \mathbb{R}$  such that  $\varphi_t^{-1} \circ R_{st}$  lies in the discriminant.*

The work of [AM13, AM18] develops an elliptic Morse theory for a pair  $(R, \varphi_t)$  whose associated spectrum of critical values is  $\text{Spec}_R(\varphi_t)$ ; their theory is based on the Rabinowitz–Floer homology (RFH) theory of [CF09].

The approaches of Sandon and Albers–Merry led to the introduction of *spectral invariants*, namely, functions  $c_R : \text{CI}(Y) \rightarrow \mathbb{R}$  satisfying:

- (R1) (*spectrality*)  $c_R(\varphi_t) \in \text{Spec}_R(\varphi_t)$ .

<sup>7</sup>A positive contact isotopy  $R_s$  is one for which the curves  $s \mapsto R_s(p)$  are positively transverse to the contact distribution

<sup>8</sup>It can be shown that each Reeb vector field satisfies  $\alpha(R) = 1$  and  $d\alpha(R, -) = 0$  for a unique contact form  $\alpha$ ; indeed, many authors define Reeb vector fields in this way.

The existence of such a spectral invariant is not guaranteed, as it implies the spectrum is non-empty, a fact which is not always true; see [Can24a].

More recently [Can23, DUZ23, Can24b, DUZ25] constructed such spectral invariants for certain contact manifolds  $Y$  using Hamiltonian Floer (co)homology in a symplectic filling; one requires that  $Y$  admits a filling  $W$  satisfying certain hypotheses; we refer the readers to [Can24b, DUZ25] for more details. This led to the construction of spectral invariants satisfying (R1) and four additional properties:

- (R2) (*monotonicity*)  $c_R(\varphi_{0,t}) \leq c_R(\varphi_{1,t})$  if  $\varphi_{0,t} \leq \varphi_{1,t}$ ,
- (R3) (*continuity from above*)  $c_R(\varphi_t) = \lim_{s \rightarrow 0^+} c_R(R_{st}\varphi_t)$ , where  $R_s$  is any Reeb flow,
- (R4) (*normalization*)  $c_R(id) = 0$ ,
- (R5) (*sub-additivity*)  $c_R(\varphi_{0,t}\varphi_{1,t}) \leq c_R(\varphi_{0,t}) + c_R(\varphi_{1,t})$ .

One of our main results (Theorem 1.10) guarantees the existence of such measurements under the assumption that  $Y$  admits a  $G$ -filling  $W$  with at least one point with non-trivial stabilizer (for the  $G$ -action).

Such a spectral invariant is sufficient to guarantee the orderability of  $Y$ :

**Proposition 1.4.** *The existence of  $c_R$  satisfying (R1) through (R5) implies  $Y$  is orderable.*

This statement is known in the literature (see, e.g., [AA23]). We quickly review the argument; the first step is to prove:

**Proposition 1.5.** *Any spectral invariant  $c_R$  satisfying axioms (R1) through (R5) also satisfies:*

$$|c_R(\varphi_{0,t}) - c_R(\varphi_{1,t})| \leq \text{dist}_\alpha(\varphi_{0,t}, \varphi_{1,t}),$$

where  $\text{dist}_\alpha$  is Shelukhin's Hofer distance [She17] associated to the contact form  $\alpha$  whose Reeb flow is  $R$ .

*Proof of Proposition 1.5.* If  $\varphi_{0,t} \leq R_{st}\varphi_{1,t}$  then:

$$c_R(\varphi_{0,t}) \leq c_R(R_{st}\varphi_{1,t}) \leq c_R(\varphi_{1,t}) + c_R(R_{st}) \leq c_R(\varphi_{1,t}) + s,$$

where the last step follows from:

$$c_R(R_{st}) \leq \lim_{k \rightarrow \infty} c_R(R_{st/k})k = (s/k)k = s;$$

here we appeal to the *spectrum gap for Reeb flows*<sup>9</sup> to conclude that  $s/k$  is the only number in the spectrum of  $R_{st/k}$  which lies in some small neighborhood of 0. Thus:

$$\begin{aligned} |c_R(\varphi_{0,t}) - c_R(\varphi_{1,t})| &\leq \min \{s : \varphi_{0,t} \leq R_{st}\varphi_{1,t} \text{ and } \varphi_{1,t} \leq R_{st}\varphi_{0,t}\} \\ &\leq \text{dist}_\alpha(\varphi_{0,t}, \varphi_{1,t}), \end{aligned}$$

where the latter inequality is well-known; see [AA23, Nak23]. □

<sup>9</sup> $R_s$  has no discriminant points if  $s \in (-\hbar, 0) \cup (0, \hbar)$  for some  $\hbar > 0$ .

*Proof of Proposition 1.4.* It follows easily from continuity and spectrality that  $c_R(R_{st}) = s$ , and hence  $\text{dist}_\alpha(1, R_{st}) \geq s$ . This implies, by work of [Hed24] that  $Y$  is orderable.  $\square$

**Remark 1.6.** The emerging picture from the spectral invariants  $c_R$  is of a contact geometry analogue to the well-known theory of spectral invariants for Hamiltonian diffeomorphisms of symplectic manifolds initiated by [Vit92, Sch00, Oh05], except there is one such structure *for each Reeb flow  $R$*  (as with the Hofer-type norms in [She17]). However, certain closed contact manifolds, e.g., the non-orderable ones, do not admit such structures (this contrasts with the symplectic setting where all closed symplectic manifolds are supposed to admit spectral invariants).

**Remark 1.7** (Hypertightness). It does not seem to be too hard to adapt the methods of this paper and [AFM15, AFM17, Oh21, Can24b, DUZ25] to show that any counterexample  $Y$  to the Weinstein conjecture admits invariants of type (R1) through (R5). The essential idea is to exploit the fact that any counterexample to the Weinstein conjecture is *hypertight*, that is, has no contractible Reeb orbits for some Reeb flow  $R$ . For such manifolds, one can “do Floer theory” directly in the symplectization  $SY$ , by appealing to the compactness results of [BEH<sup>+</sup>03].

**Remark 1.8** (Reeb orbits). Let us comment on the relationship of the invariants  $\mu$  satisfying (G1) through (G5) and the existence of Reeb orbits: any contact manifold  $Y$  admitting such a measurement  $\mu$  must admit closed Reeb orbits. Indeed, since  $\mu(R_{st})$  can only change when  $R_{st}$  crosses the discriminant, and  $\mu$  attains arbitrarily large values, so  $R_{st}$  must intersect the discriminant for infinitely many values of  $s$ . This observation and the discussion in Remark 1.7 shows the invariants of type  $\mu$  are, in a certain sense, rarer than the invariants of type  $c_R$ .<sup>10</sup>

**1.2. Statement of main results and examples.** Recall Definition 1.1 on aspherical  $G$ -fillings.

**Definition 1.9.** Let  $\varphi_t \in \text{CI}(Y)$ , and let  $W$  be a  $G$ -filling of  $Y$ . Define a  $W$ -contractible discriminant point of  $\varphi_t$  to be a discriminant point  $x$  such that  $t \mapsto \varphi_t(x)$  lifts from  $Y$  to  $\partial W$  and any lift is a contractible loop in  $W$ . The  $W$ -contractible discriminant is the subset consisting of those  $\varphi_t \in \text{CI}(Y)$  for which  $\varphi_t$  has a  $W$ -contractible discriminant point.

**Theorem 1.10.** Let  $G$  be a finite group and suppose that  $Y$  is a contact manifold admitting an aspherical  $G$ -filling  $W$ . If the  $G$ -action on  $W$  possesses at least one point with non-trivial stabilizer, then there exists a contact spectral invariant  $c_R$  satisfying (R1) through (R5) for any Reeb flow on  $Y$ .

<sup>10</sup>In fact, in certain cases, it will follow from our construction Theorem 1.11 that the values when  $\mu(R_{st})$  changes are periods of *contractible* Reeb orbits (we need to assume the map  $\pi_1(\partial W) \rightarrow \pi_1(W)$  is injective to make this deduction).

Moreover,  $c_R(\varphi_t)$  is the value of  $s$  for which  $R_{st} \circ \varphi_t^{-1}$  has a  $W$ -contractible discriminant point.

Thus we prove that all such manifolds  $Y$  admit spectral invariants for each Reeb flow, in the sense of Remark 1.6, and are, in particular, orderable.

If we assume moreover that  $\text{SH}(W) = 0$  then our construction produces a measurement  $\mu$  generalizing the Maslov class:

**Theorem 1.11.** *Assume the hypotheses of Theorem 1.10. If, in addition, the symplectic cohomology<sup>11</sup>  $\text{SH}(W; \mathbb{Z}/p\mathbb{Z})$  vanishes for some prime number  $p$  dividing the cardinality of a stabilizer of the  $G$ -action on  $W$ , then there is a measurement  $\mu : \text{CI}(Y) \rightarrow \mathbb{Z}$  satisfying properties (G1) through (G5).*

Moreover,  $\mu$  can be chosen to satisfy the strengthened version of (G5):  $\mu$  is constant on the path connected components of the complement of the  $W$ -contractible discriminant.

The obvious example of  $Y$  satisfying the hypotheses of Theorem 1.10 and 1.11 is  $\mathbb{R}P^{2n-1}$ , with its standard contact structure, since it admits the  $\mathbb{Z}/2\mathbb{Z}$ -filling  $\mathbb{C}^n$  with a fixed point, and  $\mathbb{C}^n$  is known to have vanishing symplectic cohomology (for any choice of coefficients  $\mathbf{k}$ ). Similarly, our theorem applies to the standard lens spaces  $L_p$ ,  $p > 1$ , obtained by quotienting  $S^{2n+1}$  by the  $\mathbb{Z}/p\mathbb{Z}$  action generated by  $z \mapsto e^{2\pi i/p}z$ , and we recover some of the results of [GKPS21, AAS24].

The fact that our measurement generalizes the Maslov class can be made quite precise:

**Proposition 1.12.** *Suppose  $W = \mathbb{C}^n$ , with the antipodal  $\mathbb{Z}/2\mathbb{Z}$ -action. If  $\varphi_t$  is an isotopy valued in the subgroup  $\text{Sp}(2n)$  whose time-1 map does not have 1 as an eigenvalue, then  $\mu(\varphi_t) = \text{CZ}(\varphi_t) - n$ , where the Conley-Zehnder index associated to a non-degenerate linear symplectic flow is as defined in, e.g., [PRSZ20, Chapter 8].<sup>12</sup>*

The proof of this proposition is given in §5.4.8. In the following subsections we give concrete examples of manifolds to which both Theorem 1.10 and 1.11 can be applied.

**1.2.1. Quotients of stabilized cotangent bundles.** Let  $W = T^*M \times \mathbb{C}^d$  and suppose that  $\mathbb{Z}/p\mathbb{Z}$  acts smoothly on  $M$  with isolated fixed points. There is an induced action on  $T^*M$  by canonical transformations — importantly, this

<sup>11</sup>The symplectic cohomology  $\text{SH}(W; \mathbb{Z}/p\mathbb{Z})$  is a well-studied invariant for convex-at-infinity manifolds; see, e.g., [Vit99, Sei08, Rit13]. To be concrete, we follow the conventions of [CHK23, Can24b], and work only in the free homotopy class of contractible loops. The orientation scheme needed to work with primes  $p > 2$  will be explained in §2.4

<sup>12</sup>For  $\mu$  to be normalized to zero, we require  $\text{CZ}(R_{\epsilon t}) = n$  for  $\epsilon \in (0, 1)$ , where  $R$  is the standard Reeb flow on the sphere  $S^{2n-1}$ , extended radially to  $\mathbb{C}^n$ . The quadratic Hamiltonian generating this linear isotopy is  $\epsilon\pi|z|^2$ .

action preserves the Liouville form  $\lambda_{T^*M}$ . Then  $\mathbb{Z}/p\mathbb{Z}$  acts diagonally on  $W$ , where the action on  $\mathbb{C}^d$  is by multiplication by  $e^{2\pi i/p}$ . This action preserves  $\lambda_{T^*M} + \lambda_{\mathbb{C}^d}$ , and thus has an ideal restriction; this is a free action on  $\partial W$  because the action on  $M$  has isolated fixed points. Let  $Y = \partial W/(\mathbb{Z}/p\mathbb{Z})$ . In this setting, Theorem 1.10 applies if the action on  $M$  has at least one fixed point. If, in addition,  $d > 0$ , then Theorem 1.11 applies. The case when  $M = \text{pt}$  recovers the case of  $\mathbb{R}P^{2d-1}$  and the lens spaces discussed above.

Interestingly, [EKP06, Theorem 1.16] and [HS24, Corollary 1.3] show the ideal boundary  $\partial(T^*M \times \mathbb{C}^d)$  is non-orderable (for  $d \geq 2$  by [EKP06], or  $d \geq 1$  if  $\dim M \geq 1$  by [HS24]), and hence does not admit measurements satisfying (R1) through (R5) or (G1) through (G5).

**1.2.2. Non-primitive prequantization spaces.** Let  $(B, \omega)$  be a symplectic manifold, and suppose there is a unitary line bundle  $\pi : W \rightarrow B$  satisfying<sup>13</sup>  $c_1 = -[\omega]$ . As explained in [McD91, pp. 656], there is a natural way to equip  $W$  with the structure of a symplectic manifold which is convex-at-infinity (the construction uses the unitary structure on  $W$ ). The fiberwise action of  $e^{2\pi i/p}$  is a symplectomorphism with fixed point set equal to the zero set  $B$ . If  $B$  is symplectically aspherical, it is known that the non-equivariant regular symplectic cohomology  $\text{SH}(W; \mathbb{Z}/p\mathbb{Z})$  vanishes (this is shown in, e.g., [CHK23] in the case  $p = 2$ ). Therefore Theorem 1.10 and 1.11 apply to  $Y = (\partial W)/(\mathbb{Z}/p\mathbb{Z})$ , when  $B$  is aspherical.

The construction shows that  $\partial W$  is naturally identified with the unit circle bundle in  $W$ , and that  $Y$  is simply the ideal boundary of the negative line bundle  $W^{\otimes p}$  with base space  $(B, p\omega)$ . The orderability of such  $Y$  was established already for a special class of aspherical  $B$  in [EP00, §1.3.C].

Note that in order for our methods to apply, we require that  $Y = \partial W^{\otimes p}$  for  $p > 1$ , i.e.,  $Y$  is “non-primitive,” with respect to the tensor product. Some assumption is clearly necessary, since  $Y = S^{2n-1}$  (which is a primitive prequantization space) is non-orderable [EKP06, HS24].

As a comparison, using the generating function approach of [Giv90, Giv91], the work of [BZ15, Zap20, GKPS21] establishes the existence of measurements satisfying (G1) through (G5) for certain special choices of  $Y$  — in these constructions the base  $B$  is a toric manifold; see also [San13, Ter21] which uses generating functions to show  $\text{Spec}_R(\varphi_t) \neq \emptyset$  for all  $\varphi_t$  and a specific Reeb vector field  $R$  when the base  $B$  is a closed monotone symplectic toric manifold (provided that  $Y$  is not the standard contact sphere).

<sup>13</sup>Recall that  $c_1$  is the deRham cohomology class of represented by any differential two-form  $\mathbf{c}$  solving  $d\alpha = \pi^*\mathbf{c}$ , where  $\alpha$  is a *global angular form*, i.e.,  $\alpha$  is a one-form on  $W$  which restricts to  $(2\pi)^{-1}(xdy - ydx)/(x^2 + y^2)$  in each fiber.

Floer theoretic approaches also have been developed to study the orderability (and construction of spectral invariants) for prequantization spaces; see [ASZ16], and the RFH approach<sup>14</sup> of [AF12, AM18, AK23, BKK24].<sup>15</sup>

**1.3. Infinity categorification of the Eliashberg–Polterovich relation.** The arguments used in the proofs of Theorems 1.10 and 1.11 are based on mapping cones of chain maps. We briefly preview the set-up:

- to each contact isotopy  $\varphi_t$  lying in the complement of the discriminant (see Definition 1.9), we will associate a chain complex  $\mathrm{CF}_{\mathrm{eq}}(\varphi_t)$ ;
- to each non-negative path  $\varphi_{s,t}$  (1) between  $\varphi_{0,t}$  and  $\varphi_{1,t}$  we will associate a chain map  $\mathrm{CF}_{\mathrm{eq}}(\varphi_{0,t}) \rightarrow \mathrm{CF}_{\mathrm{eq}}(\varphi_{1,t})$ ;
- the mapping cone of this chain map is related to the intersections of the path  $\varphi_{s,t}$  with the discriminant.

It is known that it is difficult to work with such cones on the level of homology groups (e.g., this leads to the theory of triangulated categories). Many arguments can be presented much more simply when one is working “on chain level.” On the other hand, it is also known to be somewhat difficult to properly keep track of chain homotopy terms when working on chain level. A modern but well-established approach to working on chain level is the theory of  $\infty$ -categories following [Lur09, Lur17].

The first goal in this section (continued in §2.1) is to introduce an  $\infty$ -category  $\mathcal{C}(Y)$  of contact isotopies. Roughly speaking, an infinity category is a collection of sets  $\mathcal{C}_k(Y)$  of  $k$ -simplices, for  $k = 0, 1, \dots$ , together with “face” relationships between these sets of simplices. The 0-simplices should be considered as the “objects” of the category and the 1-simplices should be considered as the “morphisms.” In our construction, the 0-simplices  $\mathcal{C}_0(Y)$  are contact isotopies  $\varphi_t$  lying outside the discriminant<sup>16</sup> and the 1-simplices  $\mathcal{C}_1(Y)$  are the non-negative paths  $\varphi_{s,t}$  as in (1). Each 1-simplex  $\varphi_{s,t} \in \mathcal{C}_1$  has two *faces* in  $\mathcal{C}_0(Y)$ , and these are simply the endpoints  $\varphi_{0,t}$  and  $\varphi_{1,t}$  of the non-negative path.

In a general  $\infty$ -category, the  $k$ -simplices represent homotopies between morphisms and their compositions. The definition of the set of  $k$ -simplices  $\mathcal{C}_k(Y)$  in our  $\infty$ -category is of the form:

$$(2) \left\{ \text{“coherent” cubes } [0, 1]^{k-1} \rightarrow \{ \text{non-negative paths } [0, k] \rightarrow \mathrm{CI}(Y) \} \right\}.$$

<sup>14</sup>We emphasize that here the spectral invariants we construct are monotone (R2) and sub-additive (R5), and these properties are not established in the existing RFH literature.

<sup>15</sup>The paper [AF12] generalizes Givental’s work [Giv90, Giv91] in a different direction from our paper. Their construction does not produce integer valued measurements for contact isotopies  $\varphi_t$ , but rather studies the “asymptotic growth rates” of Rabinowitz–Floer homology groups when one iterates positive paths.

<sup>16</sup>The construction admits a minor variation where  $\mathcal{C}_0(Y)$  is the complement of the  $W$ -contractible discriminant.

The precise meaning of *coherent* will be explained in §2.1. We adopt the following conventions in the edge cases when  $k = 0$ :

- $[0, 1]^{-1} = \{\text{pt}\}$ ,
- a non-negative path  $[0, 0] \rightarrow \text{CI}(Y)$  is a chosen point.

The face and degeneracy maps for coherent cubes will be explained in §2.1. Our main structural theorem is that the *equivariant Floer cohomology complex*<sup>17</sup> induces an  $\infty$ -functor:

$$(3) \quad \text{CF}_{\text{eq}} : \mathcal{C}(Y) \rightarrow \text{N}_{\text{dg}}\text{Ch}(\mathbf{k}[[x]]).$$

The right hand side is the so-called *dg-nerve* of the category of graded<sup>18</sup> chain complexes over the ring  $\mathbf{k}[[x]]$ . The only piece of information we need about this dg-nerve is *how to define a functor valued in it*. Luckily the definition is rather compact:

**Definition 1.13** (Taken from [Par16, §7.6]). *An  $\infty$ -functor (3) consists of the following assignments:*

- each zero simplex  $\sigma = \varphi_t$  is sent to a (graded)  $\mathbf{k}[[x]]$ -module:

$$V_\sigma = \text{CF}_{\text{eq}}(\varphi_t)$$

with a differential  $d_\sigma$  of degree 1;

- each  $n$ -dimensional simplex  $\sigma$  is sent to a map:

$$\mathbf{c}_\sigma : V_{\sigma|0} \rightarrow V_{\sigma|n}$$

of degree  $1 - n$ ; here  $\sigma|j$  is the  $j$ th vertex of  $\sigma$ .

These maps are required to satisfy the structural equation:

$$(4) \quad \sum_{j=1}^{n-1} (-1)^j (\mathbf{c}_{\sigma|[j\dots n]} \circ \mathbf{c}_{\sigma|[0\dots j]} - \mathbf{c}_{\sigma|[0\dots \hat{j}\dots n]}) = \mathbf{c}_\sigma \circ d_{\sigma|0} + (-1)^n d_{\sigma|n} \circ \mathbf{c}_\sigma.$$

Importantly, because 1-simplices are sent to chain maps, we can talk about their mapping cones in a simple and canonical way. We now state the theorem which will be used to prove Theorem 1.10 and 1.11:

**Theorem 1.14.** *If  $Y$  admits an aspherical  $\mathbb{Z}/p\mathbb{Z}$ -filling  $W$  with  $p$  prime, then there is an  $\infty$ -functor (3) for  $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$  satisfying the following:*

- the output  $\text{CF}_{\text{eq}}(\varphi_t)$  is a finitely generated  $\mathbf{k}[[x]]$ -module, for each zero simplex  $\varphi_t \in \mathcal{C}_0(Y)$ ,*
- the homology of the cone of the chain map associated to any 1-simplex in  $\mathcal{C}_1(Y)$  is  $x$ -torsion (i.e., multiplication by  $x^d$  on the homology of the cone acts by 0, for some positive integer  $d$ ),*

<sup>17</sup>Here we are referring to the ‘‘Borel’’ version of equivariant cohomology, as advanced by [SS10, Sei15, SZ21, GMP23, Caz24, SC25].

<sup>18</sup>Our gradings will be valued in the group  $\mathbb{Z}/2\mathbb{Z}$  (or sometimes the trivial group).

(c) if the  $\mathbb{Z}/p\mathbb{Z}$ -action on  $W$  has at least one fixed point, then the homology of  $\mathrm{CF}_{\mathrm{eq}}(\varphi_t)$  is not  $x$ -torsion, for any object  $\varphi_t \in \mathcal{C}_0(Y)$ .

The construction of the  $\infty$ -functor is performed in §2. The proof of (b) uses a local Floer homology argument in §4. The proof of (c) uses the PSS isomorphism (Theorem 1.16).

**1.3.1. Homology level functor.** Taking homology of the  $\infty$ -functor in Theorem 1.14 gives an ordinary functor:

$$(5) \quad \mathrm{HF}_{\mathrm{eq}} : \mathrm{h}\mathcal{C}(Y) \rightarrow \mathrm{Mod}(\mathbf{k}[[x]])$$

where  $\mathrm{h}\mathcal{C}(Y)$  is the so-called *homotopy category*<sup>19</sup> of  $\mathcal{C}(Y)$  and  $\mathrm{Mod}(\mathbf{k}[[x]])$  is the category of modules over  $\mathbf{k}[[x]]$ . Since  $\mathrm{HF}_{\mathrm{eq}}$  is a regular functor defined on a regular category, we can take its colimit in the usual sense (as in, e.g., [Mac71]). This leads to the definition:

**Definition 1.15.** *The equivariant symplectic cohomology  $\mathrm{SH}_{\mathrm{eq}}(W)$  is defined to be the colimit of (5) in the category of  $\mathbf{k}[[x]]$ -modules.*

The  $\mathbf{k}[[x]]$ -module  $\mathrm{SH}_{\mathrm{eq}}(W)$  will be used to define the invariants  $c_R$  and  $\mu$ . As we shall see, the invariant  $\mu$  relies on the  $\mathbf{k}[[x]]$ -module structure in an important way.

**1.4. PSS map.** There is a larger  $\infty$ -category  $\bar{\mathcal{C}}(Y)$  such that  $\bar{\mathcal{C}}_0(Y) = \mathrm{CI}(Y)$  and  $\mathcal{C}(Y) \subset \bar{\mathcal{C}}(Y)$  is the full subcategory spanned by the contact isotopies without discriminant points, (or  $W$ -contractible discriminant points, both variants work equally well); see Remark 2.6.

Inside this larger  $\infty$ -category  $\bar{\mathcal{C}}(Y)$ , we will define another  $\infty$ -category  $\mathcal{P}(Y)$ , which we call the *PSS category*.<sup>20</sup> Roughly speaking, simplices in  $\mathcal{P}(Y)$  have some of their vertices in  $\mathcal{C}(Y)$ , and the other vertices are fixed at the constant system  $id$  (see Definition 3.7 for the precise definition). For the purposes of the introduction, it suffices to say:

- any 1-simplex  $id \rightarrow \varphi_t$  in  $\bar{\mathcal{C}}(Y)$ , with  $\varphi_t \in \mathcal{C}(Y)$ , is in  $\mathcal{P}(Y)$ .

Such 1-simplices represent “continuation data” from the constant system  $id$  to some output system  $\varphi_t \in \mathrm{CI}(Y)$ , which must necessarily satisfy  $id \leq \varphi_t$ .

**Theorem 1.16.** *Assume the hypotheses of Theorem 1.14. Counting solutions of moduli spaces of type [PSS96, FS07] defines an  $\infty$ -functor:*

$$(6) \quad \mathcal{P}(Y) \rightarrow \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathbf{k}[[x]]))$$

where:

- non-identity 0-simplices  $\varphi_t$  are sent to the outputs  $\mathrm{CF}_{\mathrm{eq}}(\varphi_t)$  of the functor from Theorem 1.14;

<sup>19</sup>see [Lur09, §1.2.3] for the definition of  $\mathrm{h}\mathcal{C}$ .

<sup>20</sup>The name comes from [PSS96].

- the identity 0-simplex  $id$  is sent to the equivariant Morse cohomology complex  $CM_{\text{eq}}(X)$  associated to a  $G$ -equivariant Morse-Smale pseudogradient  $X$  which points outwards in the convex end of  $W$  (the definition is reviewed in 3.1.3).

By virtue of how  $\infty$ -functors work, the aforementioned 1-simplices in  $\mathcal{P}(Y)$  joining  $id$  to a  $\varphi_t \in \mathcal{C}(Y)$  are sent to chain maps:

$$\text{PSS} : CM_{\text{eq}}(X) \rightarrow CF_{\text{eq}}(\varphi_t).$$

If we apply this to a 1-simplex of the form  $\varphi_{s,t} = R_{\epsilon st}$ , for  $\epsilon > 0$  small enough, then the induced map on homology:

$$\text{PSS} : HM_{\text{eq}}(X) \rightarrow HF_{\text{eq}}(R_{\epsilon t}).$$

is an isomorphism which can be understood as an equivariant version of the PSS isomorphism of [PSS96, FS07].

**1.4.1. On canonicity.** The reader may wonder whether our construction is “canonical,” i.e., whether any two readers who follow our construction will obtain the “same”  $\infty$ -functor. Our invariant is not canonical, in the same way “the Morse complex” is not canonical: two readers can follow each step of the construction correctly but pick auxiliary data differently and end up with non-isomorphic complexes  $CF_{\text{eq}}(\varphi_t)$  (they should, at least, be quasi-isomorphic).

Our invariant is canonical in a weaker sense: there exists a *canonical diagram* in the category of  $\infty$ -categories:

$$\begin{array}{ccc} \mathcal{D}(W) & \xrightarrow{CF_{\text{eq}}} & N_{\text{dg}}\text{Ch}(\mathbf{k}[[x]]) \\ \downarrow \pi & & \\ \mathcal{C}(Y) & & \end{array}$$

where the vertical map is a *trivial Kan fibration*; see [Lur09, §2] or §1.7 for the definition. This diagram is canonical, in that two readers will agree on the result. This statement is of course not entirely mathematical (and borders on the philosophical). A more mathematical formulation of “canonicity” is that the diagram is well-defined up to a canonical isomorphism. A similar statement holds with  $\mathcal{C}(Y)$  replaced by the PSS category  $\mathcal{P}(Y)$ .

The  $\infty$ -functor from Theorem 1.14 is defined as a precomposition  $CF_{\text{eq}} \circ s$ , where  $s$  is a section of  $\pi$  (a similar statement holds for the  $\infty$ -functor in Theorem 1.16). Well-known results from [Lur09] imply these sections exist in abundance. Moreover, using basic properties of trivial Kan fibrations, one can relate the two functors  $CF_{\text{eq}} \circ s$  and  $CF_{\text{eq}} \circ s'$  by picking a “path” between  $s$  and  $s'$  in the “space” of sections and thereby prove the outputs are chain homotopy equivalent. Such a comparison between sections  $s, s'$  will prove the homology level functors agree up to natural isomorphism. In this

sense, the lack of canonicity is isolated into the single “contractible choice” of section  $s$ .

Of course, such geometric language as “path” and “space” needs to be interpreted properly, and we return to this more precisely in §1.7.

**1.5. Definition of the measurements.** First [Alu09, Theorem IV.2.1] implies: *if a prime number  $p$  divides the order of the stabilizer of some point  $w \in W$ , then there is a  $\mathbb{Z}/p\mathbb{Z}$ -subgroup of the stabilizer.* Then define  $Y'$  to be the quotient by the action of this  $\mathbb{Z}/p\mathbb{Z}$  subgroup. Since  $Y'$  is a covering of  $Y$ , it is sufficient to prove Theorem 1.10 and 1.11 for  $Y'$  instead of  $Y$ . Thus, for the remainder of the discussion, we may suppose that  $Y = Y'$  and  $G = \mathbb{Z}/p\mathbb{Z}$ . We can therefore apply Theorems 1.14 and 1.16.

The quantities  $c_R$  and  $\mu$  satisfying Theorems 1.10 and 1.11 are defined in terms of a *unit element*  $1 \in \mathrm{SH}_{\mathrm{eq}}$ . It is constructed as follows: the chain maps  $\mathrm{CM}_{\mathrm{eq}}(X) \rightarrow \mathrm{CF}_{\mathrm{eq}}(\varphi_t)$  from Theorem 1.16, associated to a 1-simplex  $\varphi_{s,t}$ , yield cycles  $1(\varphi_{s,t}) \in \mathrm{CF}_{\mathrm{eq}}(\varphi_t)$  corresponding to the sum of local minima in  $\mathrm{CM}_{\mathrm{eq}}(X)$ .

**Lemma 1.17.** *The image of the element  $1(\varphi_{s,t})$  in the colimit  $\mathrm{SH}_{\mathrm{eq}}$  is independent of the choice of 1-simplex  $\varphi_{s,t}$ .*

This lemma (proved in §5.1.1) furnishes a canonical element  $1 \in \mathrm{SH}_{\mathrm{eq}}$ . Following [Can23, DUZ23, Can24b, DUZ25] we define:

$$(7) \quad c_R(\varphi_t) := \inf \left\{ s : 1 \text{ is in the image of } \mathrm{HF}_{\mathrm{eq}}(\varphi_t^{-1} \circ R_{st}) \rightarrow \mathrm{SH}_{\mathrm{eq}}(W) \right\}.$$

In §5.3 we prove that this definition of  $c_R$  satisfies (R1) through (R5).

Turning to  $\mu$ , the new ingredient in Theorem 1.11 is the hypothesis that  $\mathrm{SH}(W; \mathbf{k}) = 0$ . We first explain how the classical symplectic cohomology fits into the framework of our paper. Define the *non-equivariant quotient* as a tensor product:<sup>21</sup>

$$(8) \quad \mathrm{CF}_{\mathrm{neq}}(\varphi_t) := \mathrm{CF}_{\mathrm{eq}}(\varphi_t) \otimes_{\mathbf{k}[[x]]} \mathbf{k},$$

where  $x$  acts by 0 on  $\mathbf{k}$ . Its homology is denoted by  $\mathrm{HF}_{\mathrm{neq}}(\varphi_t)$  and we define:

$$\mathrm{SH}_{\mathrm{neq}}(W) := \text{colimit of } \mathrm{HF}_{\mathrm{neq}}(\varphi_t) \text{ over } \varphi_t \in \mathrm{h}\mathcal{C}(Y).$$

We show in §5.4 that:

$$(9) \quad \mathrm{SH}(W; \mathbf{k}) = 0 \implies \mathrm{SH}_{\mathrm{neq}}(W) = 0,$$

where  $\mathrm{SH}(W; \mathbf{k})$  agrees with standard definitions of symplectic cohomology, such as the one in [Sei08].

From (8) and (9), there is a long exact sequence:

$$\cdots \longrightarrow \mathrm{SH}_{\mathrm{eq}} \xrightarrow{x} \mathrm{SH}_{\mathrm{eq}} \longrightarrow 0 \longrightarrow \cdots,$$

<sup>21</sup>Note that this is a chain level construction, which cannot be directly accessed from the homology level invariants.

The measurement  $\mu$  satisfying Theorem 1.11 is:

$$\mu(\varphi_t) := \sup \left\{ d \in \mathbb{Z} : x^{-d}1 \text{ is in the image of } \mathrm{HF}_{\mathrm{eq}}(\varphi_t) \rightarrow \mathrm{SH}_{\mathrm{eq}}(W) \right\};$$

we extend from  $\mathcal{C}_0(Y)$  (the complement of the discriminant) to all of  $\mathrm{CI}(Y)$  by continuity from above, i.e., in the unique way compatible with (G2).

In §5.4 we prove (9) and show that  $\mu$  satisfies (G1) through (G5).

**1.6. Local Floer cohomology and cones of continuation maps.** An important ingredient is part (b) of Theorem 1.14, which asserts that cones of continuation maps are always  $x$ -torsion. The key idea is that the cone of a continuation map can be understood via the *local Floer homologies* of isolated crossings with the discriminant. The rigorous development of this idea is the object of §4. Beyond the statement about cones of continuation maps being  $x$ -torsion, another important consequence of this theory is:

**Theorem 1.18.** *If  $\varphi_{s,t}$  is a 1-simplex in  $\mathcal{C}(Y)$  such that  $\varphi_{s,t}$  does not lie on the  $W$ -contractible discriminant, for each  $s$ , then the continuation map:*

$$\mathrm{CF}_{\mathrm{eq}}(\varphi_{0,t}) \rightarrow \mathrm{CF}_{\mathrm{eq}}(\varphi_{1,t})$$

*is an isomorphism.*

*Consequently, if two isotopies  $\varphi_{0,t}, \varphi_{1,t}$  can be joined in the complement of the  $W$ -contractible discriminant by any path (not assumed to be non-negative), then the morphisms  $\mathrm{HF}_{\mathrm{eq}}(\varphi_{i,t}) \rightarrow \mathrm{SH}_{\mathrm{eq}}$  are isomorphic.*

The first part of the statement goes back to [UZ22]. The method of proof of the second part is to approximate any path by a zig-zag of positive and negative paths; see [Ulj23, §9] and [DUZ25, §2.1.3] for further discussion of such zig-zags. We prove Theorem 1.18 in §4.4.4.

**1.7. On  $\infty$ -categories.** Since the  $\infty$ -categorical language is not yet completely standard in symplectic geometry we recall here the reason for its appearance in the Floer theoretic setting, as well as some basic properties; the reader comfortable with  $\infty$ -categories and their role in Floer theory may skip this section.

One of the fundamental issues in Floer theory is that of *invariance*: how to show that the chain complexes we construct do not depend (in a suitable sense) on any choices auxiliary to the geometry at hand (e.g. almost complex structures, choices of Morse-Smale pseudogradient, etc). Morally, one expects these extra choices to be ultimately irrelevant whenever the space of such choices is contractible, and more generally, certain allowable paths in the data should give rise to chain maps.

In our setting we are naturally provided with the following setup: there is a topological category  $\mathcal{C}_{\mathrm{top}}(Y)$  whose objects are contact isotopies, and whose morphism spaces consist of the non-negative paths connecting two given objects. Lying over this (via a natural forgetful functor) is a larger

topological category  $\mathcal{D}_{\text{top}}(W)$  whose objects are all data necessary to define the Floer cohomology complex, and whose morphism spaces consist of the spaces of regular continuation data with specified endpoints. However, this topological category  $\mathcal{D}_{\text{top}}(W)$  and the dg-category of chain complexes do not directly interact very nicely. The language of quasi-categories systematically developed in [Lur09] provides a setting into which both dg-categories and topological categories can be naturally integrated by means of taking the *topological and dg-nerves*,  $N_{\text{top}}, N_{\text{dg}}$  respectively.

In the present paper the aforementioned  $\infty$ -category  $\mathcal{C}(Y)$  models the topological nerve of the category  $\mathcal{C}_{\text{top}}(Y)$ , and we define in §2.2.2 an  $\infty$ -category of Floer data  $\mathcal{D}(W)$  modeling the topological nerve of  $\mathcal{D}_{\text{top}}(W)$ ; there is forgetful functor  $\mathcal{D}(W) \rightarrow \mathcal{C}(Y)$ . We prove in §2.2.6 that this forgetful map is a trivial Kan fibration:

**Definition 1.19** (See [Lur09, §2]). *A trivial Kan fibration is a morphism of simplicial sets  $\mathcal{D} \rightarrow \mathcal{C}$  such that following lifting diagram is satisfied:*

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{D} \\ \downarrow \iota & \nearrow \exists & \downarrow \pi \\ \Delta^n & \longrightarrow & \mathcal{C} \end{array}$$

for all  $n \geq 0$ , where  $\iota$  is the obvious inclusion map. If  $\mathcal{S} \rightarrow \Delta^0$  is a trivial Kan fibration, then  $\mathcal{S}$  is called a *contractible Kan complex*, and should be thought of as the simplicial set analog of a contractible space.

Trivial Kan fibrations and contractible Kan complexes enjoy many convenient properties; the most important for our purposes is the following:

**Proposition 1.20.** *Let  $\pi : \mathcal{D} \rightarrow \mathcal{C}$  be a trivial Kan fibration. Then  $\pi$  admits a section  $s$ . Moreover, the collection of such sections forms the zero simplices in a contractible Kan complex  $\mathcal{S}$ .*

*Proof.* Define an  $n$ -simplex in the space of sections  $\mathcal{S}$  to be a map

$$s : \Delta^n \times B \rightarrow E$$

such that  $\pi s = \text{pr}_B$ . There is an obvious structure of a simplicial set with these as the  $n$ -simplices. Since  $\pi$  is a trivial Kan fibration, monomorphisms (injections) of simplicial sets have the left lifting property against  $\pi$ ; see [Lur09, Example 2.0.0.2]. By using this lifting property with the monomorphisms  $\partial\Delta^n \times B \hookrightarrow \Delta^n \times B$ , we prove that this simplicial set of sections  $\mathcal{S}$  is a contractible Kan complex.  $\square$

The general strategy is then as follows: There is a canonical Floer complex functor:

$$\text{CF} : \mathcal{D}(W) \rightarrow N_{\text{dg}}(\text{Ch}(\mathbf{k}[[x]]))$$

since the left hand side consists of exactly the data necessary to define Floer theory. Then every section  $s$  of  $\pi$  induces a functor:

$$\mathrm{CF}_{\mathrm{eq}} \circ s : \mathcal{C}(Y) \rightarrow \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathbf{k}[[x]]))$$

by composition. Contractibility of the space of sections then says that any choice of  $s$  (i.e., any compatible collection of Floer data) induces homotopic Floer functors and, moreover, all such homotopies are themselves homotopic, etc. We return to this discussion in §2.2.6.

Carrying out such a “chain-level construction” has the advantage of making certain constructions (such as taking cones and tensor products<sup>22</sup>) considerably simpler. In our case, it is used in §4 to verify that a certain Floer complex may be expressed as a mapping cone for a very particular choice of data, and it then follows by the abstract machinery that any other choice of data extending the underlying system has the same property.

Such setups have been leveraged before in Floer theory to similar ends, cf., [Par16], [HLS20], [GPS20, §4], [Var21, §A].

## 2. Equivariant Floer complex as infinity functor

The goal of the first subsection §2.1 is to introduce the  $\infty$ -category of contact isotopies (continuing from the outline we gave in §1.3). In §2.2 we prove Theorem 1.14 in the case  $p = 2$ , which has various differences and simplifications from the general case  $p \geq 3$ , which we treat in §2.3 and §2.4.

**2.1. Infinity category of contact isotopies.** As an  $\infty$ -category,  $\mathcal{C}(Y)$  is, first and foremost, a special type of *simplicial set*; see [Lur09, §A.2.7] for a review of simplicial sets. This means that  $\mathcal{C}(Y)$  is a contravariant set-valued functor from the *combinatorial simplex category*, whose objects are non-negative integers  $[0], [1], [2], \dots$ , and where the morphisms  $[i] \rightarrow [j]$  are the non-decreasing maps  $\{0, \dots, i\} \rightarrow \{0, \dots, j\}$ . More prosaically,  $\mathcal{C}(Y)$  is the data of sets  $\mathcal{C}_0(Y), \mathcal{C}_1(Y), \mathcal{C}_2(Y), \dots$ , where  $\mathcal{C}_j(Y)$  is called the set of  *$j$ -simplices*, together with morphisms:

$$(10) \quad f^* : \mathcal{C}_j(Y) \rightarrow \mathcal{C}_i(Y) \text{ for every morphism } f : [i] \rightarrow [j],$$

and such that the morphisms in (10) are functorial.

**2.1.1. Definition of the simplicial set.** We define<sup>23</sup>  $\mathcal{C}_n(Y)$  to be certain coherent cubes of non-negative paths in the group of contact isotopies  $\mathrm{CI}(Y)$ ; to state the notion of coherency, we set some notation:

<sup>22</sup>We use a chain level construction in the definition of  $\mathrm{HF}_{\mathrm{neq}}(\varphi_t)$  as the homology of  $\mathrm{CF}_{\mathrm{eq}}(\varphi_t) \otimes_{\mathbf{k}[[x]]} \mathbf{k}$ .

<sup>23</sup>In this section, we primarily discuss the construction of  $\mathcal{C}(Y)$ . However, the discussion works equally well for constructing the larger category  $\bar{\mathcal{C}}(Y)$  (see §1.4), the only difference being that 0-simplices in the full-subcategory  $\mathcal{C}(Y) \subset \bar{\mathcal{C}}(Y)$  are required to be in the complement of the discriminant.

- $\Delta^n$  has  $n + 1$  vertices labelled  $v_0, v_1, \dots, v_n$ ;
- a line segment  $\ell : [a, b] \rightarrow \Delta^n$  is called a *complete straight line path segment* provided it is of the form:
  - (1)  $\ell(a) = v_{q-1}$  for some  $q = 1, \dots, n$ ,
  - (2)  $\ell(b)$  lies in the convex hull of the vertices  $\{v_q, \dots, v_n\}$ ,
  - (3)  $\ell(s) = (b - a)^{-1}((b - s)\ell(a) + (s - a)\ell(b))$ ;
- the convex hull of  $v_0, \dots, v_n$  has *barycentric coordinates*:
 
$$\theta_0, \dots, \theta_n \text{ satisfying } \theta_i \geq 0 \text{ and } \theta_0 + \dots + \theta_n = 1;$$
- the *speed one-form*  $\sigma$  is defined by the formula:

$$\sigma = \sum j d\theta_j$$

and the integral of this over a complete straight line path segment is a strictly increasing linear function.

This leads to the following definition:<sup>24</sup>

**Definition 2.1.** A straight line path from  $v_0$  to  $v_n$  is a path  $\ell : [0, n] \rightarrow \Delta^n$  together with times  $0 = \tau_0 < \tau_1 \leq \dots \leq \tau_{n-1} \leq \tau_n = n$  such that:

- (1)  $\ell(\tau_0) = v_0$
- (2)  $\ell(\tau_q)$  lies in the convex hull of  $v_q, \dots, v_n$ ; in particular  $\ell(\tau_n) = v_n$ ,
- (3) on the interval  $[\tau_{q-1}, \tau_q]$ ,  $\ell$  equals the affine reparametrization of the restriction of a complete straight line path segment starting at  $v_q$ ,
- (4) the map  $\tau \in [\tau_{q-1}, \tau_q] \mapsto \ell(\tau)$  is parametrized with unit speed, i.e., the integral of the speed one-form on an interval of parameter length  $T$  is equal to  $T$ .

We denote by  $\mathcal{M}(\Delta^n)$  the moduli space of all straight line paths in  $\Delta^n$ .

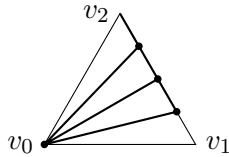


FIGURE 1. Elements of  $\mathcal{M}(\Delta^2)$ .

It is a convenient fact that:

**Lemma 2.2.** If a straight line path  $\ell$  passes through  $v_q$ , then  $\ell^{-1}(v_q) = \{q\}$ .

*Proof.* Suppose that  $\ell(\tau) = v_q$ . The total integral of the speed one-form over  $[0, \tau]$  is equal to  $q$ , since the speed one-form is exact with primitive:

$$\sum j\theta_j.$$

<sup>24</sup>This definition is closely related to other spaces of paths in the simplex, e.g., [Ada56], [GPS20, Remark 4.1], and [Par16, Definition 10.1.14].

On the other hand, since  $\ell$  is parametrized with unit speed, the length of  $\ell$  over  $[0, \tau]$  is  $\tau$ , thus  $\tau = q$ .  $\square$

There is an obvious identification  $\mathcal{M}(\Delta^1) \simeq \text{pt}$ . Moreover:

**Lemma 2.3.** *There is a canonical identification  $\mathcal{M}(\Delta^n) \simeq \mathcal{M}(\Delta^{n-1}) \times [0, 1]$ . In particular, by induction,*

$$\mathcal{M}(\Delta^n) \simeq [0, 1]^{n-1}.$$

*The coordinates furnished by the proof will be referred to as cubical coordinates.*

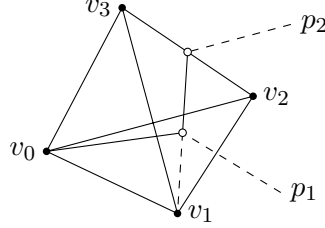


FIGURE 2. Cubical coordinates on the moduli space of straight line paths are ultimately derived from the points  $p_1$  and  $p_2$ .

*Proof.* The identification is as follows: each straight line path  $\ell$  in  $\mathcal{M}(\Delta^n)$  intersects the face opposite  $v_0$  at some time  $\tau_1(\ell)$ , at a point  $p_1 = \ell(\tau_1(\ell))$ . There is a unique straight line path  $F(\ell)$  passing through  $v_1$  such that:

- $F(\ell)(\tau)$  lies on the line segment connecting  $v_1$  and  $p_1$  for  $\tau \in [1, \tau_1(\ell)]$ .
- $F(\ell)(\tau) = \ell(\tau)$  for  $\tau \geq \tau_1(\ell)$ .

Denote by  $\mathcal{M}_{v_1}(\Delta^n)$  the moduli space of straight line paths through  $v_1$ . Then define  $G : \mathcal{M}(\Delta^n) \rightarrow \mathcal{M}_{v_1}(\Delta^n) \times [0, 1]$  by the formula:

$$G(\ell) = (F(\ell), E(p_1)) \text{ where } E(p_1) = \theta_2(p_1) + \cdots + \theta_n(p_1).$$

We claim that  $G$  is an identification; define for  $f \in \mathcal{M}_{v_1}(\Delta^n)$  the time:

$$T(f, e) := \inf \{ \tau \geq 1 : E(f(\tau)) = e \}$$

and let  $q(f, e) = f(T(f, e))$ . There is a unique straight line path  $\ell = H(f, e)$  in  $\mathcal{M}(\Delta^n)$  such that:

- $\ell|_{[0, T(f, e)]}$  parametrizes the line segment joining  $v_0$  to  $q(f, e)$ ,
- $\ell|_{[T(f, e), n]} = f|_{[T(f, e), n]}$ .

It is then easily verified that  $G(H(f, e)) = (f, e)$  and  $H(G(\ell)) = \ell$ .

To complete the proof, we only need to observe the obvious fact that  $\mathcal{M}_{v_1}(\Delta^n)$  and  $\mathcal{M}(\Delta^{n-1})$  are canonically isomorphic; the restriction of any path in  $\mathcal{M}_{v_1}$

to the interval  $[1, n]$  (and translating  $[1, n] \rightarrow [0, n - 1]$ ) gives a straight line path in the simplex spanned by  $v_1, \dots, v_n$ . This completes the proof.  $\square$

Let us denote for future use these cubical coordinates as  $x_1, \dots, x_{n-1}$ :

$$x_i(\ell) = \theta_{i+1}(p_i) + \theta_{i+2}(p_i) + \dots$$

where  $p_i$  is the first position on  $\ell$  valued in the convex hull of  $v_i, \dots, v_n$ . Recall that the times  $\tau_1, \dots, \tau_n$  from Definition 2.1 satisfy  $\ell(\tau_i) = p_i$ .

**Lemma 2.4.** *The times  $\tau_i(\ell)$  are polynomial functions of the cubical coordinates  $x_i(\ell)$  satisfying the relation  $\tau_i(\ell) = i + x_i(\tau_{i+1}(\ell) - i)$ .*

*Proof.* By the unit speed requirement,  $\tau_i = \tau_i(\ell)$  equals:

$$\tau_i = \sum j\theta_j(p_i) \text{ where } p_i \text{ is as above.}$$

Since  $p_i$  lies in the convex hull of  $v_i, \dots, v_{i+1}$ , we can replace this by:

$$\tau_i - i = \sum_{j>i} (j - i)\theta_j(p_i).$$

Now, because the line segment passing through  $p_i$  and  $p_{i+1}$  also passes through  $v_i$  it is not hard to see that:

$$\frac{\sum_{j>i} (j - i)\theta_j(p_i)}{\sum_{j>i} \theta_j(p_i)} = \frac{\sum_{j>i} (j - i)\theta_j(p_{i+1})}{\sum_{j>i} \theta_j(p_{i+1})} = \sum_{j>i} (j - i)\theta_j(p_{i+1}) = \tau_{i+1} - i.$$

Thus, for  $i = 1, \dots, n - 1$  it holds that:

$$\tau_i - i = x_i(\tau_{i+1} - i),$$

and so if  $\tau_{i+1}$  is a smooth function of  $x_1, \dots, x_{n-1}$ , then so is  $\tau_i$ . Since  $\tau_n = n$ , which is smooth, it follows that all the functions are smooth. E.g., when  $n = 3$  we have  $\tau_3 = 3$ ,  $\tau_2 = 2 + x_2$ , and  $\tau_1 = 1 + x_1(1 + x_2)$ .  $\square$

**Definition 2.5.** *An  $n$ -simplex in  $\mathcal{C}(Y)$  is a map:*

$$\Phi : \mathcal{M}(\Delta^n) \rightarrow \{\text{piecewise smooth non-negative paths } [0, n] \rightarrow \text{CI}(Y)\},$$

written  $\Phi(\ell, s) \in \text{CI}(Y)$ , satisfying the axioms:

- (N1) if  $\ell_1(q) = \ell_2(q) = v_q$  and  $\ell_1(p) = \ell_2(p) = v_p$  for  $q \leq p$ , and  $\ell_1$  and  $\ell_2$  agree on the interval  $[q, p]$ , then  $\Phi(\ell_1, s)$  and  $\Phi(\ell_2, s)$  agree for  $s \in [q, p]$ ;
- (N2) if  $\ell(q) = v_q$ , the time-1 map of  $\Phi(\ell)(q)$  has no discriminant points;
- (N3) the restriction to the  $i$ th level  $L_i\Phi(x, s)$ , defined by:

$$L_i\Phi(x(\ell), s) = \Phi(\ell)(\tau_{i-1} + s(\tau_i - \tau_{i-1}))$$

is a smooth function on the cube. Here  $x(\ell)$  refers to the cubical coordinate representation of  $\ell$ . We require the levels piece together:

$$L_i\Phi(x, 1) = L_{i+1}\Phi(x, 0).$$

(N<sub>4</sub>)  $L_i\Phi(x, s)$  is independent of  $x_i$  in a neighborhood of the faces  $x_i = 0$  and  $x_i = 1$ , for  $i = 1, \dots, n-1$ .

A zero simplex is simply a choice of element of  $\text{CI}(Y)$  whose time-1 map has no discriminant points.

**Remark 2.6.** The simplicial set  $\bar{\mathcal{C}}(Y)$  is defined in the exact same way, except we do not require axiom (N2).

We now explain how to endow  $[n] \mapsto \mathcal{C}_n(Y)$  with the structure of a simplicial set. Suppose there is a morphism  $f : [n] \rightarrow [m]$ ; this gives an affine morphism of simplices  $f : \Delta^n \rightarrow \Delta^m$  sending  $v_i \in \Delta^n$  to  $v_{f(i)} \in \Delta^m$ . Then:

**Lemma 2.7.** *For any straight line path  $\ell$  in  $\Delta^n$ , there exists a straight line path  $f_*\ell$  in  $\Delta^m$  such that:*

$$f \circ \ell(\tau) = f_*\ell(\rho(\tau))$$

for a unique piecewise affine surjection  $\rho : [0, n] \rightarrow [f(0), f(n)]$ . The restriction  $f_*\ell|_{[f(0), f(n)]}$  is uniquely determined by  $f$  and  $\ell$ .

If  $\tau_1 \leq \dots \leq \tau_{n-1}$  are the times associated to  $\ell$ , and  $\sigma_1 \leq \dots \leq \sigma_m$  are the times associated to  $f_*\ell$ , then  $\rho$  maps  $[\tau_i, \tau_{i+1}]$  linearly into one of the intervals  $[\sigma_j, \sigma_{j+1}]$  (not necessarily injectively or surjectively).

*Proof.* The idea is more or less clear when  $f$  is injective. The technical part of the argument involves the case when  $f$  is not injective. We argue as follows. Let  $\tau_0 < \tau_1 \leq \dots \leq \tau_{n-1}$  be the times for  $\ell$  as in Definition 2.1. Let  $i$  be the first index such that  $f(\ell(\tau_i))$  lies in the convex hull of the vertices  $v_q$  with  $q > f(0)$ ; if there is no such  $i$ , then  $f$  is constant, and the conclusions are trivial when  $f_*\ell$  is any straight line path passing through  $v_{f(0)} = v_{f(n)}$ .

Then we claim that  $f \circ \ell|_{[\tau_0, \tau_i]}$  is a piecewise affine reparametrization of a complete straight line path segment starting at  $v_{f(0)}$ . Indeed, it suffices to show that each segment  $f \circ \ell|_{[\tau_{j-1}, \tau_j]}$  for  $j = 1, \dots, i$  is mapped into the straight line joining  $f(v_0)$  and  $f(\ell(\tau_i))$  in a monotonic way. This holds because any complete line segment starting at  $v_{j-1}$  for  $j = 1, \dots, i$  is mapped to some line segment starting at  $f(v_0)$ .

Proceeding in this fashion, one constructs  $\rho$  and  $f_*\ell|_{[f(0), f(n)]}$ . One then extends  $f_*\ell$  to  $[0, f(0)]$  and  $[f(n), m]$  so that it is a straight line path (the extension is otherwise arbitrary).  $\square$

Given  $f, \ell$ , let  $\rho, f_*\ell$  satisfy the conclusions of Lemma 2.7. By the axioms in Definition 2.5, the values of  $\Phi(f_*\ell)$  on the interval  $[f(0), f(n)]$  are independent of the ambiguity of  $f_*\ell$ . We then define:

$$(11) \quad f^*\Phi(\ell) := \Phi(f_*\ell)|_{[f(0), f(n)]} \circ \rho.$$

Importantly, this construction gives:

**Lemma 2.8.** *The pull back  $f^*\Phi$  satisfies the axioms in Definition 2.5. Additionally, for any two such maps  $f, g$ , it holds that  $(fg)^*\Phi = g^*f^*\Phi$ .*

*Proof.* The axioms (N1) through (N4) follow more or less immediately from the properties of  $\rho$  and  $f_*\ell$  from 2.7 and the formula in (11).

To analyze  $fg$ , we combine:

- $g_*\ell \circ \rho_g = g \circ \ell$  and
- $f_*(g_*\ell) \circ \rho_f = f \circ (g_*\ell)$
- $(fg)_*\ell \circ \rho_{fg} = fg \circ \ell$

and use the uniqueness parts of Lemma 2.7 to conclude:

$$\rho_{fg} = \rho_f \rho_g \text{ and } f_*g_*\ell = (fg)_*\ell$$

where the latter equality must be restricted to  $[fg(0), fg(n)]$ . But then it follows quite easily that:

$$(fg)^*\Phi = \Phi((fg)_*\ell) \circ \rho_{fg} = \Phi(f_*g_*\ell) \circ \rho_f \rho_g = (f^*\Phi(g_*\ell)) \circ \rho_g = g^*f^*\Phi.$$

Thus (11) defines a simplicial set.  $\square$

**2.1.2. Verification of the  $\infty$ -category axiom.** An  $\infty$ -category is not merely a simplicial set, but is required to satisfy additional *horn-filling* properties; see [Lur09, Definition 1.1.2.4]. See also [Lur09, Notation A.2.7.2] and [Lur09, Example A.2.7.3].

**Proposition 2.9.** *The simplicial set  $\mathcal{C}(Y)$  is an  $\infty$ -category.*

*Proof.* Recall the  $i$ -horn  $\Lambda_i(\Delta^n) \subset \Delta^n$  is the union of all codimension 1 faces containing the vertex  $v_i$ . For a simplicial set  $\mathcal{C}$  to be an  $\infty$ -category, it must have the property that any interior  $i$ -horn can be filled: any map (in the category of simplicial sets):

$$\Lambda_i(\Delta^n) \rightarrow \mathcal{C}$$

extends to a map defined on  $\Delta^n$  provided that  $i \neq 0$  and  $i \neq n$ .

To prove the interior horn filling property holds for  $\mathcal{C}(Y)$ , we will use the cubical coordinates  $x_1, \dots, x_{n-1}$  from Lemma 2.3.

We need to build a map  $\Phi : \mathcal{M}(\Delta^n) \rightarrow \{\text{non-negative paths } [0, n] \rightarrow \text{CI}(Y)\}$ , which restricts to prescribed data on each face. In cubical coordinates:

- the face  $x_i = 0$  consists of all straight line paths through  $v_i$ ,
- the face  $x_i = 1$  consists of all straight line paths which remain in the convex hull of all vertices *not* equal to  $v_i$ .

Then, by the data prescribed on the  $i$ -horn, we see that the values of  $\Phi$  are already determined on all the faces of the cube *except* for the (interior of the) face  $x_i = 1$ . Thus we reduce the horn filling problem to a more classical problem in topology: *given a map  $\Phi$  defined on all faces of a cube, except the interior of a face, can we extend  $\Phi$  to the entire cube?*

Recall that  $\Phi$  is supposed to satisfy the constraints (N1) and (N2) on straight line paths which pass through vertices: we now explain why these constraints are unimportant.

**Claim.** *Each straight line path corresponding to an interior point of  $[0, 1]^{n-1}$ , or a point on the interior of the  $x_i = 1$  face, does not pass through any vertices except for  $v_0$  and  $v_n$ .*

The proof of the claim is easy: if a path passes through another vertex, then its cubical point  $q$  lies in a different face, say  $x_j = 0$ , and so  $q$  would not be an interior point (by definition).  $\square$

Since the constraints (N1) and (N2) on  $\Phi$  are imposed on the paths which pass through vertices  $v_1, \dots, v_{n-1}$ , our problem reduces to the problem of extending  $\Phi$  from the union of all faces, except the interior of  $x_i = 1$ , to the whole cube, remaining in the space of maps:

$$X = \{\text{non-negative paths } [0, n] \rightarrow \text{CI}(Y) \text{ from } \Phi(v_0) \text{ to } \Phi(v_n)\},$$

satisfying the smoothness conditions (N3) and (N4). Any such extension represents a valid  $n$ -simplex in  $\mathcal{C}(Y)$  filling the horn. This existence of some extension from the faces of the cube to the interior uses the Serre fibration property for the map  $X \rightarrow \text{pt}$ . Let  $Q$  denote the cube, and let  $P = \partial Q - \{x_i = 1\}$ . Using that the prescribed values of  $\Phi$  on  $P$  satisfy axiom (N4), we can extend  $\Phi$  to a small neighborhood of  $P$  so that (N4) holds for the extension. Then, one uses an appropriate vector field  $V$  which vanishes on a neighborhood of  $P$  to retract  $Q$  into the neighborhood of  $P$  on which  $\Phi$  has been extended. Pulling back the values of  $\Phi$  by this retraction defines the desired extension. It is left to the reader to show that  $V$  can be chosen so that  $\Phi$  satisfies (N4). The proof of (N3) is a straightforward verification, and uses the fact that  $V$  vanishes on a neighborhood of  $P$ . This completes the proof of the theorem.  $\square$

**Remark 2.10.** The same exact argument proves that the larger category  $\bar{\mathcal{C}}(Y)$ , where we do not require vertices lie off of the discriminant, is an infinity category.

**2.1.3. Equivalent morphisms.** In an  $\infty$ -category, two 1-simplices  $\sigma_a, \sigma_b$  are *equivalent* if there is 2-simplex  $\Sigma$  satisfying  $\Sigma|_{[0,1]} = \sigma_a$ ,  $\Sigma|_{[0,2]} = \sigma_b$ , and  $\Sigma|_{[1,2]} = id$ , where *id* means a degenerate 1-simplex; see [Lur09, pp. 39].

Express  $\sigma_a$  as  $\sigma_{a,s,t}$  and similarly with  $\sigma_b$ . In our case, we can express  $\Sigma$  as a family:

$$\Sigma_{x_1,s,t} \in \text{Cont}(Y),$$

where  $x_1 \in [0, 1]$  is the sole cubical coordinate for  $\mathcal{M}(\Delta^2)$ , satisfying:

- $\Sigma_{0,s,t} = \sigma_{a,s,t}$  for  $s \in [0, 1]$ , while  $\Sigma_{0,s,t} = \sigma_{a,1,t} = \sigma_{b,1,t}$  for  $s \in [1, 2]$ ,
- $\Sigma_{1,s/2,t} = \sigma_{b,s,t}$  for  $s \in [0, 2]$ ,
- (fixed endpoints)  $\Sigma_{x_1,0,t} = \sigma_{a,0,t}$  and  $\Sigma_{x_1,2,t} = \sigma_{a,1,t}$ , for all  $x_1$ ,

From this description we see that: *two 1-simplices  $\sigma_a, \sigma_b$  are equivalent if and only if  $\sigma_{a,s,t}$  is homotopic to  $\sigma_{b,s,t}$  in the space of non-negative paths in  $\text{CI}(Y)$  with fixed endpoints  $s = 0, 1$ .* It follows that the homotopy category  $\text{h}\mathcal{C}$  is the category considered in [CHK23, 2.2.9].

**2.1.4. On the complexity of  $\mathcal{C}(Y)$ .** In many appearances of  $\infty$ -categories in Floer theory, the  $\infty$ -category of data  $\mathcal{C}$  is a contractible Kan complex (this is the case, in, e.g., the treatment of Hamiltonian Floer theory in [Par16]). In such cases, the  $\infty$ -category itself is not of significant independent interest, and its use is solely in achieving more canonical constructions (this is not to say the use of  $\infty$ -categories is unimportant in these contexts). In other appearances, such as [GPS20, Var21] the  $\infty$ -category of data  $\mathcal{C}$  satisfies the property that the natural map to the nerve of the homotopy category:

$$(12) \quad \mathcal{C} \rightarrow \text{N}(\text{h}\mathcal{C})$$

is a trivial Kan fibration (Definition 1.19). Such categories should be considered as equivalent to their 1-skeleton; see [GPS20, §4.4] and [Var21, §3.2.2] for a related discussion. In these cases, the homotopy category  $\text{h}\mathcal{C}$  carries some amount of complexity, but no additional complexity is hidden in  $\mathcal{C}$ . It is notable that in our case, the category  $\mathcal{C}(Y)$  does not satisfy this property.

**Lemma 2.11.** *The morphism (12) is not a trivial Kan fibration for  $\mathcal{C} = \mathcal{C}(Y)$  if  $\pi_k(\text{Cont}(Y))$  is non-trivial for at least one number  $k \geq 3$ .*

*Proof.* The property we need about the nerve of the ordinary category  $\text{h}\mathcal{C}$  is that any map  $\partial\Delta^k \rightarrow \text{N}(\text{h}\mathcal{C})$  extends to a map  $\Delta^k \rightarrow \text{N}(\text{h}\mathcal{C})$ , if  $k \geq 3$ . In particular, if (12) is a trivial Kan fibration, then any map  $\partial\Delta^k \rightarrow \mathcal{C}(Y)$  extends to a map  $\Delta^k \rightarrow \mathcal{C}(Y)$ , for  $k \geq 3$ .<sup>25</sup> We will use the existence of a non-trivial class  $f \in \pi_k(\text{Cont}(Y))$  to produce a boundary  $\partial\Delta^k \rightarrow \mathcal{C}(Y)$  which cannot be filled, thereby proving the lemma.

First of all, by standard adjunction identifications, we can think of  $f$  as a  $k - 1$  simplex in  $\mathcal{C}(Y)$  (see Definition 2.5), i.e., as map:

$$[0, 1]^{k-2} \rightarrow \{[0, k-1] \rightarrow \text{CI}(Y)\},$$

written  $f_{x,s,t}$ , satisfying the properties that  $f_{x,s,t} = \text{id}$  when  $(x, s, t)$  lies on the boundary of  $[0, 1]^{k-2} \times [0, k-1] \times [0, 1]$ . We suppose that  $f$  is smooth,

<sup>25</sup>Indeed, the information of a 3-simplex in  $\text{N}(\text{h}\mathcal{C})$  is a sequence of three composable morphisms  $f, g, h$ , while the data of  $\partial\Delta^3 \rightarrow \text{N}(\text{h}\mathcal{C})$  is the four sequences

- $f, g$  on the  $\{0, 1, 2\}$  face,
- $f, gh$  on the  $\{0, 1, 3\}$  face,
- $fg, h$  on the  $\{0, 2, 3\}$  face,
- $g, h$  on the  $\{1, 2, 3\}$  face.

One can simply take  $f, g$  from the  $\{0, 1, 2\}$  face and  $h$  from the  $\{1, 2, 3\}$  face, and prove that the three simplex  $(f, g, h)$  fills the map  $\partial\Delta^3 \rightarrow \text{N}(\text{h}\mathcal{C})$ ; the verification uses the fact that the faces of the constituents of a map  $\partial\Delta^3$  satisfy various equations amongst each other.

so that it represents a  $(k-1)$ -simplex in  $\mathcal{C}(Y)$  (crucially, the non-negativity assumption is automatic, since  $f_{x,s,1} = id$  holds for all  $x, s$ ).

Now consider  $\Gamma(f) = \partial\Delta^k \rightarrow \mathcal{C}(Y)$  which agrees with  $f$  on the  $[1 \dots k]$  face, and agrees with  $id$  on each other face. Then:

**Claim.** *If  $\Gamma(f)$  can be filled to a  $k$ -simplex in  $\mathcal{C}(Y)$ , then  $f$  represents the trivial element in  $\pi_k(\text{Cont}(Y))$ .*

Indeed, if so, then there exists  $F_{x,s,t}$  so that:

$$x_1 = 0 \implies F_{x,s,t} = \begin{cases} id & \text{for } s \in [0, 1], \\ f_{x_2, \dots, x_{k-1}, s-1, t} & \text{for } s \in [1, k+1]. \end{cases}$$

and so that,  $F_{x,s,t} = id$  on all other faces of  $[0, 1]^{k-1} \times [0, k] \times [0, 1]$ . The only “tricky” face to establish is the face  $t = 1$ . Because  $s \mapsto F_{x,s,1}$  is a non-negative loop based at  $id$ , and there do not exist  $C^0$  small non-constant and non-negative loops,<sup>26</sup> it must hold that  $s \mapsto F_{x,s,1}$  is constant on an open and closed set of values of  $x$ . Since the loop is constant for some values of  $x$ , it holds by connectivity that  $s \mapsto F_{x,s,1}$  is constant for all values of  $x$ . This completes the proof of the claim, and therefore of the lemma.  $\square$

**Remark 2.12.** In [CS16, Theorem 1] it is shown that  $i : U(n) \rightarrow \text{Cont}(S^{2n-1})$  is injective on homotopy groups. Thus the above result is not vacuous.

**2.2. Equivariant Floer cohomology over characteristic two.** We split the proof of Theorem 1.14 into the two cases  $p = 2$  and  $p \geq 3$ . There are differences between the cases:

- in the case  $G = \mathbb{Z}/2\mathbb{Z}$ , we consider  $EG$  as  $S^\infty \subset \mathbb{R}^\infty$  with the antipodal  $\mathbb{Z}/2\mathbb{Z}$ -action, and use the shift self-symmetry of  $\mathbb{R}^\infty$  :

$$\tau : (\eta_0, \eta_1, \dots) \mapsto (0, \eta_0, \eta_1, \dots);$$

- in the case  $G = \mathbb{Z}/p\mathbb{Z}$  with  $p \geq 3$ , we consider  $EG$  as  $S^\infty \subset \mathbb{C}^\infty$  with the  $\mathbb{Z}/p\mathbb{Z}$ -action inherited from the circle action on  $\mathbb{C}^\infty$ , and we use the shift self-symmetry of  $\mathbb{C}^\infty$  instead of  $\mathbb{R}^\infty$ . This leads to various differences in the recipe for constructing equivariant Floer complexes.
- in the case  $p = 2$ , so  $\mathbf{k} = \mathbb{Z}/2\mathbb{Z}$ , we do not need to discuss orientations in Floer theory, whereas the case  $p \geq 3$  requires such discussion.

In the remainder of this subsection §2.2 we focus solely on the case  $p = 2$ .

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<sup>26</sup>The non-existence of  $C^0$  small non-negative and non-constant loops of contactomorphisms follows from the existence of generating functions and action selectors for Legendrian isotopies in 1-jet spaces due to [Che96] see also [CS15].

**2.2.1. Reminder of the Borel equivariant cohomology complex.** Our approach to defining equivariant Floer cohomology complexes follows [SS10, Sei15]. Their strategy is to emulate the Borel construction [Bor60] in the language of Morse theory in such a way that it carries over to Floer theory. We briefly recall the idea in the context of Morse theory: given a compact manifold  $M$  with a  $\mathbb{Z}/2\mathbb{Z}$ -action, one considers pseudogradients on  $M \times_{\mathbb{Z}/2\mathbb{Z}} S^\infty$  of a specific type (some care is needed due to the non-compactness of  $S^\infty$ ). Appropriately counting the flow lines of such a pseudogradient defines the equivariant Morse differential.

One thinks of a pseudogradient  $P$  on  $M \times_{\mathbb{Z}/2\mathbb{Z}} S^\infty$  as a  $\mathbb{Z}/2\mathbb{Z}$ -equivariant vector field on  $M \times S^\infty$ . It is convenient to fix a standard pseudogradient  $V$  on  $S^\infty$ , and then only consider  $P$  which are related to  $V$  under the projection  $M \times S^\infty \rightarrow S^\infty$ . Such vector fields can be thought of as  $S^\infty$ -families of vector fields on  $M$  satisfying a certain  $\mathbb{Z}/2\mathbb{Z}$ -equivariance. In the Floer theoretic setting, one similarly considers  $S^\infty$ -families of auxiliary data as the necessary input to obtain a chain complex.

**2.2.2. Auxiliary category of Floer data.** As explained in §1.7, we do not directly define an infinity functor  $\mathcal{C}(Y) \rightarrow \mathbf{N}_{\text{dg}}(\text{Ch})$ , but rather construct a diagram of the form:

$$(13) \quad \begin{array}{ccc} \mathcal{D}^*(W) & \xrightarrow{\text{CF}_{\text{eq}}} & \mathbf{N}_{\text{dg}}(\text{Ch}) \\ \text{forget} \downarrow & & \\ \mathcal{C}(Y) & & \end{array}$$

where  $\mathcal{D}^*(W)$  is a larger  $\infty$ -category whose objects consist of all the necessary data needed to define the Floer complex; for similar set-up, we refer the reader to [Par16, Eqn. 7.0.1]. We will first define  $\mathcal{D}(W)$  and then define  $\mathcal{D}^*(W) \subset \mathcal{D}(W)$  as a subcategory of “regular” simplices. The goal of the present section is to construct  $\mathcal{D}(W)$  and the forgetful map. In §2.2.6, we show that the forgetful map in (13) is a *trivial Kan fibration* (see [Par16, Eqn. 7.5.3]). Precomposing  $\text{CF}_{\text{eq}}$  with a section of this fibration induces an  $\infty$ -functor  $\mathcal{C}(Y) \rightarrow \mathbf{N}_{\text{dg}}(\text{Ch})$ ; this is the  $\infty$ -functor satisfying Theorem 1.14.

The definition is in terms of *Borel data*, namely pairs  $(\psi_{\eta,t}, J)$  where:<sup>27</sup>

- $J$  is a  $G$ -invariant,  $\omega$ -tame and Liouville-equivariant-at- $\infty$  almost complex structure on  $W$ ,
- $\psi_{\eta,t}$  is a family of Hamiltonian isotopies of  $W$  depending on a parameter  $\eta \in S^\infty$ ;

Borel data is supposed to satisfy the following:

- (B1) (*equivariance*)  $g^{-1}\psi_{g\eta,t}g = \psi_{\eta,t}$ ,
- (B2) (*shift invariance*)  $\psi_{\tau(\eta),t} = \psi_{\eta,t}$ , where  $\tau : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$  shifts by 1,

<sup>27</sup>We refer the reader to [SS10, pp. 1468] for a similar set-up.

- (B3) (*polar*)  $\psi_{\eta,t}$  is independent of  $\eta$  in fixed neighborhoods of the poles,  
 (B4) (*ideal restriction*) the Hamiltonian generator  $X_{\eta,t}$  of  $t \mapsto \psi_{\eta,t}$  is  $\eta$ -independent and Liouville-equivariant outside of a compact set.

Consequently there is a contact isotopy arising as the ideal restriction, denoted  $\varphi_t = \text{IR}(\psi_{\eta,t})$ . A piecewise smooth path of Borel data:<sup>28</sup>

$$\tau \in [0, n] \mapsto (\psi_{\tau,\eta,t}, J_\tau)$$

is said to be *non-negative* provided  $\tau \in [0, n] \mapsto \text{IR}(\psi_{\tau,\eta,t})$  is non-negative. Then an  $n$ -simplex  $\Phi \in \mathcal{D}_n(W)$  is defined as a map:

$$(14) \quad \Phi : \mathcal{M}(\Delta^n) \rightarrow \{\text{non-negative paths } [0, n] \rightarrow \text{Borel data}\}$$

satisfying the axioms of Definition 2.5, except for a minor rewording of the second axiom:

- (N2) if  $\ell(q) = v_q$ , the time-1 map of the ideal restriction of  $\Phi(\ell)(q)$  has no discriminant points.

The same proof of the ‘‘horn-filling axioms’’ given in Theorem 2.9 works here to show  $\mathcal{D}(W)$  is an  $\infty$ -category. Moreover, there is an obvious ‘‘ideal restriction’’  $\infty$ -functor  $\mathcal{D}(W) \rightarrow \mathcal{C}(Y)$ .

Inside of  $\mathcal{D}(W)$  we will define a subcategory  $\mathcal{D}^*(W)$  consisting of those simplices which achieve *regularity*, i.e., which render a certain (countable) collection of moduli spaces transverse. We defer the definition of  $\mathcal{D}^*(W)$  and the proof that it is an  $\infty$ -category to the end of the next section §2.2.3.

**2.2.3. Moduli spaces.** First of all, we fix a pseudogradient  $V$  on  $\mathbb{R}P^\infty$ . In projective coordinates, the time  $s$  flow of  $V$  is given by:

$$[x_0 : x_1 : x_2 : \dots] \mapsto [x_0 : e^s x_1 : e^{2s} x_2 : \dots].$$

This pseudogradient on  $\mathbb{R}P^\infty$  has one critical point  $v_k$  of each index  $k$ , and we refer to these as *poles*. The lift of  $V$  to the double cover  $S^\infty$  will also be denoted by  $V$ , and it has two critical points  $v_{k,\pm}$  of each index, which we also call poles. Let  $\mathcal{P}(v_{k,\pm})$  denote the space of parametrized flow lines of  $V$  on  $S^\infty$  joining  $v_{0,+}$  to  $v_{k,\pm}$ . These form open manifolds of the expected dimension, namely  $k$ , and can be compactified to manifolds with corners in a reasonable way; see, e.g., [Sei15, §3.1].

Each triple  $(\Phi, k, \pm)$ , where  $\Phi$  is an  $n$ -simplex in  $\mathcal{D}(W)$ ,  $k = 0, 1, 2, \dots$ , and  $\pm$  is a sign, determines a moduli space  $\mathcal{M}(\Phi, k, \pm)$  of triples  $(\ell, \pi, u)$  of the following type:

- $\ell \in \mathcal{M}^{\text{int}}(\Delta^n) \simeq (0, 1)^{n-1}$ ,
- $\pi \in \mathcal{P}(v_{k,\pm})$ , and,
- $u$  solves a suitable version of Floer’s equation on  $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ .

<sup>28</sup>The axioms (B3) and (B4) involve open neighborhoods, and these neighborhoods are assumed to be uniform along any piecewise smooth family.

**Remark 2.13.** As is typical in Floer theory, it is necessary to show  $\mathcal{M}(\Phi, k, \pm)$  has certain compactness properties, which ultimately boil down to establishing a priori estimates on certain energy integrals; see, e.g., [Can24b, §2.2.6] and [CHK23, §2.2.5] for some related discussion.

For technical use, we also fix a smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that:

- $f'(t)$  is 1-periodic and non-negative,
- $f'(t)$  vanishes in a neighborhood of 0, and,
- $f(0) = 0$  and  $f(1) = 1$ .

and a  $[0, 1]$ -valued cut-off function  $\rho$  satisfying:

- $\rho'(t) \leq 0$ ,
- $\rho'(t)$  is supported in the interior of the region where  $f = 1$ ,
- $\rho(0) = 1$  and  $\rho(1) = 0$ ;

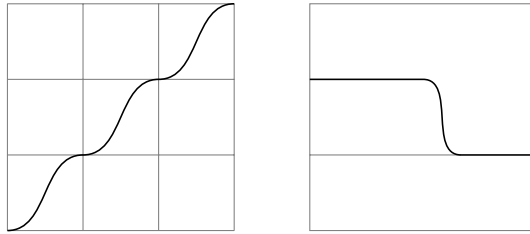


FIGURE 3. Graph of auxiliary functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $\rho : \mathbb{R} \rightarrow \mathbb{R}$ . We require the derivative  $\rho'(t)$  to be supported in the interior of the interval where  $f = 1$ .

For similar use of such auxiliary functions  $f, \rho$  in setting up the continuation equation, we refer the reader to [Can23, CHK23, Can24b].<sup>29</sup>

The role of  $f$  will be the following: if  $\psi_{\eta,t}$  is a Hamiltonian isotopy, extended to all times  $t \in \mathbb{R}$  by the rule  $\psi_{\eta,t+1} = \psi_{\eta,t}\psi_{\eta,1}$ , then we will consider the time-reparametrization  $\psi_{\eta,f(t)}$ . We will also use  $f$  as a cut-off function: given  $[a, b]$  and  $[c, d]$ , the function:

$$(15) \quad s \mapsto \tau(s) = c + (d - c)f\left(\frac{b - s}{b - a}\right)$$

parametrizes  $[c, d]$  by  $[a, b]$  in a decreasing way.

Let us first consider  $\mathcal{M}(\Phi, k, \pm)$  in the case when  $n = 0$ , so  $\mathcal{M}^{\text{int}}(\Delta^n) = \text{pt}$  and  $\ell$  is unimportant. Let  $X_{\eta,t}$  be the generator of  $\psi_{\eta,f(t)}$ , and consider the

<sup>29</sup>In order for our constructions to be canonical, in the sense that any two readers of the paper will agree on all constructions, it is necessary that the readers pick the same cut-off functions  $f, \rho$ . To make this choice canonical, we should give a precise formula for the cut-off functions  $f$  and  $\rho$ . Such formulas can be found in any advanced calculus text.

following equation for a pair  $(\pi, u)$ :

$$(16) \quad \begin{cases} u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow W \text{ and } \pi : \mathbb{R} \rightarrow S^\infty, \\ \pi'(s) = -V(\pi(s)) \text{ with } \pi(-\infty) = v_{k,\pm} \text{ and } \pi(+\infty) = v_{0,+}, \\ \partial_s u + J(u)(\partial_t u - X_{\pi(s),t}(u)) = 0, \\ \text{the integral of } \omega(\partial_s u, \partial_t u - X_{\pi(s),t}(u)) \text{ over the cylinder is finite.} \end{cases}$$

See Figure 4 for an illustration:



FIGURE 4. Illustration of a solution to (16). With our cohomological conventions,  $\gamma_+$  will be considered as the “input.”

Let us call Borel data *non-degenerate* provided the time-1 map  $\psi_{\eta,1}$  has only non-degenerate fixed points when  $\eta$  is a pole. In this case, it follows from (B3) that the solutions  $u$  are asymptotic to stationary solutions over non-degenerate orbits of  $X_{v_{k,\pm},t}$  and  $X_{v_{0,+},t}$  at the two ends.

At this stage, we impose our first regularity assumption for an  $n$ -simplex  $\Phi$ :

- (\*1) If  $\ell(j) = v_j$ , then  $\Phi(\ell)(j)$  is a non-degenerate Borel data.

This ensures the asymptotics of solutions to (16) behave well for all 0-simplices.

We next define  $\mathcal{M}(\Phi, k, \pm)$  in the case of a one-simplex  $\Phi$ , which we write in the one form  $(\psi_{\eta,\tau,t}, J_\tau)$  where  $\tau \in [0, 1]$ . Write  $X_{\eta,s,t}$  and  $Y_{\eta,s,t}$  for the generators of  $\psi_{\eta,f(1-s),f(t)}$  with respect to  $t$  and  $s$ , and (abusing notation) let  $J_s = J_{f(1-s)}$ . Extend  $X_{\eta,s,t}, Y_{\eta,s,t}, J_s$  from  $s \in [0, 1]$  to all of  $s \in \mathbb{R}$  by requiring that  $\partial_s X_{\eta,s,t} = \partial_s Y_{\eta,s,t} = \partial_s J_s = 0$  holds for  $s \notin [0, 1]$ .

Then  $\mathcal{M}(\Phi, k, \pm)$  is defined to be the solutions  $(\pi, u)$  to:

$$(17) \quad \begin{cases} u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow W \text{ and } \pi : \mathbb{R} \rightarrow S^\infty, \\ \pi'(s) = -V(\pi(s)) \text{ with } \pi(-\infty) = v_{k,\pm} \text{ and } \pi(+\infty) = v_{0,+}, \\ (\partial_s u - \rho(t)Y_{\pi(s),s,t}(u)) + J_s(u)(\partial_t u - X_{\pi(s),s,t}(u)) = 0, \\ \text{the integral of } \omega(\partial_s u - \rho(t)Y_{\pi(s),s,t}(u), \partial_t u - X_{\pi(s),s,t}(u)) \text{ is finite.} \end{cases}$$

Let us note that (\*1) implies that each solution  $u$  is asymptotic to non-degenerate orbits at its two ends.

In general, we need to define  $\mathcal{M}(\Phi, k, \pm)$  for an  $n$ -simplex  $\Phi$  for  $n > 1$ . The tricky part is “distributing” the continuation data onto the cylinder in a way which relates the Floer theoretic breakings to the composition law of the dg-nerve explained in (4).

Define, for each  $\ell \in \mathcal{M}(\Delta^n)$ , the following intervals  $I_1, \dots, I_n$  of  $\mathbb{R}$  by:

$$(18) \quad I_i(\ell) = [s_i, s_i + w_i] \text{ where } w_i = 1 - x_{i-1} \text{ and } s_i = \sum_{j < i} x_j - x_j^{-1}$$

in cubical coordinates  $\ell = (x_1, \dots, x_{n-1})$ , with the convention that  $s_1 = 0$  and  $w_1 = 1$ ; see Figure 5. It is important to note that:

**Lemma 2.14.** *The interiors of the intervals  $I_1(\ell), \dots, I_n(\ell)$  are pairwise disjoint.*  $\square$

We will define a version of Floer's equation in a piecewise fashion by specifying it on each interval. Recall Definition 2.1 introduced:

$$0 = \tau_0 < \tau_1 \leq \dots \leq \tau_{n-1} \leq \tau_n = n$$

where  $\tau_i$  is the first time for which  $\ell(\tau_i)$  lies in the convex hull of  $v_i, \dots, v_n$ . Also recall that Definition 2.5 introduced the  $i$ th level:

$$L_i \Phi(x; \sigma) = \Phi(x; \tau_{i-1} + \sigma(\tau_i - \tau_{i-1})) \text{ defined for } (x, \sigma) \in [0, 1]^n.$$

Write  $L_i \Phi(x, \sigma) = (\psi_{x, \eta, \sigma, t}^i, J_\sigma)$ . Let  $X_{x, \eta, s, t}^i$  and  $Y_{x, \eta, s, t}^i$  generate:

$$\psi_{\eta, f((s_i + w_i - s)/w_i), f(t)}^i$$

considered as defined on  $I_i \times [0, 1]$ ; here we are simply using the cut-off formula given in (15). Similarly let  $J_s^i = J_{f((s_i + w_i - s)/w_i)}^i$  (abusing notation in the same manner as above).

Note that  $X_{\eta, s, t}^i, \rho(t)Y_{\eta, s, t}^i, J_s^i$  are smooth on  $I_i \times \mathbb{R}/\mathbb{Z}$ . The different vector fields and almost complex structures for  $i = 1, \dots, n$  can be patched together by a piecewise function:

$$X_{\eta, s, t}^\ell = X_{\eta, s, t}^i \text{ and } \rho(t)Y_{\eta, s, t}^\ell = \rho(t)Y_{\eta, s, t}^i \text{ and } J_s^\ell = J_s^i \text{ if } s \in [s_i, s_i + w_i],$$

and these can be extended to all of  $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$  by requiring that:

$$\partial_s X_{\eta, s, t}^\ell = \partial_s Y_{\eta, s, t}^\ell = \partial_s J_s^\ell = 0$$

holds outside of  $(I_n \cup \dots \cup I_1) \times \mathbb{R}/\mathbb{Z}$ . The reader may worry about smoothness when the widths  $w_i$  tend to zero. However, since  $w_i = 1 - x_{i-1}$ , when the widths  $w_i$  are close to zero, the  $i$ th level  $L_i \Phi(x; \sigma)$  is independent of  $x_{i-1}$  coordinate. Thus  $L_i \Phi(x; \sigma) = L_i \Phi(\hat{x}; \sigma)$  where  $\hat{x}_{i-1} = 1$ . But  $L_i \Phi(\hat{x}; \sigma)$  is independent of  $\sigma$ , because  $\tau_{i-1} = \tau_i$  if  $\hat{x}_{i-1} = 1$ . In particular, as  $w_i \rightarrow 0$ ,  $L_i \Phi(x, \sigma)$  becomes independent of  $\sigma$ , and the above  $s$ -derivatives vanish.

Finally, we define  $\mathcal{M}(\Phi, k, \pm)$  as the moduli space of triples  $(\ell, \pi, u)$  such that:

$$(19) \quad \begin{cases} \ell \in \mathcal{M}^{\text{int}}(\Delta^n), \pi : \mathbb{R} \rightarrow S^\infty, \text{ and } u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow W, \\ \pi'(s) = -V(\pi(s)) \text{ with } \pi(-\infty) = v_{k, \pm} \text{ and } \pi(+\infty) = v_{0, +}, \\ (\partial_s u - \rho(t)Y_{\pi(s), s, t}^\ell(u)) + J_s^\ell(u)(\partial_t u - X_{\pi(s), s, t}^\ell(u)) = 0, \\ \text{the integral of } \omega(\partial_s u - \rho(t)Y_{\pi(s), s, t}^\ell(u), \partial_t u - X_{\pi(s), s, t}^\ell(u)) \text{ is finite.} \end{cases}$$

Practically speaking,  $\mathcal{M}(\Phi, k, \pm)$  is a parametric moduli space of continuation cylinders, over the parameters  $(\ell, \pi) \in \mathcal{M}^{\text{int}}(\Delta^n) \times \mathcal{P}(v_{k, \pm})$ . As one varies the parameter  $\ell$ , the sub-cylinders  $I_i \times \mathbb{R}/\mathbb{Z}$  travel around on the cylinder. As  $\ell$  approaches the boundary  $x_{i-1} = 0$ , the sub-cylinder  $I_i \times \mathbb{R}/\mathbb{Z}$  drifts off to  $s = -\infty$  (leading to a Floer theoretic breaking). As  $\ell$  approaches the boundary  $x_{i-1}(\ell) = 1$ , the sub-cylinder  $I_i \times \mathbb{R}/\mathbb{Z}$  shrinks. Ultimately, such considerations lead to the equation (4); we return to this point in §2.2.5.

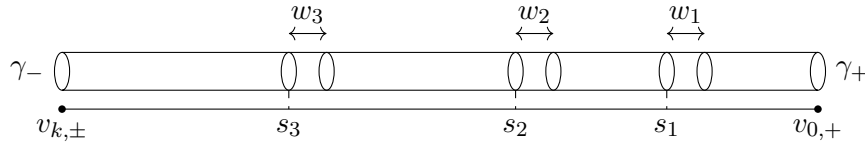


FIGURE 5. Illustration of the domain of a solution in  $\mathcal{M}(\Phi, k, \pm)$ , when  $\Phi$  is a  $n$ -simplex. The equation a solution  $(\ell, \pi, u)$  must solve is expressed in terms of  $n$  sub-intervals  $[s_i, s_i + w_i]$  determined by  $\ell$ , and the values of  $\Phi(\ell)$ .

This leads to the second regularity requirement imposed on  $\Phi$ :

- (\*2) the parametric moduli space of continuation cylinders  $\mathcal{M}(\Phi, k, \pm)$  is cut transversally (in the parametric sense).

See, e.g., [Sei15, §3.2] for related discussion. We define  $\mathcal{D}^*(W) \subset \mathcal{D}(W)$ :

**Definition 2.15.** *An  $n$ -simplex is regular provided (\*1) and (\*2) hold for  $\Phi$  and for  $i^*\Phi$  for all morphisms  $i$  in the combinatorial simplex category (this imposes countably many transversality constraints). The collection of regular simplices is denoted  $\mathcal{D}^*(W)$*

**Proposition 2.16.** *The subset of regular simplices is a sub- $\infty$ -category.*

*Proof.* This follows from standard results asserting “transversality is generic;” indeed, the horn filling problem can be solved in  $\mathcal{D}(W)$ , and simplices can be perturbed so as to lie in  $\mathcal{D}^*(W)$  (the perturbation can be made relative the horn if all of the faces in the horn are already in  $\mathcal{D}^*(W)$ ).

We should comment on how exactly we achieve transversality for the moduli space  $\mathcal{M}(\Phi, k, \pm)$ . As it is a parametric moduli space over  $(0, 1)^{n-1}$ , and solutions “break” into solutions of the lower parametric families as one approaches  $\partial[0, 1]^{n-1}$ , one can inductively achieve transversality by perturbing only the data in a compactly supported way in the interior of the cube  $(0, 1)^{n-1}$ ; moreover, we only perturb in a compactly supported way in  $W$  to avoid influencing the ideal restriction. Since the perturbation is allowed to depend on the location in the parameter space  $(0, 1)^{n-1}$ , the problem boils down to one in [SS10, pp. 1468] and [Sei15, §4] where transversality for families over the flow line spaces  $\mathcal{P}(v_{k, \pm})$  is considered, (one also appeals to standard techniques such as those in [MS12, §8]).  $\square$

**2.2.4. Definition of the equivariant chain complex.** Let  $(\psi_{\eta,t}, J)$  be a single Borel datum in  $\mathcal{D}_0^*(W)$ . Following [SS10, Sei15], we define:

$$\mathrm{CF}_{\mathrm{eq}}(\psi_{\eta,t}, J) := \mathrm{CF}(\psi_t) \otimes \mathbf{k}[[x]],$$

where  $\psi_t = \psi_{v_{0,+},t}$  and where  $\mathrm{CF}(\psi_t)$  is the  $\mathbf{k}$ -vector space generated by contractible 1-periodic orbits of  $\psi_t$ . The differential is defined as a power series:

$$d_{\mathrm{eq}} = \sum x^k d_k = \sum x^k (d_{k,+} + d_{k,-})$$

where  $d_{k,\pm}$  counts solutions in the family  $\mathcal{M}(\Phi, k, \pm)$ , as follows:

$$d_{k,+}(\gamma_+) := \sum \{u(-\infty) : (\pi, u) \in \mathcal{M}_1(\Phi, k, +)/\mathbb{R} \text{ and } u(+\infty) = \gamma_+\},$$

$$d_{k,-}(\gamma_+) := \sum \{g(u(-\infty)) : (\pi, u) \in \mathcal{M}_1(\Phi, k, -)/\mathbb{R} \text{ and } u(+\infty) = \gamma_+\}.$$

Note that the  $d_{k,-}$  term twists the orbit by  $g$ , which is necessary for the count to make sense. Note as well that  $d_{0,+}$  is the usual Floer differential and  $d_{0,-} = 0$ .

The fact that  $d_{\mathrm{eq}}^2 = 0$  is essentially an argument in family Floer cohomology, similar to some discussion in [BC25, Lemma 42], and is explained in [SS10] and [Sei15]. We briefly review the argument:

*Proof that the equivariant differential squares to zero.* As is usual in Floer theory, we prove  $d_{\mathrm{eq}}^2 = 0$  by considering the non-compact ends of the 1-dimensional moduli spaces  $\mathcal{M}_2(\Phi, k, \pm)/\mathbb{R}$ . Here an *end* is a proper embedding of  $[0, \infty)$  into the moduli space, understood via the unbounded sequences  $(\ell_n, \pi_n, u_n)$  which remain in the end. Of course, in this case  $\Phi$  is a single choice of Borel data so  $\ell_n \in \mathcal{M}(\Delta^n)$  is entirely irrelevant. We consider two kinds of ends  $(\pi_n, u_n)$ :

- (i)  $[\pi_n] \in \mathcal{P}(v_{k,\pm})/\mathbb{R}$  converges (but  $u_n$  does not converge),
- (ii)  $[\pi_n]$  breaks into a limit in  $\mathcal{P}(v_{l_1,\pm})/\mathbb{R} \times \mathcal{P}(v_{l_2,\pm})/\mathbb{R}$  where  $l_1 + l_2 = k$  and both  $l_i > 0$ .

As in [SS10, Sei15] the fact that the pseudogradient is invariant under the self-similarity map and the involution implies that:

$$\{\text{flows lines from } v_{l_1,\epsilon_1} \text{ to } v_{l_2,\epsilon_2}\} \simeq \mathcal{P}(v_{l_2-l_1,\epsilon_1\epsilon_2}),$$

and it is with respect to this identification that only spaces of flow lines starting at  $v_{0,+}$  appear in the formula (ii).

The linearization of the parametric equation is a linear map:

$$T\mathcal{P}(v_{k,\pm}) \times W^{1,p}(u_n^*TW) \rightarrow L^p(u_n^*TW),$$

and the restriction to  $W^{1,p}$  is a Cauchy-Riemann operator  $D_{u_n}$  whose index depends on the asymptotic orbits  $\gamma_-, \gamma_+$  and the homology class of  $[u]$  relative  $\gamma_-, \gamma_+$ . We record the formula for its index:

$$(20) \quad \mathrm{index}(D_{u_n}) = 2c_1^{\mathfrak{s}}(u) + \mathrm{CZ}_{\mathfrak{s}}(\gamma_+) - \mathrm{CZ}_{\mathfrak{s}}(\gamma_-),$$

where:

- $\mathfrak{s}$  is any generic section of  $\det_{\mathbb{C}}(TW)$  non-vanishing along  $\gamma_{\pm}$ ,
- $c_1^{\mathfrak{s}}(u)$  is the signed intersection number  $\text{PD}(\mathfrak{s}^{-1}(0)) \cap [u]$ ,
- the Conley-Zehnder indices  $\text{CZ}_{\mathfrak{s}}(\gamma_{\pm})$  are computed using trivializations of  $TW|_{\gamma_{\pm}}$  which render  $\mathfrak{s}$  constant;

see, e.g., [Sal97]; we also refer the reader to [Can22] and the references therein for further details on this formula. In particular, along any sequence in  $\mathcal{M}_2(\Phi, k, \pm)$  it holds that  $\text{index}(D_{u_n}) = 2 - k$ . In case (i), one picks representatives:

$$(\pi_n, u_n) \in \mathcal{M}_2(\Phi, k, \pm)/\mathbb{R}$$

so that  $\pi_n$  converges (after passing to a subsequence). Then  $u_n$  converges on compact subsets by standard Floer theory to a solution  $u_{\infty}$  in  $\mathcal{M}_d(\Phi, k, \pm)$  with  $d \leq 2$ . By (\*2) it holds that:

$$(21) \quad \cdots = \mathcal{M}_{-1}(\Phi, k, \pm) = \mathcal{M}_0(\Phi, k, \pm) = 0,$$

if  $k \neq 0$ , while if  $k = 0$  then  $\mathcal{M}_0(\Phi, 0, +)$  consists of only the stationary cylinders. Note that if  $k = 0$ , then  $\pi_n$  is irrelevant, and the argument boils down to Floer's original argument and shows  $d_0^2 = 0$ . Assume  $k > 0$  for the rest of the proof.

If  $u_n$  does not converge uniformly, then it must break off non-stationary Floer differential cylinders (solutions to the  $s$ -independent equation) at one of the ends. Each Floer differential cylinder must have a positive index (20), by (\*1) and (\*2). It then follows from (21) that only one non-stationary Floer cylinder of index 1 breaks off, and the other limit  $u_{\infty}$  is in  $\mathcal{M}_1(\Phi, k, \pm)$ . For each  $k \geq 1$ , the count of these breakings (interpreted in the usual Floer theoretic sense) gives:

$$d_0 d_k + d_k d_0.$$

Standard Floer theory gluing proves each hypothetical breaking in the fiber product inside of  $\mathcal{M}_1/\mathbb{R} \times \mathcal{M}_1/\mathbb{R}$  actually does appear as the end of the moduli space  $\mathcal{M}_2/\mathbb{R}$ . In general we will suppress discussion of gluing.

Turning now to the other non-compact ends (ii). These breakings only occur when  $k \geq 2$ . In this case, if  $[\pi_n]$  breaks into  $[\pi_-], [\pi_+]$ , we claim that  $[\pi_n, u_n]$  must converge (after a subsequence) into a broken configuration  $[\pi_-, u_-]$  and  $[\pi_+, u_+]$  in  $\mathcal{M}_1/\mathbb{R} \times \mathcal{M}_1/\mathbb{R}$ . This is the only possibility compatible with (ii) and the index formula (20). There is some small technicality regarding how the breakings are interpreted, when the self-similarity and involution maps are taken into account; see [Sei15, pp.972] for related discussion. It suffices to say that this technicality explains that the breakings of type (ii) correspond to the sum of compositions  $d_{l_1} d_{l_2}$  where  $l_1 + l_2 = k$  and  $l_1, l_2 > 0$ , extending the case  $d_k d_0 + d_0 d_k$  to the intermediate values. In total, one shows:

$$\sum_{l_1 + l_2 = k} d_{l_1} d_{l_2} = 0, \text{ for each } k$$

which is equivalent to  $d_{eq}^2 = 0$ .  $\square$

**2.2.5. Chain homotopies associated to higher simplices.** We continue from §1.3. Recall that a map  $\mathcal{D}^*(W) \rightarrow \text{N}_{\text{dg}}(\text{Ch}(\mathbf{k}[[x]]))$  is the data of:

- (a) a finitely generated graded<sup>30</sup>  $\mathbf{k}[[x]]$ -module  $V_\Phi$  with differential  $d_\Phi$  of degree 1 for each zero simplex  $\Phi$ ,
- (b) a map  $\mathbf{c}_\Phi : V_{\Phi|_0} \rightarrow V_{\Phi|_n}$  of degree  $1 - n$  for each  $n$ -simplex  $\Phi$ .

In the present case where  $\text{char}(\mathbf{k}) = 2$ , these are supposed to satisfy:

$$(22) \quad \sum_{j=1}^{n-1} \mathbf{c}_{\Phi|[j\dots n]} \circ \mathbf{c}_{\Phi|[0\dots j]} + \mathbf{c}_{\Phi|[0\dots \hat{j}\dots n]} = \mathbf{c}_\Phi \circ d_{\Phi|_0} + d_{\Phi|_n} \circ \mathbf{c}_\Phi.$$

The first stage (a) is done: each zero simplex  $\Phi = (\psi_{\eta,t}, J)$  is sent to its equivariant Floer cohomology complex  $V_\Phi = \text{CF}_{\text{eq}}(\psi_{\eta,t}, J)$  with differential  $d_\Phi = d_{eq}$  as defined in §2.2.4.

The second stage (b) is satisfied by the following definition:

$$\mathbf{c}_\Phi = \sum x^k \mathbf{c}_k = \sum x^k (\mathbf{c}_{k,+} + \mathbf{c}_{k,-})$$

where  $\mathbf{c}_{k,\pm}$  are defined by:

$$\begin{aligned} \mathbf{c}_{k,+}(\gamma_+) &= \sum \{u(-\infty) : u \in \mathcal{M}_0(\Phi, k, +) \text{ with } u(+\infty) = \gamma_+\}, \\ \mathbf{c}_{k,-}(\gamma_+) &= \sum \{g(u(-\infty)) : u \in \mathcal{M}_0(\Phi, k, -) \text{ with } u(+\infty) = \gamma_+\}. \end{aligned}$$

It remains only to verify the equation (22). Continuing the discussion §2.2.3, we observe a relationship between the equation (22) and the 1-dimensional component  $\mathcal{M}_1(\Phi, d, \pm)$ . The moduli space admits a map:

$$\ell : \mathcal{M}_1(\Phi, d, \pm) \rightarrow (0, 1)^{n-1},$$

which records the underlying straight-line path on  $\Delta^n$ ; this is used to assign algebraic interpretations to the non-compact ends:

- (i)  $\mathbf{c}_{\Phi|[j\dots n]} \circ \mathbf{c}_{\Phi|[0\dots j]}$ : ends where  $x_j(\ell_n)$  converges to zero,
- (ii)  $\mathbf{c}_{\Phi|[0\dots \hat{j}\dots n]}$ : ends where  $x_j(\ell_n)$  converges to one,
- (iii)  $\mathbf{c}_\Phi \circ d_{\Phi|_0}$  and  $d_{\Phi|_n} \circ \mathbf{c}_\Phi$ : ends where  $\ell_n$  converges in  $(0, 1)^{n-1}$ .

In the remainder of this section we make this informal sketch more precise.

*Proof of equation (22).* Let us recall how the parametric moduli space:

$$\mathcal{M}(\Phi, k, \pm) \rightarrow (0, 1)^{n-1}$$

is defined. A given straight line path  $x \in (0, 1)^{n-1}$  determines  $n$  sub-cylinders  $I_i \times \mathbb{R}/\mathbb{Z}$  where  $I_i = [s_i, s_i + w_i]$  are defined in (18). These sub-intervals are mutually disjoint (except possibly at their boundaries), and

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<sup>30</sup>We need to use  $\mathbb{Z}/2\mathbb{Z}$ -gradings in certain signs when the characteristic of the field is  $p \geq 3$ ; in the case  $p = 2$  we can use  $\mathbb{Z}/1\mathbb{Z} = 0$  gradings, i.e., ungraded modules.

ordered,  $I_n \leq \dots \leq I_1 = [0, 1]$ . On the  $i$ th subinterval  $I_i$ , we implant a continuation equation using the values of the  $i$ th level  $\sigma \mapsto L_i \Phi(x; \sigma)$ , as described in (19). For  $x$  fixed, this gives a family of equations parameterized by  $\eta \in \mathcal{P}(v_{k, \pm})$ , but as  $x \in (0, 1)^{n-1}$  also varies the intervals move around and change widths; in total we have the parametric moduli space  $\mathcal{M}(\Phi, k, \pm)$  over  $\mathcal{P}(v_{k, \pm}) \times (0, 1)^{n-1}$ .

Referring to the Fredholm index in (20), it holds that:

$$(23) \quad \text{index}(u) + k + n - 1 = d \text{ for } (\ell, \pi, u) \in \mathcal{M}_d(\Phi, k, \pm),$$

As in (i) through (iii), let us study the parameter map  $\ell : \mathcal{M}_1 \rightarrow (0, 1)^{n-1}$ . We begin with:

**Claim 2.17.** *Along any end  $(\ell_n, \pi_n, u_n) \in \mathcal{M}_1(\Phi, k, \pm)$  such that the  $x_i(\ell_n)$  converges to 1 but the other coordinates converge in  $(0, 1)^{n-1}$ , and  $\pi_n$  converges, some subsequence will converge to a solution  $(\ell_\infty, \pi_\infty, u_\infty)$  for the restriction  $\mathcal{M}_0(\Phi|_{[0 \dots \hat{i} \dots n]}, k, \pm)$ .*

*Proof of claim.* Indeed, by definition, the  $n - 1$  simplex  $\Phi|_{[0 \dots \hat{i} \dots n]}$  is determined by restricting  $\Phi$  to only those paths which remain in the face opposite vertex  $v_i$ ; these paths have  $L_{i+1} \Phi = \text{const}$ , and the levels of  $\Phi|_{[0 \dots \hat{i} \dots n]}$  are given by:

$$L_j(\Phi|_{[0 \dots \hat{i} \dots n]}) = \begin{cases} L_j \Phi & \text{if } j \leq i, \\ L_{j+1} \Phi & \text{if } j > i. \end{cases}$$

In particular, it follows from this and the fact  $s_{i+1} - s_i \rightarrow 0$  and  $w_{i+1} \rightarrow 0$  that the PDE which  $u_n$  solves converges to the one for the restricted simplex; and then, by standard compactness results,  $u_n$  also converges (after passing to a subsequence) in the stated sense, yielding the claim. The dimension drops by 1 by inspecting (23).  $\square$

These ends lead to the terms of the form  $\mathbf{c}_{\Phi|_{[0 \dots \hat{i} \dots n]}}$  appearing in (22). All such contributions do appear as ends of one-dimensional moduli spaces by standard Floer theoretic “gluing” (here we do not actually glue together two solutions, similar in this regard to [BC24, §4]).

The same argument used for the claim works if  $x_i(\ell_n) \rightarrow 1$  holds for multiple values of  $i$ , (i.e., all coordinates converge in  $(0, 1]$  and  $\pi_n$  converges). Then the index will drop by the number of coordinates which converged to 1. For generic data, we can therefore assume that only one coordinate ever converges to 1, provided the other coordinates converge in  $(0, 1]$  and  $\pi_n$  converges.

Let us now analyze the case when one of the coordinates, say  $x_i$ , converges to 0, but the other coordinates converge in  $(0, 1]$ . Then  $s_i$  remains bounded but  $s_{i+1}$  diverges to  $-\infty$  and  $w_{i+1}$  converges to 1; recall (18).

This enables us to extract two limits. The first  $u_+$  is just the limit of the original sequence on compact subsets. The second  $u_-$  is the limit of the

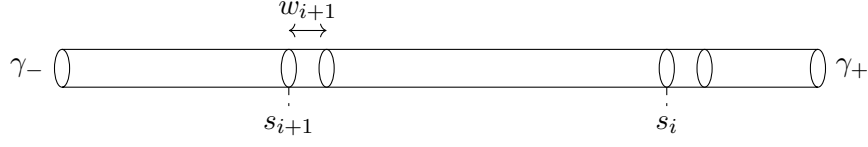


FIGURE 6. The case when  $x_i(\ell_n)$  converges to 0 but the other coordinates converge in  $(0, 1]$ ,  $s_i$  remains bounded but  $s_{i+1}$  converges to  $-\infty$ ; all of the differences  $s_{i+1} - s_j$  remain bounded if  $j \geq i + 1$ . Moreover,  $w_{i+1}$  converges to 1.

retranslations  $u_n(s + s_{i+1}(\ell_n), t)$ . Moreover,  $\ell_n$  converges to a straight-line path which passes through  $v_i$ , and we consider the limit  $\ell_\infty$  as split into the concatenation  $\ell_+, \ell_-$  of its restrictions to  $[0, i]$  and  $[i, n]$  which meet at  $v_i$ . Moreover, using the same parametrizations,  $\pi$  also converges to some pair of flow lines  $\pi_-, \pi_+$  of indices  $k_-, k_+$  on  $S^\infty$ . We claim:

- $(u_-, \ell_-, \pi_-) \in \mathcal{M}_{d_-}(\Phi|_{[i\dots n]}, k_-, \pm)$ ,
- $(u_+, \ell_+, \pi_+) \in \mathcal{M}_{d_+}(\Phi|_{[0\dots i]}, k_+, \pm)$ ,

for appropriate  $\pm$  signs modulo applying the involution and using the shift self-symmetry of  $S^\infty$ , as in [SS10, Sei15]. Here  $d_+ + d_- \leq 0$ . The only possibility is  $d_+ = d_- = 0$ , and  $k_- + k_+ = k$  (rather than  $k_- + k_+ \leq k$ ). This implies that  $\pi_n$  actually converges in the generalized Morse sense to the configuration  $\pi_-, \pi_+$  we already found. Moreover, by regularity, the lines  $\ell_\pm$  actually have all coordinates in  $(0, 1)$ , rather than  $(0, 1]$ .

Therefore, the pair  $(u_-, \ell_-, \pi_-), (u_+, \ell_+, \pi_+)$  contributes to the term:

$$\mathbf{c}_{\Phi|_{[i\dots n]}} \mathbf{c}_{\Phi|_{[0\dots i]}};$$

all such contributions actually do appear as ends of one-dimensional moduli spaces by standard Floer theoretic gluing for parametric families of continuation cylinders, similarly to [Sei15, pp.972].

Essentially the same argument implies that one cannot have two coordinates which converge to zero, without contradicting regularity.

All of the other ends are of the form where  $x(\ell_n)$  converges in  $(0, 1)^{n-1}$ . If  $\pi_n$  converges to  $(\pi_-, \pi_+)$  with positive indices  $k_-, k_+$  in the Morse theoretic sense, then one concludes the terms of the form:

$$\mathbf{c}_{k_-} d_{k_+} + d_{k_-} \mathbf{c}_{k_+}.$$

In all other cases permitted by regularity,  $\pi_n$  converges in  $\mathcal{P}(v_{k,\pm})/\mathbb{R}$ , and one concludes the terms of the form  $\mathbf{c}_0 d_k + d_k \mathbf{c}_0$ . Summing together all of these possible ends gives the desired relation (22).  $\square$

**2.2.6. Definition of the  $\infty$ -functor.** Most of the arguments needed in this subsection were already explained in §1.7. The main technical result of this subsection is:

**Lemma 2.18.** *The forgetful map  $\mathcal{D}^*(W) \rightarrow \mathcal{C}(Y)$  is a trivial Kan fibration.*

*Proof.* One shows that  $\mathcal{D}(W) \rightarrow \mathcal{C}(Y)$  is a trivial Kan fibration by direct construction. Using a  $G$ -invariant radial function  $r$ , one can cut-off the generating Hamiltonians functions from the symplectization end to the filling and explicitly construct extensions; e.g., one can use a formula of the form  $f(r)H$  where  $f$  vanishes outside the symplectization end. Extensions constructed in this way will not be regular, but they can be perturbed on the compact part to be made regular.  $\square$

Therefore, by Proposition 1.20, there is a contractible Kan complex of sections  $\Sigma : \mathcal{C}(Y) \rightarrow \mathcal{D}^*(W)$  of this forgetful map. We define:

$$\mathrm{CF}_{\mathrm{eq}} \circ \Sigma : \mathcal{C}(Y) \rightarrow \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]]),$$

as the  $\infty$ -functor satisfying Theorem 1.14. While this depends on a choice of section, this choice is “contractible” in the  $\infty$ -category sense, and any two choices give equivalent functors (in an appropriate sense).

It is perhaps reassuring to observe the following: for any  $(\psi_{\eta,t}, J) \in \mathcal{D}_0^*(W)$ , there is a section  $\Sigma$  such that  $\Sigma(\varphi_t) = (\psi_{\eta,t}, J)$ , where  $\varphi_t$  is the ideal restriction of  $\psi_{\eta,t}$ . Indeed, this follows from the lifting property for the diagram:

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & \mathcal{D}^*(W) \\ \downarrow & \nearrow \Sigma & \downarrow \\ \mathcal{C}(Y) & \xrightarrow{id} & \mathcal{C}(Y). \end{array}$$

This means that, given any desired regular extension of  $\varphi_t$  to Borel data  $(\psi_{\eta,t}, J)$ , one can assume that the output of  $\mathrm{CF}_{\mathrm{eq}} \circ \Sigma(\varphi_t)$  agrees with the equivariant Floer complex  $\mathrm{CF}_{\mathrm{eq}}(\psi_{\eta,t}, J)$  from §2.2.4 “on the nose.” In any case, the chain level constructions we will appeal to are insensitive to the precise choice of  $\Sigma$ .

**2.3. The case when  $p \geq 3$ .** For the rest of this section we fix a prime  $p \geq 3$  and let  $G = \mathbf{k} = \mathbb{Z}/p\mathbb{Z}$ .

The main contents of the construction are similar to the  $\mathbb{Z}/2\mathbb{Z}$ -case. The principal difference lies in the underlying Morse model for  $BG$ . As noted at the start of §2.2, in the case  $p \geq 3$ , we work on  $S^\infty \subset \mathbb{C}^\infty$  with the diagonal  $G \subset U(1)$  action, and use the shift operator  $\tau$  associated to the standard  $\mathbb{C}$ -valued coordinates. The usual perfect Morse function on  $\mathbb{C}P^\infty$  pulls back to a  $U(1)$  invariant Morse–Bott function on  $S^\infty$  given by:

$$(24) \quad f(z_0, z_1, \dots) = \sum_{n=0}^{\infty} n |z_n|^2.$$

Well chosen  $G$ -equivariant perturbations turn this into a Morse function on  $BG$  (see [SZ21, §4] for discussion of this construction, indeed we are happy

to work with the same model). Moreover, this may be done compatibly with the action of  $\tau$ .

Ultimately we are concerned with the pseudogradient as in §2.2.3. We take the pseudogradient  $B$  on  $\mathbb{C}P^\infty$  whose time- $s$  flow is given by:

$$[z_0 : z_1 : z_2 : \dots] \mapsto [z_0 : e^s z_1 : e^{2s} z_2 : \dots].$$

Fixing a connection on the line bundle  $S^\infty \rightarrow \mathbb{C}P^\infty$ , we obtain a unique lift to a vector field  $B$  on  $S^\infty$  taking values in the horizontal distribution of the connection; this vector field is Morse-Bott with Lyapunov function given by (24). One “Morsifies”  $B$  by adding a vector field  $F$  tangent to the fibers of the map  $S^\infty \rightarrow \mathbb{C}P^\infty$ . One considers  $V = B + F$  as the  $p > 2$  analogue of the vector field  $V$  considered in §2.2.3. For convenience, one can assume that the connection is flat in a neighborhood of the poles of  $\mathbb{C}P^\infty$ .

One picks  $F$  so that:

- its restriction to each critical circle of  $B$  is as in Figure 7;
- $F$  is  $\mathbb{Z}/p\mathbb{Z}$  equivariant;
- the *total vector field*  $V = B + F$  is Morse-Smale;
- $F$  is invariant under the shift map:

$$\tau : (z_0, z_1, z_2, \dots) \mapsto (0, z_0, z_1, \dots).$$

We leave the construction of such  $F$  to the reader, and refer to [SZ21] for further discussion.

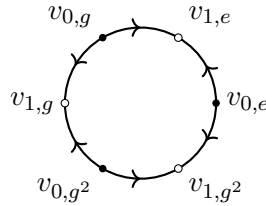


FIGURE 7. Restriction of  $F$  to the critical circle has  $2p$  critical points (shown for  $p = 3$ ). Here  $v_{1,e}$  is required to be a positive pole.

This  $V$  then has exactly  $p$  zeros of index  $n$  for each  $n \geq 0$ , which are related to each other by the  $G$  action, and  $\tau$  acts by shifting the index by 2. Choosing distinguished critical points  $v_{0,e}, v_{1,e}$ , of index 0 and 1 respectively, we have a natural labeling of the rest  $v_{n,g}$  by the data of their index  $n$ , and the element  $g \in G$  which sends  $v_{n,g}$  to the appropriate shift of  $v_{0,e}$  or  $v_{1,e}$ . Coherent orientations of the finite dimensional spaces of flow lines completes the Morse setup; we describe this in greater detail in the sequel. To keep things shorter we fix such a model on  $BG$  once and for all as input to the rest of the construction.<sup>31</sup>

<sup>31</sup>Unlike the case of  $B\mathbb{Z}/2\mathbb{Z} = \mathbb{R}P^\infty$ , the Morse-Smale gradient on  $BG$  is not canonically defined, as we have written it. This is a minor loss of canonicity in our construction, and

Having fixed a Morse model on  $BG$ , the category of auxiliary data  $\mathcal{D}(W)$ , and moduli spaces are essentially as in §2.2, with the following modifications:

- In addition to conditions (B1) through (B4) above, auxiliary data must be identical near  $v_{0,e}, v_{1,e}$ .
- Associated to a  $k$ -simplex of Borel data  $\Phi$  we have moduli spaces:

$$\mathcal{M}(\Phi, k, g, i)$$

where  $i \in \{0, 1\}$  identifies the input point (otherwise as in §2.2.3).

Fixing a Borel datum  $\Phi = (\psi_{\eta,t}, J)$ , the equivariant chain complex now has underlying  $\mathbf{k}[[x]]$  module:

$$\mathrm{CF}_{eq}(\Phi) := \mathrm{CF}(\psi_t) \otimes_{\mathbf{k}} (\mathbf{k}[[x]] \oplus \mathbf{k}[[x]][1]).$$

Here  $\mathrm{CF}(\psi_t)$  is the direct sum of the  $\mathbf{k}$ -orientation lines  $\mathfrak{o}(\gamma)$  of contractible 1-periodic orbits  $\gamma$  of  $\psi_t = \psi_{(1,0,0,\dots),t}$ . This module carries a differential, given by:

$$(25) \quad d_{eq}(\gamma \otimes 1) = \sum_{k \geq 0} x^k d_0^{2k}(\gamma) \otimes 1 + x^k d_0^{2k+1}(\gamma) \otimes \theta$$

$$(26) \quad d_{eq}(\gamma \otimes \theta) = \sum_{k \geq 0} x^k d_1^{2k+1}(\gamma) \otimes \theta + x^{2k+2} d_1^{2k+2}(\gamma) \otimes 1,$$

where  $1, \theta$  denote the generators of  $\mathbf{k}[[x]] \oplus \mathbf{k}[[x]][1]$ , and:

$$d_i^k(\gamma) = \sum_{g \in G} \sum_{u \in \mathcal{M}(\Phi, k, g, i)} \mathfrak{o}(u).$$

The inner sum is over all  $u \in \mathcal{M}(\Phi, k, g, i)$  which are rigid up to translation and which are asymptotic to  $\gamma$  at their input end. The symbol  $\mathfrak{o}(u)$  can be considered in two ways:

- (1)  $\mathfrak{o}(u) = \pm \gamma'$ , where  $\gamma'$  is the output asymptotic of  $u$ , and the sign  $\pm$  is determined by comparing the orientation lines of  $\mathcal{M}(\psi, k, g, i)$  with a system of coherent orientations, as in [FH93];
- (2)  $\mathfrak{o}(u) \in \mathrm{Hom}(\mathfrak{o}(\gamma), \mathfrak{o}(\gamma'))$  is a canonical isomorphism constructed by the linear gluing theory of [FH93] (see, e.g., [Abo15, Par16]).

That  $d_{eq}$  squares to 0 follows from an analysis of the moduli spaces identical to that of 2.2.4 above, replacing the  $\mathbb{Z}/2$  counting of that section with the usual orientation line formalism (see 2.4).

Chain homotopies from higher simplices, and the definition of the  $\infty$ -functor are also built as above, *mutatis mutandis*.

In the next section, we digress and explain the set-up of orientations; the reader who is comfortable with the framework of [Abo15, pp. 290], and/or [Par16, §C.13.2] can safely skip to §3.

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it seems the optimal resolution would be to encode the remaining ambiguity as auxiliary data to be included in the category  $\mathcal{D}(W)$ . For reasons of space, we leave the details of this optimization to the reader.

**2.4. Gradings, asymptotic operators, and orientation lines.** The essential ingredient in the construction is [FH93, Theorem 10] and their kernel gluing lemma. We will also appeal to [Par16, §C.13.1] for orientations on “spaces of flow lines on  $\Delta^n$ ,” and [SZ21, §A] for discussion of the orientations on the spaces of Morse flow lines on  $BG$ , for  $G = \mathbb{Z}/p\mathbb{Z}$ .

**2.4.1. Asymptotic operators.** An *asymptotic operator* is a first order differential operator  $\mathfrak{A}$ , acting on  $\mathbb{R}^{2n}$ -valued functions on  $\mathbb{R}/\mathbb{Z}$ , of the form:

$$\mathfrak{A} = -J\partial_t - S(t)$$

where  $t \in \mathbb{R}/\mathbb{Z} \mapsto S(t)$  is a smooth loop of symmetric matrices and  $J$  is the standard complex structure on  $\mathbb{R}^{2n}$ . Such operators are elliptic and self-adjoint. If  $\mathfrak{A}$  induces an isomorphism  $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \rightarrow C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$  then we say  $\mathfrak{A}$  is *non-degenerate*.

**2.4.2. Cauchy-Riemann operators with asymptotics.** Given non-degenerate asymptotic operators  $\mathfrak{A}_-, \mathfrak{A}_+$ , we denote by  $C(\mathfrak{A}_-, \mathfrak{A}_+)$  the space of operators on the cylinder  $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$  of the form:

$$\partial_s + J\partial_t + S(s, t).$$

such that  $-J\partial_t - S(s, t)$  converges to  $\mathfrak{A}_\pm$  as  $s \rightarrow \pm\infty$ . The convergence should be in a suitable topology whose details are not so relevant to the present discussion.

Then  $C(\mathfrak{A}_-, \mathfrak{A}_+)$  is a convex space of Fredholm operators  $D$ , see [Sal97], all of which have the same Fredholm index  $i(\mathfrak{A}_-, \mathfrak{A}_+)$ . Moreover there is a continuous real-line bundle:

$$\det(\mathfrak{A}_-, \mathfrak{A}_+) \rightarrow C(\mathfrak{A}_-, \mathfrak{A}_+)$$

whose restriction to any stratum with fixed kernel dimension agrees with:

$$\det(\ker D \oplus (\operatorname{coker} D)^*),$$

The fact that these bundles patch together across the different strata is a fundamental property of Fredholm operators, and is discussed in [MS12].

**2.4.3. Orientation lines.** The real line bundle  $\det(\mathfrak{A}_-, \mathfrak{A}_+)$  admits two orientations  $O, \bar{O}$  (since the base space is contractible). Let us denote by  $\mathfrak{o}(\mathfrak{A}_-, \mathfrak{A}_+)$  the *orientation line* generated by  $O, \bar{O}$  modulo the relation that  $O + \bar{O} = 0$ . This object  $\mathfrak{o}(\mathfrak{A}_-, \mathfrak{A}_+)$  is a free  $\mathbb{Z}$ -module of rank 1.

**2.4.4. Automatic isomorphism.** In [Sal97] it is shown that  $\partial_s - \mathfrak{A}$  is an isomorphism for any non-degenerate asymptotic operator  $\mathfrak{A}$ . Thus  $\mathfrak{o}(\mathfrak{A}, \mathfrak{A})$  is *canonically* identified with  $\mathbf{k}$ , because the determinant line at  $D = \partial_s - \mathfrak{A}$  is the determinant line of zero, which is canonically oriented.

2.4.5. **Kernel gluing.** The kernel gluing theorem of [FH93] gives isomorphisms:

$$\bullet \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \otimes \mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_2) \rightarrow \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_2)$$

which are associative, and such that “multiplication” by the canonical positive generator of  $\mathfrak{o}(\mathfrak{A}, \mathfrak{A})$  acts by the identity.

Moreover, the pairing:

$$\mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \otimes \mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_0) \rightarrow \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_0) \simeq \mathbb{Z}$$

canonically identifies  $\mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \rightarrow \mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_0)$ , as follows: one sends a generator of  $\mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1)$  to the unique generator of  $\mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_0)$  which pairs with it to produce 1. Associativity proves the composition of the canonical identifications  $\mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \rightarrow \mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_0) \rightarrow \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1)$  is the identity map.

2.4.6. **Twisting identifications.** Given  $\mathfrak{A}$  and a loop  $\varphi : \mathbb{R}/\mathbb{Z} \rightarrow \mathrm{U}(n)$  consider the conjugation:

$$\varphi_* \mathfrak{A} := \varphi \mathfrak{A} \varphi^{-1}.$$

Multiplication of kernel and cokernel elements by  $\varphi$  yields an identification:

$$(27) \quad \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \rightarrow \mathfrak{o}(\varphi_* \mathfrak{A}_0, \varphi_* \mathfrak{A}_1).$$

Unpacking the construction from [FH93] shows (27) is compatible with the kernel gluing maps from §2.4.5, in the sense that the diagram commutes:

$$\begin{array}{ccc} \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \otimes \mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_2) & \longrightarrow & \mathfrak{o}(\varphi_* \mathfrak{A}_0, \varphi_* \mathfrak{A}_1) \otimes \mathfrak{o}(\varphi_* \mathfrak{A}_1, \varphi_* \mathfrak{A}_2) \\ \downarrow & & \downarrow \\ \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_2) & \longrightarrow & \mathfrak{o}(\varphi_* \mathfrak{A}_0, \varphi_* \mathfrak{A}_2) \end{array}$$

where the vertical maps are the kernel gluing maps, and the horizontal maps are the twisting identifications.

The twisting identification between  $\mathfrak{o}(\mathfrak{A}, \mathfrak{A})$  and  $\mathfrak{o}(\varphi_* \mathfrak{A}, \varphi_* \mathfrak{A})$  preserves the canonical generators from §2.4.4, since the empty basis is conjugated to the empty basis.

2.4.7. **Parallel transport.** Suppose that  $\mathfrak{A}_{0,s}, \mathfrak{A}_{1,s}$ ,  $s \in [0, 1]$ , are families of asymptotic operators which remain non-degenerate. Then there is a *parallel transport map*:

$$(28) \quad \mathfrak{o}(\mathfrak{A}_{0,0}, \mathfrak{A}_{1,0}) \rightarrow \mathfrak{o}(\mathfrak{A}_{0,1}, \mathfrak{A}_{1,1}),$$

defined by considering any continuous family  $D_s \in C(\mathfrak{A}_{0,s}, \mathfrak{A}_{1,s})$  and the associated determinant line bundle over this family.

**2.4.8. Reference asymptotic operator.** Let  $\mathfrak{R} = -J\partial_t - \delta$  for  $\delta \in (0, 2\pi)$ . This acts as a reference operator.

It will be important for us to consider the *complex linear* orientation:

$$\mathfrak{o}(\varphi_*\mathfrak{R}, \mathfrak{R}) \rightarrow \mathbb{Z},$$

The space of complex linear  $D$  is an affine subspace, so the resulting orientation of  $\det(\varphi_*\mathfrak{R}, \mathfrak{R})$  is independent of the choice of complex linear  $D$ .

Consequently, we obtain canonical isomorphisms:

$$(29) \quad \mathfrak{o}(\mathfrak{A}, \mathfrak{R}) \rightarrow \mathfrak{o}(\varphi_*\mathfrak{A}, \varphi_*\mathfrak{R}) \rightarrow \mathfrak{o}(\varphi_*\mathfrak{A}, \mathfrak{R}),$$

where in the latter step we multiply by the generator of  $\mathfrak{o}(\varphi_*\mathfrak{R}, \mathfrak{R})$  corresponding to the complex linear orientation.

**2.4.9. Linearization procedure.** Given a non-degenerate orbit  $\gamma$  of a Hamiltonian system in a symplectic manifold  $W$ :

$$\gamma(t) - X_t(\gamma(t)) = 0.$$

Let us consider travelling symplectic coordinates  $e_t : B(\epsilon) \rightarrow W$ , where  $B(\epsilon)$  is the standard symplectic ball of capacity  $\epsilon$ , such that  $e_t(0) = \gamma(t)$ . Then nearby loops can be expressed as  $e_t(\eta(t))$ , and the non-linear functional is represented by:

$$\eta'(t) + de_t(\eta(t))^{-1} [\partial_t e_t(\eta(t)) - X_t(e_t(\eta(t)))].$$

The linearization of this is equal to  $J\mathfrak{A}_e$  where  $\mathfrak{A}_e$  is an asymptotic operator. Changing  $e$  amounts to conjugating  $J\mathfrak{A}_e$  by a linear symplectic map  $\Psi(t)$ ,

$$J^{-1}\Psi(t)J\mathfrak{A}_{e'}\Psi(t)^{-1} = \mathfrak{A}_e.$$

Because  $\mathrm{Sp}(2n)$  deformation retracts to  $\mathrm{U}(n)$ , one obtains a canonical parallel transport isomorphism:

$$(30) \quad \mathfrak{o}(\mathfrak{R}, \varphi_*\mathfrak{A}_{e'}) \rightarrow \mathfrak{o}(\mathfrak{R}, \mathfrak{A}_e).$$

This can be precomposed with the morphism constructed in §2.4.8:

$$(31) \quad \mathfrak{o}(\mathfrak{R}, \mathfrak{A}_{e'}) \rightarrow \mathfrak{o}(\mathfrak{R}, \varphi_*\mathfrak{A}_{e'}).$$

**Claim 2.19.** *The assignment  $e \mapsto \mathfrak{o}(\mathfrak{R}, \mathfrak{A}_e)$ , with the morphisms obtained by composing (31) and (30) defines a functor from the indiscrete groupoid spanned by choices of symplectic coordinates.  $\square$*

**Definition 2.20.** *The orientation line  $\mathfrak{o}(\gamma)$  is the limit of  $e \mapsto \mathfrak{o}(\mathfrak{R}, \mathfrak{A}_e)$ .*

A similar linearization procedure can be done at any solution of the Floer-type PDEs considered in this text (without varying parameters), and one obtains canonical identifications:

$$\mathfrak{o}(D_{u,e}) \rightarrow \mathfrak{o}(\mathfrak{A}_{-,e}, \mathfrak{A}_{+,e}) \rightarrow \mathfrak{o}(\gamma_-) \otimes \mathfrak{o}(\gamma_+),$$

where  $e$  is a travelling family of symplectic coordinates defined along the domain of  $u$  which restrict to families of coordinates at the asymptotic ends  $\gamma_-, \gamma_+$  (in general, one tensors the orientation lines of all asymptotics).

**Claim 2.21.** *The identifications  $\mathfrak{o}(D_{u,e}) \rightarrow \mathfrak{o}(\gamma_-) \otimes \mathfrak{o}(\gamma_+)$  are compatible with the conjugation isomorphisms arising from changing  $e$ .*  $\square$

**2.4.10. Note on parametric moduli spaces.** Recall that in the construction of the equivariant cohomology infinity-functor, we count solutions  $(\ell, \pi, u)$  of parametric moduli spaces where:

- $\ell$  is a straight line path in  $\Delta^n$ ,
- $\pi$  is a flow line on  $S^\infty$  for a suitable  $G$ -invariant pseudogradient,
- $u$  solves Floer's continuation cylinder equation for data which depends on  $\ell$  and  $\pi$ ;

see (19) from §2.2.3 for the precise details.

Let us denote by  $D_{\ell,\pi,u}$  the linearization associated to the parametric problem. Parallel transport from  $D_{\ell,\pi,u}$  to  $D_u$  in the space of Fredholm operators produces an identification :

$$\mathfrak{o}(D_{\ell,\pi,u}) \rightarrow \mathfrak{o}(T_\ell) \otimes \mathfrak{o}(T_\pi) \otimes \mathfrak{o}(D_u) = \mathfrak{o}(T_\ell) \otimes \mathfrak{o}(T_\pi) \otimes \mathfrak{o}(\gamma_-) \otimes \mathfrak{o}(\gamma_+),$$

where  $T_\ell, T_\pi$  are the (finite-dimensional) tangent spaces to:

- the space of straight line paths §2.1 at  $\ell$ , and,
- the space of flow lines on  $BG$  with the same asymptotics as  $\pi$ .

We comment here that [Par16, §C.13.1] considers orientations on  $T_\ell$ , in a slightly different context, while [SZ21, §A] considers orientations on  $T_\pi$ .

**2.4.11. The definition of the infinity-functor.** Fix the Morse model on  $BG$  as described in §2.3. Given a zero-simplex  $(\varphi_{\eta,t}, J)$  in the  $\infty$ -category of regular Borel data  $\mathcal{D}(W)$ , we define:

$$\mathrm{CF}_{\mathrm{eq}}(\varphi_{\eta,t}, J) = \bigoplus \mathfrak{o}(\gamma) \otimes \mathfrak{o}(y) \otimes \mathbf{k}[[x]],$$

where the direct sum is over pairs  $\gamma, y$  where:

- $y$  is a critical point in the fiber of  $BG$  over the minimum on  $\mathbb{C}P^\infty$ ;
- $\gamma$  is an orbit of the system  $H_{y,t} = H_t$  over the distinguished lift of  $y$ ; see Figure 7 for details on distinguished lifts  $v_{0,e}$  and  $v_{1,e}$ .

Here  $\mathfrak{o}(y)$  is the orientation line for the unstable manifold of  $y$ . Standard Morse theory and the self-similarity transformations give isomorphisms:

$$\mathfrak{o}(T_\pi) \rightarrow \mathfrak{o}(y_-) \otimes \mathfrak{o}(x^k y_+) \simeq \mathfrak{o}(y_-) \otimes \mathfrak{o}(y_+) \otimes \mathfrak{o}(x^k),$$

where we abuse notation and let  $x^k$  denote the critical point of index  $2k$  on the base space  $\mathbb{C}P^\infty$ , and let  $x^k y$  denote the “shift” of  $y$  by the self-similarity map  $k$  times.

The count of rigid-up-to-translation trajectories can be considered as:

$$d_{eq} : \text{CF}_{eq}(\varphi_{\eta,t}, J, \mathbf{k}) \rightarrow \text{CF}_{eq}(\varphi_{\eta,t}, J, \mathbf{k})$$

provided we pick orientations for  $\mathfrak{o}(x^k)$  (and provided we follow the overall “twisting asymptotics by the group action” scheme of [SS10, Sei15, SZ21], similarly to in §2.2).

**Claim 2.22.** *If one picks the complex linear orientations of  $\mathfrak{o}(x^k)$ , then it holds that  $d_{eq}^2 = 0$ .*

*Proof.* This is a simplified version of what is shown in [SZ21, §A].  $\square$

This complex produces the first half of an infinity-functor. The second half concerns the morphisms associated to  $n$ -simplices  $\Phi$  for  $n \geq 1$ . Counting the rigid elements  $(\ell, \pi, u)$  produces a map:

$$\mathfrak{c}(\Phi) : \text{CF}_{eq}(\Phi|_{v_0}) \rightarrow \text{CF}_{eq}(\Phi|_{v_k}),$$

provided one also picks orientations of  $T_\ell$  (which amounts to picking orientations of the open  $n - 1$  cube). Then:

**Claim 2.23.** *If one orients the  $j$ -dimensional cubes in the standard way, so that  $\partial_1, \dots, \partial_j$  is an oriented basis, for  $j < n$ , it holds that:*

$$\sum_{j=1}^{n-1} (-1)^j (\mathfrak{c}_{\Phi|[j\dots n]} \circ \mathfrak{c}_{\Phi|[0\dots j]} - \mathfrak{c}_{\Phi|[0\dots\hat{j}\dots n]}) = \mathfrak{c}_{\Phi} \circ d_{\Phi|0} + (-1)^n d_{\Phi|n} \circ \mathfrak{c}_{\Phi}.$$

*Proof.* This follows from the boundary classification and orientation properties established by Pardon in [Par16, §C]. We leave the verification to the reader.  $\square$

This completes our description of the infinity-functor in the case  $p \geq 3$ .

### 3. PSS isomorphism and localization in Morse theory

In this section we prove Theorem 1.16, and part (c) of Theorem 1.14.

Let  $W$  be an aspherical  $G$  filling of a closed contact manifold  $Y$ , for some group  $G$  of prime order  $p$ . Throughout we use the field  $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$ , which we generally suppress from the notation (since it is determined by the group).

The usual PSS map in symplectic cohomology, as constructed in [FS07], yields an isomorphism:

$$\text{HM}^*(W) \cong \text{HF}(R_{\epsilon t})$$

where the left hand side is the Morse cohomology of  $W$ , and the right hand side is the Floer cohomology of any linear-at-infinity Hamiltonian whose slope  $\epsilon$  with respect to a fixed Reeb flow  $R$  on  $\partial W$  is sufficiently small and positive. We’ll implement this in our equivariant setup, and lift the construction to the chain level statement stated in Theorem 1.16.

The strategy to prove Theorem 1.16 is four-fold:

- (1) define the Morse equivariant cohomology complex  $\text{CM}_{\text{eq}}(X_\eta)$  for suitable “Borel families” of pseudogradients  $X_\eta$  which point outwards along the convex end of  $W$ ; see §3.1;
- (2) Prove a localization result in §3.2, where fixed points of the  $G$ -action in  $W$  are shown to imply the non-vanishing of free summands of the equivariant cohomology  $\text{HM}_{\text{eq}}(X_\eta)$ ;
- (3) construct the  $\infty$ -category  $\mathcal{P}(W)$  and the  $\infty$ -functor appearing in the statement of Theorem 1.16; see §3.3;
- (4) prove the resulting chain maps  $\text{PSS} : \text{CM}_{\text{eq}}(X_\eta) \rightarrow \text{CF}_{\text{eq}}(R_{\epsilon t})$  are quasi-isomorphisms, if  $R$  is a Reeb flow and  $\epsilon$  is sufficiently small; this is achieved by constructing maps in the reverse direction (from the Floer side to the Morse side), and showing they are chain-homotopy inverses to the PSS map, as in [FS07]; see §3.4.

**3.1. The equivariant Morse complex.** In this section, we are concerned with defining the  $G$ -equivariant Morse complex  $\text{CM}_{\text{eq}}(X_\eta)$  and developing its properties. As above,  $G = \mathbb{Z}/p\mathbb{Z}$ , and we work over a base field  $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$ .

**3.1.1. Morse–Borel data.** The auxiliary data is analogous §2.2.2.

**Definition 3.1.** A Morse–Borel datum is a family of vector fields  $X_\eta$  on  $W$  parameterized by  $\eta \in EG = S_\infty$  such that:

- (M1)  $g_*X_\eta = X_{g\eta}$ , for all  $g \in G$ ,
- (M2)  $X_\eta$  agrees with the Liouville vector field  $Z$  outside a compact set,<sup>32</sup>
- (M3)  $X_{\tau(\eta)} = X_\eta$ , where  $\tau$  is the self-similarity map,
- (M4)  $X_\eta$  is independent of  $\eta$  on neighborhoods of the critical points of the model  $G$ -invariant Morse pseudogradient  $V$  of  $S^\infty = EG$ .
- (M5)  $X_\eta$  is a Morse pseudogradient, whenever  $\eta \in EG$  is a zero of  $V$ .

Here the model  $G$ -invariant Morse pseudogradient on  $EG$  is as described in §2.2 in the case  $p = 2$ , or in §2.3 in the case  $p \geq 3$ .

**3.1.2. Spaces of flow lines.** Associated to such Morse–Borel data  $X_\eta$  one can form the moduli spaces  $\mathcal{N}$  of flowlines  $(q, \pi) : \mathbb{R} \rightarrow W \times S^\infty$  satisfying:

$$\begin{cases} q'(s) = -X_{\pi(s)}(q(s)), \\ \pi'(s) = -V(\pi(s)), \\ \lim_{s \rightarrow \pm\infty} q(s) \text{ exists.} \end{cases}$$

Any such flow line has well-defined asymptotics  $(q_\pm, \eta_\pm)$ .

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<sup>32</sup>We use the Liouville flow here only because it is a canonical outwards pointing vector field; any other outwards pointing vector field would suffice.

**Definition 3.2.** *If the moduli spaces  $\mathcal{N}$  are smooth manifolds of the expected dimension given by:*

$$\text{index}(q_-) - \text{index}(q_+) + \text{index}(\eta_-) - \text{index}(\eta_+),$$

*then we say that  $X_\eta$  is regular Morse–Borel data.*

The existence and genericity of regular data follow from the usual Morse-theoretic transversality arguments.

**3.1.3. Definition of the Borel-equivariant Morse complex.** The equivariant Morse complex associated to Morse–Borel data  $X_\eta$  is defined to be the direct sum  $\text{CM}_{\text{eq}}(X_\eta) := \bigoplus \mathfrak{o}(q) \otimes \mathfrak{o}(y) \otimes \mathbf{k}[[x]]$ , similarly to the Floer case (see §2.2.4 and §2.4.11), where the direct sum runs over pairs  $(q, y)$  such that:

- $y$  is a critical point of the pseudogradient on  $BG$  lying above the pole  $[1 : 0 : \dots]$  in projective space,<sup>33</sup>
- $q$  is a critical point of  $X_\eta$ , where  $\eta$  is the distinguished lift of  $y$  from  $BG$  to  $EG$ . The word “distinguished” is to be understood vis-a-vis Figure 7 from §2.3: the distinguished lifts were denoted  $v_{0,e}, v_{1,e}$ .

The differential  $d_{eq}$  on this complex is defined by counting the 1-dimensional components in  $\mathcal{N}$  and interpreting the count as follows: if  $\pi(s)$  is asymptotic to  $g\tau^k(\eta_-)$  and  $\eta_+$ , and  $q(s)$  is asymptotic to  $gq_-, q_+$ , then  $(q, \pi)$  is registered as contributing towards the coefficient:

$$\mathfrak{o}(q_+) \otimes \mathfrak{o}(y_+) \mapsto x^k \mathfrak{o}(q_-) \otimes \mathfrak{o}(y_-),$$

where  $y_\pm$  are the projections of  $\eta_\pm$  to  $BG$ . This is all in a manner similar to §2.4.11. The differential squares to zero by the same arguments used in [SS10, Sei15, SZ21] and in §2.2, §2.3, and §2.4.

**3.1.4. Invariance statement.** Let us denote the homology of  $\text{CM}_{\text{eq}}(X_\eta)$  by  $\text{HM}_{\text{eq}}(X_\eta)$ . Continuation maps endow  $X_\eta \mapsto \text{HM}_{\text{eq}}(X_\eta)$  with the structure of a functor on an indiscrete groupoid, and we define  $\text{HM}_{\text{eq}}(W)$  as the limit of this functor. We defer additional discussion of continuation maps to §3.3.4 and §3.3.6.

**3.1.5. The unit element in equivariant Morse cohomology.** Let us comment on the construction of a class  $1 \in \text{HM}_{\text{eq}}(W)$ . Fixing Borel–Morse data  $X_\eta$ , each generating pair  $(q, y)$  where  $\text{index}(q) = \text{index}(y) = 0$  admits canonical isomorphisms:

$$\mathbb{Z} \simeq \mathfrak{o}(q) \text{ and } \mathbb{Z} \simeq \mathfrak{o}(y),$$

because the orientation line of a zero-dimensional space has a canonical generator (corresponding to the empty basis). Taking their tensor product

<sup>33</sup>With the correct interpretation of “projective space” (as either  $\mathbb{R}P^\infty$  or  $\mathbb{C}P^\infty$ ), this handles both cases  $p = 2$  and  $p \geq 3$ ; in the former case, there is a unique choice of  $y$ , while in the latter case, one can arrange the construction so there are two choices of  $y$ .

with  $1 \in \mathbf{k}[[x]]$ , and direct summing over such pairs  $(q, y)$  produces an element in the complex  $\mathrm{CM}_{\mathrm{eq}}(X_\eta)$ , see §3.1.3, denoted by the symbol  $1$ .

**Lemma 3.3.** *The element  $1$  satisfies  $d_{\mathrm{eq}}(1) = 0$ , and is preserved under continuation maps, and consequently induces an element in  $\mathrm{HM}_{\mathrm{eq}}(W)$ .*

*Proof.* The proof is based on two standard ideas in Morse theory:

- the sum of all local minima is a cycle,
- the continuation map from one Morse complex to another sends the sum of all local minima to the sum of all local minima.

The details are left to the reader. □

**3.1.6. Equivariant pseudogradients.** Let us now focus on a special class of Borel–Morse data, namely, data  $X = X_\eta$  which is globally independent of  $\eta$ , and which satisfy the following additional axiom:

- (EP) around each critical point  $q$  in the fixed submanifold  $F \subset W$  of the  $G = \mathbb{Z}/p\mathbb{Z}$  action, there is a coordinate chart  $B^{2r} \times B^{2n-2r}$  so that  $F$  is identified with  $B^{2r} \times \{0\}$ , and  $g \in G$  acts by  $(z_1, z_2) \mapsto (z_1, gz_2)$  for some unitary representation of  $G$  on  $\mathbb{C}^{n-r}$ , and so that  $X$  appears as a direct sum  $X_1 + X_2$  of two linear vector fields, and  $X_2$  points radially outwards.

**Lemma 3.4.** *There exist vector fields  $X$  satisfying (EP) which define regular Borel–Morse data.*

*Proof.* See [SS10, §2.2]. The idea is to pick  $X$  first on the fixed submanifold  $F \subset W$ , and then extend it to a tubular neighborhood of  $F$  by requiring it points outwards (from the perspective of Lyapunov functions, we add a positive definite quadratic function on each fiber of the normal bundle  $NF$ ). Then we extend  $X$  equivariantly to the rest of  $W$  (generically), bearing in mind it should agree with  $Z$  in the convex end.

If the choice of extension from  $NF$  to  $W$  is done sufficiently generically, then all flow lines asymptotic to a critical point in  $W \setminus F$  will be cut transversally. Moreover, if the initial choice of  $X$  on  $F$  is done sufficiently generically, then all flow lines contained entirely in  $F$  will be cut transversally *inside of*  $F$ . However, the assumption that  $X$  points outwards along fibers of  $NF$  ensures that such flow lines are also cut transversally when considered in  $W$  (there is no change in the index difference).

The cases covered in the two preceding paragraphs cover all possible flow lines, and the proof is complete. □

**3.2. Localization in equivariant Morse theory.** *Localization* refers to the phenomenon that the ring-theoretic localization of the equivariant cohomology ring of a  $G$ -space at certain elements recovers the regular cohomology

of the fixed point set. This statement can be made quite sophisticated (we refer the reader to [AB84] for a refined result). In our paper, we will require only a small input from this theory (for Theorem 1.14 part (c)):

**Proposition 3.5.** *If the action of  $G = \mathbb{Z}/p\mathbb{Z}$  on a  $G$ -filling  $W$  has at least one fixed point, then the unit element  $1 \in \mathrm{HM}_{\mathrm{eq}}(W)$  is not  $x$ -torsion.*

The group  $\mathrm{HM}_{\mathrm{eq}}(W)$  is a finitely generated  $\mathbf{k}[[x]]$ -module, so it can be decomposed into a free part and a torsion part; the proposition guarantees the free part is non-zero (in the presence of at least one fixed point). More refined localization results identify exactly the rank of this free part.

The somewhat involved Morse-theoretic arguments of [SS10, §2], which are carried out in the case of a closed manifold with a  $\mathbb{Z}/2$ -action, prove something stronger. We will take a simpler approach<sup>34</sup> than that of [SS10].

**3.2.1. Proof of Proposition 3.5.** We assume  $p \geq 3$ , as the case  $p = 2$  is simpler (and is essentially established in [SS10]).

Without loss of generality, we pick  $X_\eta = X$  satisfying (EP). Since we assume there is at least one fixed point, the fixed point submanifold  $F$  is non-empty, and, by construction of  $X$ , there is at least one index 0 critical point  $q_0$  in the fixed point submanifold.

Let us introduce symbols  $v_0, v_1$  as the index 0 and index 1 critical points on  $BG$ ; all other critical points are of the form  $\tau^k(v_0)$  or  $\tau^k(v_1)$ , and each such critical point has  $p$  lifts to critical points on  $EG$  (shown in Figure 7 using notation  $v_{0,g}$  and  $v_{1,g}$  for the lifts of  $v_0, v_1$ ).

Recall from §3.1.3 that:

$$(32) \quad \mathrm{CM}_{\mathrm{eq}}(X) := \bigoplus_q (\mathfrak{o}(v_0) \otimes \mathfrak{o}(q) \otimes \mathbf{k}[[x]]) \oplus (\mathfrak{o}(v_1) \otimes \mathfrak{o}(q) \otimes \mathbf{k}[[x]]),$$

where the sum is over all critical points  $q$  (zeros of  $X$ ). Let us consider the projection map onto the summand:

$$Q(X, q_0) := (\mathfrak{o}(v_0) \otimes \mathfrak{o}(q_0) \otimes \mathbf{k}[[x]]) \oplus (\mathfrak{o}(v_1) \otimes \mathfrak{o}(q_0) \otimes \mathbf{k}[[x]])$$

associated to the chosen index 0 critical point  $q_0 \in F$ . This map is a quotient by a subcomplex because any flow line  $(q(s), \pi(s))$  which ends with  $q(-\infty) = q_0$  must also start with  $q(+\infty) = q_0$ .

The induced differential on  $Q(X, q_0)$  yields a complex which computes the equivariant Morse cohomology of a point  $\mathrm{HM}_{\mathrm{eq}}(\mathrm{pt})$  (this is obvious, since

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<sup>34</sup>Another reasonable approach is to essentially translate each step of [AB84] into Morse theoretic terms. This requires four ingredients: *embedding functoriality* (i.e., pull-back maps), *the relative cohomology of a pair*, *the Thom isomorphism*, and *the ring structure on (equivariant) cohomology*. The interpretation of these steps within Morse theory is more or less standard. The recipe in [AB84] then gives a refined localization result.

the quotient differential on  $Q(X, q_0)$  only considers flow lines  $(q(s), \pi(s))$  for which  $q(s) = q_0$  while  $\pi(s)$  is allowed to vary arbitrarily on  $EG$ .<sup>35</sup>

It is also clear from the definitions that this quotient map sends  $1 \in \mathrm{HM}_{\mathrm{eq}}(W)$  to  $1 \in \mathrm{HM}_{\mathrm{eq}}(\mathrm{pt})$ . We now invoke the following lemma:

**Lemma 3.6.** *The element  $1 \in \mathrm{HM}_{\mathrm{eq}}(\mathrm{pt})$  is not  $x$ -torsion; here it is important to assume that  $G = \mathbf{k} = \mathbb{Z}/p\mathbb{Z}$ .*

*Proof.* This is obvious provided one knows that  $\mathrm{HM}_{\mathrm{eq}}(q_0)$  is the ordinary Borel  $G$ -equivariant cohomology  $H_G^*(q_0, \mathbf{k})$ . However, it also follows from a direct computation of the differentials for the special pseudogradient  $BG$ ; see, e.g., [SZ21, §4].  $\square$

Thus 1 is not  $x$ -torsion in  $\mathrm{HM}_{\mathrm{eq}}(W)$ . This proves Proposition 3.5.  $\square$

**3.3. The equivariant PSS construction.** The goal is to give a precise description of the  $\infty$ -category  $\mathcal{P}(Y)$  introduced in §1.4, construct an  $\infty$ -functor:

$$\mathcal{P}(Y) \rightarrow \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]])$$

with the stated properties.

**3.3.1. The PSS category.** In this section we define the PSS category  $\mathcal{P}(Y)$ . Recall from §1.4 and Remark 2.6 the  $\infty$ -category  $\bar{\mathcal{C}}(Y)$ .

**Definition 3.7.** *Define  $\mathcal{P}(Y) \subset \bar{\mathcal{C}}(Y)$  to be the sub- $\infty$ -category such that  $\mathcal{P}_n(Y)$  consists of the simplices  $\Sigma : \Delta^n \rightarrow \bar{\mathcal{C}}(Y)$  satisfying the following properties for some  $j \in \{-1, \dots, n\}$ :*

- $\Sigma|_{\Delta_{\{j+1, \dots, n\}}}$  is mapped into  $\mathcal{C}(Y)$  (the non-discriminant part),
- $\Sigma|_{\Delta_{\{0, \dots, j\}}}$  is sent to the constant simplex at  $id$ .

Here  $id$  denotes the identity system  $\varphi_t = id$ . If  $j = -1$ , then we mean that all of  $\Sigma$  is sent into  $\mathcal{C}(Y)$ . We also use the notation  $\Delta^I \subset \Delta^n$  to refer to the subsimplex associated with the inclusion  $I \subset \{0, \dots, n\}$  of ordered subsets.

The definition of  $\mathcal{P}(Y)$ , and the notation  $\Delta^I$  used, is highly reminiscent of the *join operation* of simplicial sets explained in [Lur09, §1.2.8]. One should think of  $\mathcal{P}(Y)$  as the join of  $\Delta^0$  and  $\mathcal{C}(Y)$ . However,  $\mathcal{P}(Y)$  is not *actually* defined as this join, but is rather defined using the category  $\bar{\mathcal{C}}(Y)$ .

**Proposition 3.8.** *The simplicial set  $\mathcal{P}(Y)$  is an  $\infty$ -category.*

*Proof.* The argument is close to the one in [Lur09, Proposition 1.2.8.3]. We will prove it as follows. To prove  $\mathcal{P}(Y)$  is an  $\infty$ -category, we need to show that any “inner horn”  $p : \Lambda_i^n \rightarrow \mathcal{P}(Y)$  can be filled to a map  $\Delta^n \rightarrow \mathcal{P}(Y)$ .

<sup>35</sup>The reader will notice the ghost of the aforementioned “embedding functoriality.” Specifically, the quotient map  $\mathrm{CM}_{\mathrm{eq}}(X) \rightarrow Q(X, q_0)$  is the Morse-theoretic realization of the cohomological pullback  $H_G^*(W) \rightarrow H_G^*(\{q_0\})$  induced by the inclusion  $\{q_0\} \rightarrow W$ .

Case 1. If  $p$  sends all of  $\Lambda_i^n$  into  $id$ , then we can just extend  $p$  to be the constant simplex at  $id$ .

Case 2. Otherwise, just pick any filling  $p$  to a map  $\Delta^n \rightarrow \bar{\mathcal{C}}(Y)$  (which is possible since  $\bar{\mathcal{C}}(Y)$  is an  $\infty$ -category; see Remark 2.10). Because  $p$  already takes values in  $\mathcal{P}(Y)$  on the face  $\Delta^{\{0, \dots, n-1\}}$  and the vertex  $\Delta^{\{n\}}$ , the extension of  $p$  takes values in  $\mathcal{P}(Y)$  (since the conditions in Definition 3.7 only constrain the face  $\Delta^{\{0, \dots, n-1\}}$  and the vertex  $\Delta^{\{n\}}$  when not in Case 1).  $\square$

**3.3.2. Auxiliary category.** We construct an auxiliary  $\infty$ -category  $\mathcal{Q}(W)$ , depending on the  $G$ -filling  $W$  of  $Y$ , which (similarly to  $\mathcal{D}(W)$  from §2.2.2) encodes all of the data needed to define the PSS map (in our equivariant context).

Just as the category  $\mathcal{C}(Y)$  admits a natural enlargement  $\bar{\mathcal{C}}(Y)$  where 0-simplices  $\varphi_t$  are allowed to lie on the discriminant, so there is a larger category  $\bar{\mathcal{D}}(W)$  containing  $\mathcal{D}(W)$ : it is defined exactly as  $\mathcal{D}(W)$  is except we do not require property (N2). In particular, 0-simplices of  $\bar{\mathcal{D}}(W)$  are Borel data  $(\psi_{\eta,t}, J)$ , and the “maximally degenerate” Borel data  $(id, J)$  is allowed (unlike in  $\mathcal{D}(W)$ ). Note that, exactly as with  $\mathcal{D}(W)$  and  $\mathcal{C}(Y)$ , there is an ideal restriction  $\infty$ -functor  $\text{IR} : \bar{\mathcal{D}}(W) \rightarrow \bar{\mathcal{C}}(Y)$ .

The auxiliary category  $\mathcal{Q}(W)$  we define will also need to incorporate the choice of Borel–Morse data from §3.1.1. To achieve this, we define (yet) another  $\infty$ -category  $\mathcal{G}(W)$  whose  $n$ -simplices are smooth maps:

$$\Delta^n \times EG \rightarrow \{\text{vector fields on } W\},$$

written  $v, \eta \mapsto X_{v,\eta}$ , such that, for each  $v$  fixed,  $X_{v,\eta}$  satisfies (M1) through (M4); we refer the reader to §3.1.1 for the statements of these axioms.

In a loose sense, the category  $\mathcal{Q}(W)$  we will define should be interpreted as the join of  $\mathcal{G}(W)$  and  $\mathcal{D}(W)$ . However this is not quite the correct thing to do, and the precise definition uses the larger category  $\bar{\mathcal{D}}(W)$ , in a similar manner to how  $\mathcal{P}(Y)$  was defined using  $\bar{\mathcal{C}}(Y)$  in §3.3.1.

**Definition 3.9.** *An  $n$ -simplex in  $\mathcal{Q}(W)$  is defined to be the data of an  $n$ -simplex  $\Phi \in \bar{\mathcal{D}}_n(W)$ , a number  $j \in \{-1, \dots, n\}$ , a  $j$ -simplex  $X \in \mathcal{G}_j(W)$  if  $j \geq 0$ , that satisfy the following properties:*

- $\Phi|_{\Delta^{\{j+1, \dots, n\}}}$  is a simplex in  $\mathcal{D}(W)$ ,
- $\Phi|_{\Delta^{\{0, \dots, j\}}}$  is sent to the constant simplex at  $(id, J)$ , for some  $J$ .

An  $n$ -simplex in  $\mathcal{Q}(W)$  will be denoted by the symbol  $\Sigma = (\Phi, X)$ .

This  $\mathcal{Q}(W)$  is a simplicial set in a fairly obvious way (indeed, the same arguments which explain why the join construction of [Lur09, §1.2.8] produces simplicial sets apply).<sup>36</sup> We provide an example in Figure 8.

<sup>36</sup>Briefly,  $\mathcal{Q}(W) \rightarrow \bar{\mathcal{D}}(W)$  is required to be an  $\infty$ -functor; this fact entirely dictates how the structure maps act on the  $\Phi$  factor. The action on the  $X$  factor is a bit more

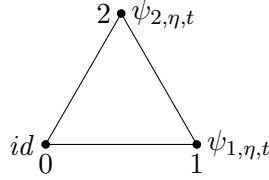


FIGURE 8. A 2-simplex in  $\mathcal{Q}(W)$  with  $j = 0$ . The  $[01]$  and  $[02]$  faces both have  $j = 0$ , while the  $[12]$  face has  $j = -1$ . This simplex in  $\mathcal{Q}(W)$  also has the extra data of a 0 simplex in  $\mathcal{G}(W)$  (one should imagine the  $id$  vertex is decorated with a vector field on  $W$ ).

**Proposition 3.10.** *The simplicial set  $\mathcal{Q}(W)$  is an  $\infty$ -category.*

*Proof.* The argument is only mildly more complicated than that of Proposition 3.8, and we leave the proof to the reader.  $\square$

**3.3.3. Moduli space of cylinders associated to  $n$ -simplices.** Let  $\Sigma = (\Phi, X)$  be an  $n$ -simplex in  $\mathcal{Q}(W)$  with  $j \leq n$ . By following the exact same prescription as in §2.2.3, we obtain a moduli space  $\mathcal{M}_{j,n}(\Phi)$  of pairs  $(\ell, \pi, u)$  where:

- $\ell \in \mathcal{M}^{\text{int}}(\Delta^n)$ , the moduli space of straight line paths,
- $\pi \in \mathcal{P}$ , some space of flow lines<sup>37</sup> on  $BG$ ,
- $u$  solves the continuation map equation associated to the non-negative path:

$$\Phi(\ell) : [0, n] \rightarrow \{\text{Borel Data } (\psi_{\eta,t}, J)\}.$$

The precise details are as follows:  $\ell$  determines subintervals:

$$I_{n-1} \leq I_{n-2} \leq \cdots \leq I_1,$$

where  $I_i = [s_i, s_i + w_i]$  as explained in (18), with  $I_1 = [0, 1]$ . If  $n = 0$ , then there are no intervals, and  $(\pi, u)$  solve the equivariant differential equation. In general, on the  $I_i \times \mathbb{R}/\mathbb{Z}$  sub-cylinder, we insert the  $i$ th level determined by  $\Phi$ , and formulate the joint equation for  $(u, \pi)$  exactly as in (19).

Let us note that  $\mathcal{M}_{-1,n}$  is exactly the moduli space considered in (19). By the standard Gromov removal of singularities result from [Gro85], if  $\Sigma$  is an  $n$ -simplex with  $j \geq 0$ , then each  $u \in \mathcal{M}_{j,n}(\Phi)$  has a well-defined input asymptotic  $u(+\infty) \in W$  which is a removable singularity. This aspect of the story is standard amongst all treatments of PSS; see, e.g., [Can24b, §3.5], [BC25, §3.4.6], and the references therein.

subtle: if  $f : \{0, \dots, m\} \rightarrow \{0, \dots, n\}$  is a simplicial map, then  $f^*X$  is defined to be the  $k$ -simplex, where  $k = \max f^{-1}(\{0, \dots, j\})$ , obtained by pulling back  $X$  by  $f|_{\Delta^k}$ .

<sup>37</sup>There is a clash of notation;  $\mathcal{P}$  refers to spaces of flow lines on  $BG$ , and  $\mathcal{P}(Y)$  also refers to the PSS-category. These can be distinguished by the context and the  $Y$  symbol.

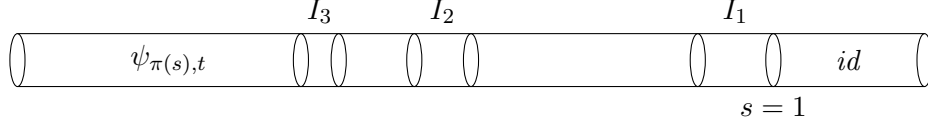


FIGURE 9. The input asymptotic of the continuation cylinder is degenerate, if  $j \geq 0$ , and  $u$  solves the holomorphic curve equation on the region  $s \geq 1$ . If  $j = -1$  then the input asymptotic is non-degenerate Borel data. If any of the cubical coordinates  $x_i(\ell)$ , with  $i \leq j$ , are sufficiently close to zero, then the curve will be genuinely holomorphic in more regions, by axiom (N4), some of the levels we stabilize to be the identity  $id$ .

**3.3.4. Moduli spaces of flow lines.** Suppose that  $X$  is a  $j$ -simplex,  $j \leq n$ . We will construct a moduli space  $\mathcal{N}_{j,n}(X)$  consisting of triples  $(\ell, \pi, q)$  where  $\ell \in \mathcal{M}^{\text{int}}(\Delta^n)$ ,  $\pi \in \mathcal{P}$  are as above, and  $q : \mathbb{R} \rightarrow W$  solves a particular ODE depending on  $\ell, \pi$ , and the values of  $X$  along the straight line path.

The construction is similar to §3.3.3, in that we arrange travelling intervals (depending on  $\ell$ ) on the real line. We use the same intervals  $I_i = [s_i, s_i + w_i]$  from (18); recall that these satisfy:

$$s_i = \sum_{k < i} x_k - x_k^{-1}, \quad w_i = 1 - x_{i-1},$$

with  $x_0 = 0$ , so  $w_1 = 1$ . When  $j < n$ , we also add a special time:

$$(33) \quad s_* = s_{j+1} + 1,$$

which, crucially, lies to the *right* of  $I_{j+1}$ . In the sequel, we will consider our flow lines as only defined on  $[s_*, \infty)$ . This time  $s_*$ , similarly to the intervals, is a function of the underlying straight-line path  $\ell$ ; see Figure 10.

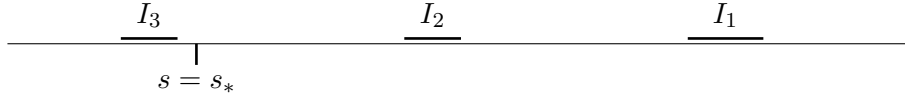


FIGURE 10. Shown with  $j = 2$  and  $n = 3$  and cubical coordinates  $(x_1, x_2) = (1/4, 1/4)$ . The time  $s_*$  is always to the right of the interval  $I_{j+1}$ .

A straight-line path  $\ell \in \mathcal{M}^{\text{int}}(\Delta^n)$  has cubical coordinates  $x_1, \dots, x_{n-1}$ . There is a unique straight line path  $\ell^{(j)} \in \mathcal{M}^{\text{int}}(\Delta^j)$  whose cubical coordinates are  $x_1, \dots, x_{j-1}$ . The times  $\tau_1, \dots, \tau_j$ , as in §2.1.1, of  $\ell^{(j)}$  determine line segments  $\ell^{(j)}|_{[\tau_{i-1}, \tau_i]}$  on the simplex  $\Delta^j$ . Evaluating  $X_\eta$  along the  $i$ th segment and using the affine reparametrization  $[0, 1] \rightarrow [\tau_{i-1}, \tau_i]$  gives a family of vector fields  $L_{\ell,i} X_{\eta,s}$  defined for  $s \in [0, 1]$ ,  $\ell \in \mathcal{M}^{\text{int}}(\Delta^n)$ , and  $\eta \in EG$ . They satisfy (M1) through (M4).

We distribute these “levels”  $L_{\ell,i}X_{\eta,s}$  onto the real line by:

- $(\ell X)_{\eta,s} := L_{\ell,i}X_{\eta,\rho_i(s)}$  if  $s \in I_i$ ,

where  $\rho_i : I_i \rightarrow [0, 1]$  is the unique affine diffeomorphism which reverses orientation (because the inputs to our chain level operation are situated on the right end), and which satisfies:

- $\partial_s(\ell X)_{\eta,s} = 0$  outside of the intervals  $I_j, I_{j-1}, \dots, I_1$ .

In particular,  $(\ell X)_{\eta,s} = X_{v_0}$  for  $s$  sufficiently large.

**Definition 3.11.** *The moduli space  $\mathcal{N}_{j,n}(X)$  is defined as the solutions of:*

$$(34) \quad \begin{cases} \ell \in \mathcal{M}^{\text{int}}(\Delta^n), \pi : \mathbb{R} \rightarrow EG, \text{ and } q : \mathbb{R} \rightarrow W, \\ \pi'(s) = -V(\pi(s)), \\ q'(s) = -(\ell X)_{\pi(s),s}(q(s)). \end{cases}$$

compare with (19). In the case  $j < n$ , we will only consider the restriction of  $q$  to  $[s_*, \infty)$ .

The case of  $\mathcal{N}_{n,n}(X)$  is slightly special, as it corresponds to  $\Sigma = (X, \Phi)$  where  $\Phi$  is the constant simplex at  $id$ . The case of  $\mathcal{N}_{j,n}(X)$  with  $j < n$  will be “paired” with the moduli space  $\mathcal{M}_{j,n}(\Phi)$  from §3.3.3; this is the subject of the next subsection.

**3.3.5. Hybrid moduli spaces and regular simplices.** Let  $\Sigma = (\Phi, X)$  be an  $n$ -simplex in  $\mathcal{Q}(W)$  with parameter  $j \leq n$  as above. Define  $\mathcal{M}_{j,n}(\Sigma)$  to be the *hybrid moduli space*, for  $0 \leq j < n$ , as the solutions to the problem:

$$(35) \quad \begin{cases} \ell \in \mathcal{M}^{\text{int}}(\Delta^n), \pi : \mathbb{R} \rightarrow EG, q : \mathbb{R} \rightarrow W, \text{ and } u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow W, \\ (\ell, \pi, u) \in \mathcal{M}_{j,n}(\Phi) \text{ from §3.3.3,} \\ (\ell, \pi, q) \in \mathcal{N}_{j,n}(X) \text{ from §3.3.4, and,} \\ u(+\infty) = q(s_*) \text{ (we ignore the values } q(s) \text{ for } s < s_*). \end{cases}$$

We define special cases when  $j = -1$  or  $j = n$ :

- $\mathcal{M}_{-1,n}(\Sigma) = \mathcal{M}_{-1,n}(\Phi)$  as in §3.3.3, and
- $\mathcal{M}_{n,n}(\Sigma) = \mathcal{N}_{n,n}(X)$ .

As in §2.2.3, we define a subcategory  $\mathcal{Q}^*(W) \subset \mathcal{Q}(W)$  of *regular simplices*  $\Sigma$ ; briefly:

**Definition 3.12.** *An  $n$ -simplex  $\Sigma$  is regular provided the moduli spaces  $\mathcal{M}_{j,n}(\Sigma')$  associated to all sub-simplices  $\Sigma' \subset \Sigma$  are cut transversally.*

Definition 3.12 is to be understood in the usual Floer theoretic sense (common to all treatments of parametric moduli spaces of continuation/PSS cylinders). In particular, we require that:

- the  $i$ th vertex of  $X \in \mathcal{G}(W)$ , with  $i \leq j$ , is regular Borel–Morse data, in the sense of §3.1.2;
- the restriction  $\Phi|_{\Delta_{j+1, \dots, n}}$ , is a regular  $n - 1 - j$  simplex in  $\mathcal{D}^*(W)$ , in the sense of Definition 2.15 from §2.2.3.

Standard transversality results (similar to those cited in §2.2.3) ensure the abundance of regular simplices. Indeed, we claim the following:

**Proposition 3.13.** *The subset  $\mathcal{Q}^*(W) \subset \mathcal{Q}(W)$  is an  $\infty$ -category, and there is a commutative diagram involving the ideal restriction maps:*

$$\begin{array}{ccccc} \mathcal{D}^*(W) & \longrightarrow & \mathcal{Q}^*(W) & \longleftarrow & \mathcal{G}^*(W) \\ \downarrow \text{IR} & & \downarrow \text{IR} & & \downarrow \\ \mathcal{C}(Y) & \longrightarrow & \mathcal{P}(Y) & \xleftarrow{id} & \Delta^0, \end{array}$$

where the vertical morphisms are trivial Kan fibrations (Definition 1.19) and the horizontal morphisms are inclusions ( $\mathcal{G}^*(W)$  is just the subcategory of  $\mathcal{Q}^*(W)$  with  $j = n$  and  $\mathcal{D}^*(W)$  is the subcategory with  $j = -1$ ).

*Proof.* That the subset  $\mathcal{Q}^*(W)$  and the subset  $\mathcal{G}^*(W)$  form  $\infty$ -categories (i.e., satisfy the horn filling axioms) follows from the same transversality arguments used for  $\mathcal{D}^*(W)$  in Proposition 2.16.

The diagram commutes by its construction in §3.3.1 and §3.3.2. It remains to show the vertical maps are trivial Kan fibrations:

- $\mathcal{D}^*(W) \rightarrow \mathcal{C}(Y)$  is a trivial Kan fibration by Lemma 2.18;
- $\mathcal{G}^*(W) \rightarrow \Delta^0$  is a trivial Kan fibration because the space of vector fields satisfying the Morse–Borel conditions (M1) through (M4) is convex, and general “transversality is generic” results can be used (similarly to the family Morse theory of [Hut08]) to construct regular extensions of any boundary  $\partial\Delta^n \rightarrow \mathcal{G}^*(W)$ ;
- $\pi : \mathcal{Q}^*(W) \rightarrow \mathcal{P}(Y)$  is a trivial Kan fibration follows essentially the same argument as 2.18, with straightforward variations due to the vector fields  $X$ .

This completes the proof.  $\square$

**3.3.6. PSS as infinity functor.** In this section, we explain how the counts of rigid solutions of elements in  $\mathcal{M}_{j,n}(\Sigma)$  associated to  $\Sigma \in \mathcal{Q}^*(W)$  can be packaged into an infinity functor:

$$(36) \quad \mathcal{Q}^*(W) \rightarrow \text{N}_{\text{dg}}\text{Ch}(\mathbf{k}[[x]]).$$

The argument follows the lines of §2.2.5 and §2.4.11.

First of all, we explain how (36) acts on zero simplices. There are two types of zero simplices: those with  $j = 0$  and those with  $j = -1$ ; the former includes the data of a zero simplex  $X \in \mathcal{G}^*(W)$ , which consists of a single (regular) Morse–Borel datum  $X_\eta$ . We send this to the equivariant

Morse complex  $\text{CM}_{\text{eq}}(X_\eta)$  as defined in §3.1.3. The zero simplices of type  $j = -1$  only consist of a zero simplex  $\Phi \in \mathcal{D}^*(W)$ , and we send these to the equivariant Floer complex  $\text{CF}_{\text{eq}}(\Phi)$ , as in §2.2.4 and §2.4.11.

Let us abbreviate, for the purposes of this section, by  $\text{C}(\Sigma)$  the chain complex associated to a zero simplex.

For an  $n$ -simplex  $\Sigma = (\Phi, X)$ , with  $n \geq 1$  and  $j \leq n$ , we package the count of the rigid  $G$ -orbits in  $\mathcal{M}_{j,n}$ , modulo the self-similarity map  $\tau$ , as an operation:

$$(37) \quad \mathbf{c}_\Sigma : \text{C}(\Sigma|0) \rightarrow \text{C}(\Sigma|n),$$

similarly to §2.2.5.

We digress and explain this “count of the rigid  $G$ -orbits.” By the axioms of Borel data from §2.2.2, §2.3 and §3.1.1, there is a diagonal  $G$ -action on the moduli space  $\mathcal{M}_{j,n}$ , where  $g$  replaces  $(\ell, \pi, u, q)$  by  $(\ell, g\pi, gu, gq)$ . Because the action on  $EG$  is free, one can “break” this symmetry by requiring that  $\pi$  is asymptotic to the distinguished lift of a critical point in  $BG$  (“distinguished lifts” are explained in Figure 7).

One can also shift solutions by the self-similarity  $(\ell, \pi, u, q) \mapsto (\ell, \tau \circ \pi, u, q)$ . We only count those solutions for which the input of  $\pi$  lies above the pole  $[1 : 0 : \dots]$  in  $\mathbb{C}P^\infty$  (or  $\mathbb{R}P^\infty$ , if  $G = \mathbb{Z}/2\mathbb{Z}$ ). The other asymptotic of  $\pi$  lies above some pole  $\tau^k([1 : 0 : \dots])$ , and we record this as the power  $x^k$ .

With these identifications in place, the count defines a map (37), via the orientation line framework similarly to §2.4.11 and §3.1.3.

**Claim 3.14.** *The map  $\mathbf{c}_\Sigma$  so constructed is a morphism of  $\mathbf{k}[[x]]$ -modules, and satisfies the  $\infty$ -functor relation:*

$$\sum_{k=1}^{n-1} (-1)^k (\mathbf{c}_{\Sigma|[k\dots n]} \circ \mathbf{c}_{\Sigma|[0\dots k]} - \mathbf{c}_{\Sigma|[0\dots \hat{k}\dots n]}) = \mathbf{c}_\Sigma \circ d_{\Sigma|0} + (-1)^n d_{\Sigma|n} \circ \mathbf{c}_\Sigma.$$

*Proof.* See §2.2.5 for the main geometric argument when  $j = -1$  (the only difference here is the difference in working with  $G = \mathbb{Z}/p\mathbb{Z}$ ). The case  $j = n$  is essentially the same (using Morse continuation lines  $q$  rather than Floer continuation cylinders  $u$ ).

The hybrid moduli spaces  $\mathcal{M}_{j,n}$  with  $0 \leq j < n$  require some additional discussion. The main geometric ideas follow standard lines, going back to [PSS96], but we need to check the equations formulated in §3.3.5 degenerate in the correct way as  $\ell$  approaches the boundary of  $\mathcal{M}^{\text{int}}(\Delta^n) \simeq (0, 1)^{n-1}$  in order for the  $\infty$ -functor equation to hold on the nose.

To illustrate the main points, we consider the case  $j = 1$  and  $n = 2$ . Then  $\mathcal{M}_{1,2}(\Sigma)$  is a parametric moduli space lying over  $\mathcal{M}^{\text{int}}(\Delta^2) \simeq (0, 1)$ . To prove the  $\infty$ -functor relation, we analyze the non-compact ends, as is typical in Floer theory. There are different failures of compactness which can occur in a

1-dimensional<sup>38</sup> component of  $\mathcal{M}^{\text{int}}(\Delta^2)$ . Let us fix a sequence  $(\ell_n, \pi_n, u_n, q_n)$  approaching a non-compact end. The possibilities are:

- (a)  $\ell_n$  converges in  $\mathcal{M}^{\text{int}}(\Delta^2)$ ; these contribute to  $\mathfrak{c}_\Sigma \circ d_{\Sigma|_0}$  or to  $d_{\Sigma|_2} \circ \mathfrak{c}_\Sigma$ ;
- (b)  $x_1(\ell_n)$  converges to 0; these contribute to  $\mathfrak{c}_{\Sigma|_{12}} \circ \mathfrak{c}_{\Sigma|_{01}}$ ,
- (c)  $x_1(\ell_n)$  converges to 1; these ends contribute to  $\mathfrak{c}_{\Sigma|_{02}}$ .

Let us check that these interpretations of the ends actually agree with the set-up of (34) and (35). The case of (a) is typical of Floer theoretic arguments, and is not sensitive to the precise placement of the intervals  $K_j \geq \dots \geq K_1$ , so we only discuss (b) and (c).

In case (b), the interval  $I_2$  is located very far to the left, and has length close to 1 (since  $x_1 - x_1^{-1} \approx -\infty$  and  $1 - x_1 \approx 1$ ); see Figure 11. There are two limits to extract as  $x_1 \rightarrow 0$ :

- the equation for  $u$  converges on compact subsets to the equation for a holomorphic cylinder; by the aspherical assumption, this holomorphic cylinder must be a constant map. Indeed, once  $x_1$  is small enough, the equation is genuinely holomorphic to the right of  $I_2$  (see Figure 11). On the other hand, the equation for  $q(s)$  converges on compact subsets to the equation for a Morse continuation line between  $X_0$  and  $X_1$ . After passing to a subsequence, this will converge to a solution in  $\mathcal{M}_{1,1}(\Sigma|_{01})$ .
- if one reparametrizes by translation so that  $s_*$  is moved to position  $s = 0$ , then the equation converges to a solution in  $\mathcal{M}_{0,1}(\Sigma|_{12})$ . Here it is important that the right endpoint of  $I_2$  converges to  $s_*$ , and the length of  $I_2$  converges to 1.

By standard consideration of Fredholm indices and PSS gluing results (as in [PSS96]), one concludes that such configurations of solutions in  $\mathcal{M}_{1,1}(\Sigma|_{01})$  and  $\mathcal{M}_{0,1}(\Sigma|_{12})$  are in 1-to-1 bijection with the non-compact ends of type (b). Let us note that, along each sequence, the underlying flow line  $\pi$  on  $BG$  will break into a configuration of two flow lines, similarly to the discussion §2.2.5, and in the correct sense to obtain  $\mathfrak{c}_{\Sigma|_{12}} \circ \mathfrak{c}_{\Sigma|_{12}}$ .

Turning now to the ends of type (c), when  $x_1 \rightarrow 1$ , the interval  $I_2$  has length  $1 - x_1 \approx 0$  and is located at position  $x_1 - x_1^{-1} \approx 0$ . Similarly the evaluation point  $s_* = x_1 - x_1^{-1} + 1 \approx 1$ . In the limit  $x_1 \rightarrow 1$ , the evaluation point  $s_*$  converges to 1, and a subsequence converges to a solution in  $\mathcal{M}_{0,1}(\Sigma|_{02})$ . Here it is crucial that  $I_1$  lies in  $(-\infty, s_*]$ , so there is no Morse continuation data used in the limit. Standard “gluing” results (applications of the implicit function theorem similar to those in [BC24, §4]), prove that it is precisely the solutions in  $\mathcal{M}_{0,1}(\Sigma|_{02})$  contributing to  $\mathfrak{c}_{\Sigma|_{02}}$  which comprise the ends of type (c).

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<sup>38</sup>technically, we mean a 1-dimensional component after quotient by translation.

This completes the discussion of the non-compact ends of the 1-dimensional components of  $\mathcal{M}_{1,2}(\Sigma)$ . The general analysis of the 1-dimensional components of  $\mathcal{M}_{j,n}(\Sigma)$  follows the same lines, and is similar to §2.2.5; we leave the details to the reader.

For questions concerning orientations, we refer the reader Claim 2.23 (which refers the reader to [Par16]). For details on orientation lines for such hybrid moduli spaces we refer the reader to [Abo15] and [BCS25].  $\square$

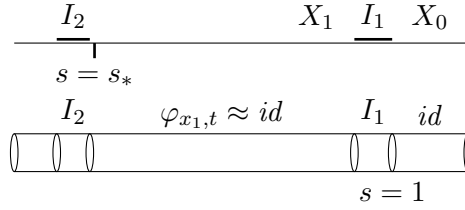


FIGURE 11. The set-up for the moduli space  $\mathcal{M}_{1,2}(\Sigma)$  and coordinate  $x_1 = 1/8$  shown. The symbols  $X_0, X_1$  represent the Borel–Morse data (on the interval  $I_1$ , one has a continuation from  $X_0$  to  $X_1$ ). The contact isotopy  $\varphi_{x_1,t}$  is the endpoint of the first level, and eventually equals  $id$  as  $x_1 \rightarrow 0$ .

Having constructed the infinity functor  $\mathcal{Q}^*(W) \rightarrow \text{N}_{\text{dg}}\text{Ch}(\mathbf{k}[[x]])$ , the construction of the functor with domain  $\mathcal{P}(Y)$  satisfying Theorem 1.16 follows the same lines as §2.2.6 (picking sections of the trivial Kan fibration  $\mathcal{Q}^*(W) \rightarrow \mathcal{P}(Y)$  from Proposition 3.13, etc).

**3.4. The low-slope PSS isomorphism.** We explain why the maps:

$$\text{PSS} : \text{HM}_{\text{eq}}(W) \rightarrow \text{HF}_{\text{eq}}(R_{\epsilon t})$$

associated to specific 1-simplices  $\varphi_{s,t} = R_{s\epsilon t}$  joining  $id$  to  $R_{\epsilon t}$ , for small positive slopes  $\epsilon$ , are chain-homotopy equivalences. The argument we will use follows [FS07, pp.24-25]; the idea is to construct a chain-homotopy inverse. Due to the similarity with [PSS96, FS07], we only sketch the results.

**3.4.1. Geometric set up for the inverse of PSS.** Let  $r$  be a radial coordinate on the convex end of  $W$ , so that  $\Omega = \{r \leq 1\}$  is an aspherical symplectic domain with contact-type boundary, and so that  $r \geq 1$  is identified with the positive half of the symplectization of  $\partial W$ . We suppose that  $r$  is  $G$ -invariant. Then  $X_r$  has an ideal restriction to  $\partial W$  as a Reeb flow  $R$  which lifts a Reeb flow on  $Y = \partial W/G$ .

Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a convex cut-off function so that:

- $h(r) = r$  for  $r \geq 2$ ,
- $h'(r) > 0$  for  $r > 1$ ,
- $r = 3/2$  on  $\Omega = \{r \leq 1\}$ ,

and let  $H = \epsilon h(r)$ . We assume  $\epsilon$  is small enough that the only orbits of  $X_H$  are the constant orbits in  $\Omega$ . For a choice of almost complex structure  $J$  as in §2.2.2, the pair  $(H, J)$  is equivariant Borel data (certainly not regular) whose ideal restriction is  $R_{\epsilon t}$ .

We also pick a vector field  $Z$  which agrees with the Liouville vector field  $Z$  in the convex end (i.e., we extend  $Z$  to the compact part) and is a regular equivariant Morse–Borel data as in §3.1.6.<sup>39</sup>

**3.4.2. Reverse PSS cylinders.** Let  $(\psi_{\eta,t}, J)$  be regular Borel data whose ideal restriction is  $R_{\epsilon t}$ . As in §2.2.3, let us denote by  $X_{\eta,t}$  the generator of  $\psi_{\eta,f(t)}$ , where  $f$  is the cut-off function from Figure 3. We may assume, without loss of generality, that  $X_{\eta,t} = f'(t)X_H$  outside  $\Omega$  (i.e., we assume  $\psi_{\eta,t} = R_{\epsilon t}$  outside  $\Omega$ ). We use this  $f(t)$  only to be consistent with the set-up of §2.2.3, rather than for any purpose internal to this section.

Introduce the continuation data:

$$X_{\eta,s,t} := (1 - f(s))f'(t)X_H + f(s)X_{\eta,t} + P_{s,t},$$

for  $s \in [0, 1]$ , extended to the rest of the line by  $s$ -independence, where:

- $P_{s,t}$  is a  $C^\infty$ -small Hamiltonian perturbation, supported in  $\Omega$ .

Introduce the moduli space  $\mathcal{R}$  of *reverse PSS continuation cylinders*:

$$(38) \quad \begin{cases} u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow W, \quad q : (-\infty, 0] \rightarrow W, \quad \text{and } \pi : \mathbb{R} \rightarrow EG, \\ \pi'(s) = -V(\pi(s)), \quad q'(s) = -Z(q(s)), \\ \partial_s u + J(u)(\partial_t u - X_{\pi(s),s,t}(u)) = 0, \\ u(-\infty) = q(0). \end{cases}$$

This solution is asymptotic at the input to a Hamiltonian orbit, and has a flow line of  $-Z$  connected at the output end.

Importantly, because  $X_{\pi(s),s,t}(u)$  is independent of  $s$ , outside of  $\Omega$ , the finite energy solutions obey the necessary a priori energy estimates needed for the usual Floer theory compactness results.

We only consider those solutions  $(u, q, \pi) \in \mathcal{R}$  so that:

- $u$  has finite energy, which implies the left asymptotic is a removable singularity  $u(-\infty) \in \Omega$  (see [FS07, Figure 3]);<sup>40</sup>

<sup>39</sup>We use the symbol  $Z$  rather than  $X$ , as the symbol  $X$  will be used extensively as a Hamiltonian vector field in the sequel.

<sup>40</sup>This point is slightly subtle since  $f'(t)X_H$  has every point in the domain  $\Omega$  as a constant 1-periodic orbit, and has no other orbits. For any sequence  $s_n \rightarrow -\infty$ , there is a subsequence of  $u(s_n, t)$  which converges to a point  $q$ . Then, by some maximum principle, e.g., the one of [AS10, Lemma 7.2], we conclude that  $u$  takes values entirely in  $\Omega$ . It follows that  $u$  is a genuine holomorphic curve on the half-cylinder  $s \leq 0$ ; having established this, the analysis follows the usual PSS arguments of [PSS96].

- $q(-\infty)$  converges to a zero of  $Z$  (i.e., we do not consider solutions where  $q(-\infty)$  drifts off in the non-compact end);

By counting the rigid such solutions in  $\mathcal{R}$  in the same way as in §3.3.6 (for generic choice of the perturbation term  $P_{s,t}$ ), we obtain a map:

$$(39) \quad \text{CF}_{\text{eq}}(\psi_{\eta,t}, J) \rightarrow \text{CM}_{\text{eq}}(Z);$$

consideration of the 1-dimensional part of  $\mathcal{R}$  proves this is a chain map.

**3.4.3. The chain homotopy inverse.** We now explain why (39) is the chain homotopy inverse of the PSS map of §3.3.6.

Recall that, to define the PSS map, one uses the continuation map equation (19) from §2.2.3, as explained in §3.3.3. In the case of the 1-simplex  $R_{est}$ , one needs to pick a path Borel data  $s \mapsto (\psi_{\eta,s,t}, J)$  whose ideal restriction is  $R_{est}$ . As in §3.4.2, we pick  $\psi_{\eta,s,t}$  agreeing with  $R_{est}$  outside of  $\Omega$ .

Unpacking §3.3.3 leads to a family of vector fields  $Y_{\eta,s,t}$  and  $X_{\eta,s,t}$  so that:

- (i)  $Y_{\eta,s,t} = 0$  outside of  $s \in [0, 1]$ ,
- (ii)  $X_{\eta,s,t} = X_{\eta,t}$  for  $s \leq 0$ , where  $X_{\eta,t}$  is as in §3.4.2,
- (iii)  $X_{\eta,s,t} = 0$  for  $s \geq 1$ ,
- (iv) both  $X_{\eta,s,t}, Y_{\eta,s,t}$  are  $\eta$  independent and lie in the line spanned by the Reeb vector field  $R$ , outside of  $\Omega$ ,
- (v) the curvature of the Hamiltonian connection<sup>41</sup> on the complement of  $\Omega$  determined by  $X_{s,t}, Y_{s,t}$  is non-positive.

One counts the rigid solutions of the parametric moduli space where  $\eta$  is constrained to equal  $\pi(s)$  for flow lines  $\pi : \mathbb{R} \rightarrow EG$ , as usual, to define the chain map. This PSS chain map can be post-composed with (39). By standard Floer theory, the composition is chain homotopic to the count of rigid cylinders solving Floer's equation for glued Hamiltonian connections (with flow lines attached at both ends, in the PSS sense).



FIGURE 12. Cartoon of the composition of PSS followed by (39)

The glued Hamiltonian connections are similarly determined by vector fields  $X_{R,\eta,s,t}, Y_{R,\eta,s,t}$ , where  $R$  is a gluing parameter. These glued connections still satisfy (iv) and (v). In addition, they satisfy the property that:

- (vi)  $X_{R,\eta,s,t} = f'(t)X_H$  for  $s \leq -s_R$ , and  $X_{R,\eta,s,t} = 0$  for  $s \geq s_R$ , for some  $s_R$ .

<sup>41</sup>see [BC25, §3.2] for an introduction to Hamiltonian connections

The key idea is that the space of connections on the cylinder (depending on the auxiliary parameter  $\eta$ ) satisfying (iv), (v) and (vi) is a convex space.<sup>42</sup> Thus we can deform the glued data  $X_{R,\eta,s,t}, Y_{R,\eta,s,t}$  in a one parameter family until it agrees with the following:

- $X_{s,t} = (1 - f(s))X_H$  on  $s \in [0, 1]$ , and extended by  $s$ -independence.

Indeed, we can simply travel in a straight line through the space of affine Hamiltonian connections.

The solutions for this final Hamiltonian connection are as follows:

- (1) a cylinder  $u$  solving  $\partial_s u + J(u)(\partial_t u - X_{s,t}(u)) = 0$ ,
- (2) a flow line of  $-Z$  on the interval  $[1, \infty)$  starting at the removable singularity  $u(+\infty)$ ,
- (3) a flow line of  $Z$  on the interval  $(-\infty, 0]$  ending at the removable singularity  $u(-\infty)$ ,
- (4) an underlying flow line of  $\pi : \mathbb{R} \rightarrow EG$ , which does not influence the equation because we use equivariant data.

By the maximum principle, e.g., [AS10, Lemma 7.2],  $u$  must lie entirely in the domain  $\Omega$ , and so must be a holomorphic sphere, and therefore must be constant. Thus all we are doing is counting the flow lines of  $Z$ . The index zero requirement implies the only solutions we will count are the constant solutions, and so the resulting map is the identity map  $\text{CM}_{\text{eq}}(Z) \rightarrow \text{CM}_{\text{eq}}(Z)$ .

Thus, by the usual chain homotopy argument, we conclude the composition of the PSS map with (39) is chain homotopic to the identity map, as desired.

Thus we have proved (39) is the left inverse of the PSS map. A similar argument to the one given in [PSS96, FS07] proves that (39) is also the right inverse. We leave the details of this half of the argument to the reader. This completes the proof of Theorem 1.16.  $\square$

#### 4. Local Floer cohomology as the cone of a simple crossing

*Local Floer cohomology* refers to a homology group associated to a suitable set  $C$  of orbits. What is important is that Floer cylinders which start and end in  $C$  occupy a certain minimal amount of energy (in a way which persists under perturbations). This is the perspective developed in [Flo89]. One often specializes to the cases:

- $C$  is a single isolated fixed point; see [GG10, Zha19, SZ21, She22];
- $C$  is an  $S^1$  family of orbits; see, e.g., [CFHW96, McI12].

In our setting, the families of orbits to which we will develop a local Floer cohomology are associated to crossings with the discriminant.

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<sup>42</sup>this uses (iv) in a crucial way, otherwise having non-positive curvature is a non-linear condition; for related discussion, see [BC25, §3.4.2]

4.1. **Extenders.** To any compact contact manifold  $Y$  we will define a class of “simple” morphisms  $\mathcal{E} \rightarrow \mathcal{C}_1(Y)$  and a map:

$$\mathrm{CF}_{\mathrm{loc}} : \mathcal{E} \rightarrow \mathrm{Ch}(\mathbf{k})$$

by applying local Floer cohomology theory in the symplectization  $SY$ .

Define an *extender*, denoted  $\psi_{s,t} \in \mathcal{E}$ , to be a Hamiltonian isotopy on  $SY$  with  $s, t$ -generating Hamiltonians  $K_{s,t}, H_{s,t}$  satisfying:

- (E1)  $H_{s,t} = H_{0,t}$  and  $K_{s,t} = 0$  on the negative end,
- (E2)  $X_{H_{0,t}}$  is Liouville equivariant everywhere,
- (E3)  $X_{H_{s,t}}$  is Liouville equivariant on the positive end,
- (E4)  $K_{s,1}$  is non-negative,
- (E5)  $X_{H_{1,t}}$  generates a Hamiltonian diffeomorphism with a compact set of fixed points, all of which have negative action,
- (E6) there is at most a single action value attained in (E5).

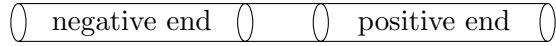


FIGURE 13. Domain of an extender is the symplectization  $SY$  decomposed into a negative end, a positive end, and a compact part in between.

There is an ideal restriction  $\mathcal{E} \rightarrow \mathcal{C}_1(Y)$  given by the flow generated by  $H_{s,t}$  on the positive end.

Given an extender  $\psi_{s,t} \in \mathcal{E}$ , and a subset  $\kappa \in \pi_0(\Lambda Y)$  of free homotopy classes of loops, we will define in §4.4 the *local Floer chain homotopy type*<sup>43</sup>  $\mathrm{CF}_{\mathrm{loc}}(\psi_{s,t}; \kappa)$  for  $H_{1,t}$  following [Flo89]. Then:

**Theorem 4.1.** *Consider the infinity functor constructed in §2:*

$$\mathrm{CF}_{\mathrm{eq}} : \mathcal{C}(Y) \rightarrow \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]]),$$

*assuming the hypotheses of Theorem 1.14. For  $\psi_{s,t} \in \mathcal{E}$  with ideal restriction  $\varphi_{s,t} \in \mathcal{C}_1(Y)$ , the cone of the continuation map associated to  $\varphi_{s,t}$  lies in the chain homotopy class  $\mathrm{CF}_{\mathrm{loc}}(\psi_{s,t}; \kappa)$  where  $\kappa$  is the collection of  $W$ -contractible orbits in  $Y$  (namely, those orbits in  $Y$  which lift to orbits in  $\partial W$  which are contractible in  $W$ ).*

With this result in mind, we say an extender  $\psi_{s,t}$  is *trivial relative  $W$*  if  $X_{H_{1,t}}$  has no  $W$ -contractible 1-periodic orbits.

The proof of Theorem 4.1 is given in §4.4.3. In order to apply the theorem, it is necessary to show that there is an abundance of such extenders:

**Theorem 4.2.** *For any  $\varphi_{s,t} \in \mathcal{C}_1(Y)$  such that  $\varphi_{s,1}$  is strictly positive, there exists an  $n$ -simplex in  $\mathcal{C}_n(Y)$  whose  $[0, n]$  edge is  $\varphi_{s,t}$  and whose  $[i - 1, i]$  edge is one of two types:*

<sup>43</sup>Here a chain homotopy type is a chain complex up to chain homotopy equivalence.

- (i) the ideal restriction of an extender, or,
- (ii) an isomorphism in  $\mathcal{C}_1(Y)$ , i.e., projects to an isomorphism in the homotopy category,

If  $\varphi_{s,1}$  never intersects the  $W$ -contractible discriminant, then one can pick the extenders in (i) to be trivial relative  $W$ , and the continuation map associated to  $\varphi_{s,t}$  is a quasi-isomorphism.

This result together with Theorem 4.1 implies part (b) of Theorem 1.14. It also implies, together with §4.2.6, Theorem 1.18.

**4.2. Decompositions of continuation data.** The goal of this section is to prove Theorem 4.2 on the decomposability of 1-simplices into simpler pieces.

**4.2.1. Decompositions.** We say that a 1-simplex  $\sigma$  is a *composition* of simplices  $\sigma_1 \dots \sigma_n$  provided there is an  $n$ -simplex whose  $[i-1, i]$  edge equals  $\sigma_i$  and whose  $[0, n]$  edge equals  $\sigma$ . Then  $[\sigma]$  is equal to the composition of  $[\sigma_1], \dots, [\sigma_n]$  in the homotopy category  $\mathrm{h}\mathcal{C}$ . We refer to this process as *decomposition* of  $\sigma$ .

Our first observation is rather basic, but useful nonetheless:

**Lemma 4.3.** *Let  $\sigma, \sigma'$  be two 1-simplices with the same source and target. The following are equivalent:*

- (a)  $\sigma_{s,t}$  and  $\sigma'_{s,t}$  are homotopic in the space of paths  $[0, 1] \rightarrow \mathrm{CI}(Y)$ , relative endpoints, and the homotopy can be chosen to be through non-negative paths;
- (b)  $[\sigma'] = [\sigma]$  in the homotopy category  $\mathrm{h}\mathcal{C}(Y)$ ;
- (c)  $\sigma'$  can be decomposed into the sequence  $\sigma_1 = 1, \sigma_2 = \sigma$ ;
- (d)  $\sigma'$  can be decomposed into the sequence  $\sigma_1 = \sigma, \sigma_2 = 1$ ;

here 1 is the degenerate 1-simplex (the identity element).

*Proof.* The equivalence of (b) through (d) can be found in [Lur09, §1.2.3]. The equivalence of (a) and (d) follows from §2.1.3.  $\square$

Mapping cones also interact nicely with decompositions:

**Lemma 4.4.** *Let  $\mathcal{C}$  be an  $\infty$ -category and let  $F : \mathcal{C} \rightarrow \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]])$  be an  $\infty$ -functor. Suppose that  $\sigma$  is the composition of  $\sigma_1, \dots, \sigma_n$  and the homology of the mapping cones of the chain maps  $F(\sigma_i)$  is  $x$ -torsion for each  $i$ . Then the homology of the mapping cone of  $F(\sigma)$  is also  $x$ -torsion.*

*Proof.* It suffices to prove the case  $n = 2$ , by induction. Each of the three vertices of  $\sigma$  is mapped by  $F$  to a chain complex over  $\mathbf{k}[[x]]$ ; let us call these  $C_0, C_1, C_2$ . The three faces are mapped to chain maps  $f, g, h$ . The equation for the  $\infty$ -functor:

$$h - gf = kd + dk$$

where  $k : C_0[1] \rightarrow C_2$  is the map associated to the 2-simplex. Throughout we use gradings modulo two.

Introduce:

$$\Gamma = C_2 \oplus C_0[1] \quad \Delta = C_1 \oplus C_0[1] \oplus C_2 \oplus C_1[1]$$

with morphisms:

$$d_\Gamma = \begin{bmatrix} d & h \\ 0 & d \end{bmatrix} \quad d_\Delta = \begin{bmatrix} d & f & 0 & 1 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & g \\ 0 & 0 & 0 & d \end{bmatrix} \quad p = \begin{bmatrix} g & k & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad i = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & k \\ 0 & f \end{bmatrix},$$

Some care is needed to explain the sign conventions, but the basic rule is that the “shifting” symbol  $[1]$  introduces  $\pm$  signs; we leave this subtle point to the reader and work modulo 2. We claim:

- $d_\Gamma^2 = d_\Delta^2 = 0$ ,
- $p, i$  are chain maps.
- $p, i$  are chain homotopy inverses of each other.

The non-obvious part is that  $ip$  is chain homotopic to 1; for this we have:

$$1 - ip = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & f & 0 & 1 \end{bmatrix} = \begin{bmatrix} d & f & 0 & 1 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & g \\ 0 & 0 & 0 & d \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d & f & 0 & 1 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & g \\ 0 & 0 & 0 & d \end{bmatrix}$$

Thus we have constructed a complex  $\Delta$  which is chain homotopy equivalent to  $\Gamma$  (the cone of  $h$ ). On the other hand,  $\Delta$  has a two term filtration whose associated graded complex is isomorphic to the direct sum of the cone of  $f$  and the cone of  $g$ . Thus, by a simple spectral sequence argument, if the cones of  $f$  and  $g$  are  $x$ -torsion, then so is the cone of  $h$ .  $\square$

Our next lemma gives an explicit formula representing the composition in the case of  $\mathcal{C}(Y)$ :

**Lemma 4.5.** *Let  $\sigma_1, \dots, \sigma_n$  be a composable sequence. Writing  $\sigma_i = \sigma_{i,s,t}$ , the 1-simplex:*

$$(40) \quad (\sigma_{n,s,t} \circ \sigma_{n,0,t}^{-1}) \circ \dots \circ (\sigma_{1,s,t} \circ \sigma_{1,0,t}^{-1}) \circ \sigma_{1,0,t}$$

*represents the composition of  $\sigma_1, \dots, \sigma_n$ .*

*Proof.* It suffices to prove the case  $n = 2$ , as the rest follows by induction. Then the expression reduces to:

$$\sigma_{2,s,t} \circ \sigma_{2,0,t}^{-1} \circ \sigma_{1,s,t}.$$

A two-simplex is a map  $\Psi : [0, 1] \rightarrow \text{CI}(Y)^{[0,2]}$  satisfying certain properties; see Definition 2.5. Write this as  $\Psi_{r,s,t} \in \text{Cont}(Y)$ . In order for  $\Psi$  to witness the stated composition, it is necessary and sufficient that:

- $\Psi_{0,s,t} = \sigma_{1,s,t}$  for  $s \in [0, 1]$ ,
- $\Psi_{0,s,t} = \sigma_{2,s,t}$  for  $s \in [1, 2]$ ,
- $\Psi_{1,s,t} = \sigma_{2,s/2,t} \circ \sigma_{2,0,t}^{-1} \circ \sigma_{1,s/2,t}$  for  $s \in [0, 2]$ .

These can be achieved by the formula:

$$\Psi_{r,s,t} = \sigma_{2,f(r,s),t} \circ \sigma_{2,0,t}^{-1} \circ \sigma_{1,g(r,s),t}$$

provided  $f, g$  are functions defined on  $r \in [0, 1]$  and  $s \in [0, 2]$  satisfying:

- $f(0, s) = \max\{0, s - 1\}$  and  $g(0, s) = \min\{s, 1\}$ ;
- $f(1, s) = g(1, s) = s/2$ .

This can be achieved in a way compatible with the axioms (N1) through (N4) for 2-simplices.<sup>44</sup>  $\square$

**4.2.2. Isomorphisms in the homotopy category.** We characterize those 1-simplices which project to isomorphisms in the homotopy category.

**Lemma 4.6.** *The following are equivalent conditions on a 1-simplex  $\sigma$ :*

- (a)  $[\sigma]$  is an isomorphism in the homotopy category;
- (b)  $\sigma_{s,t}$  satisfies  $\sigma_{s,1} = \sigma_{0,1}$  for all  $s \in [0, 1]$ .

*Proof.* That (b) implies (a) follows from the fact that  $\sigma_{1-s,t}$  remains a well-defined 1-simplex, and that the composition:

$$\sigma_{1-s,t} \circ \sigma_{1,t}^{-1} \circ \sigma_{s,t}$$

is homotopic through non-negative paths to the identity morphism based at  $\sigma_{0,t}$ . The implication that (a) implies (b) is slightly more subtle. Assume that  $\sigma_{s,t}$  is an isomorphism; then there is an inverse map  $\psi_{s,t}$  and the composition:

$$\psi_{s,t} \circ \psi_{0,t}^{-1} \circ \sigma_{s,t}$$

is homotopic to the identity morphism. However, this implies that the non-negative *and non-constant* loop  $\psi_{s,1} \circ \psi_{0,1}^{-1} \circ \sigma_{s,1}$  is contractible through positive loops (that the stated loop is non-constant can be seen by taking the derivative with respect to  $s$ ). But it is well-known that there are no  $C^1$  small positive loops of contactomorphisms on  $Y$ , by the orderability of the zero section in the 1-jet space (see, e.g., [Che96, CS15]).  $\square$

**Remark 4.7.** Interestingly enough, the space of automorphisms of any 0-simplex  $\varphi_t$  is canonically identified with  $\pi_2(\text{Cont}(Y), id)$  via:

$$g_{s,t} \mapsto g_{s,t}\varphi_t,$$

where  $g_{s,t}$  is a map  $[0, 1]^2 \rightarrow \text{Cont}(Y)$  which equals  $id$  on  $\partial[0, 1]^2$ . That this is surjective follows from the above lemma; that this is injective follows from similar arguments on the non-existence of  $C^1$  small non-negative and non-constant loops.

<sup>44</sup>To achieve the piecewise smoothness requirements we use convolution with mollifiers and linear interpolation to construct  $f(r, s), g(r, s)$  for  $r > 0$

**4.2.3. Time reparametrization trick.** In this subsection, we will use the explicit formula for the composition (40) to decompose morphisms into simpler pieces. Let us introduce the following special class of 1-simplices, which have certain benefits vis-à-vis the construction of extenders.

**Definition 4.8.** A 1-simplex  $\sigma_{s,t}$  is said to be of type (M) provided that there is a positive isotopy  $\psi_s$  and a contact isotopy  $\varphi_\tau$  such that:<sup>45</sup>

$$\sigma_{s,t} = \psi_{s\beta(2t-1)} \circ \varphi_{\beta(2t)} = \begin{cases} \varphi_{\beta(2t)} & \text{for } t \leq 1/2 \\ \psi_{s\beta(2t-1)} \circ \varphi_1 & \text{for } t \geq 1/2 \end{cases}$$

If, in addition,  $\psi_s$  is autonomous, we say that  $\sigma_{s,t}$  is of type (AM). We implicitly assume that  $\varphi_1$  and  $\psi_1\varphi_1$  do not lie on the discriminant, in order for  $\sigma_{s,t}$  to be considered as a 1-simplex in  $\mathcal{C}(Y)$ .

Not every morphism  $\sigma$  can be decomposed into morphisms of type (M) or type (AM), because of the assumption that  $\psi$  must be positive. However, if we restrict to those  $\sigma$  which are positive then:

**Lemma 4.9.** Any positive morphism  $\sigma$  can be decomposed into a sequence  $\sigma_1, \sigma_2, \sigma_3$  where  $\sigma_1$  and  $\sigma_3$  are isomorphisms and  $\sigma_2$  is of type (M).

*Proof.* We will show that  $\sigma_{s,t}$  can be homotoped relative the sets  $t = 1$ ,  $t = 0$  to a morphism  $\sigma_2$  of type (M). During this homotopy the restrictions to the lines  $s = 0$  and  $s = 1$  change by a homotopy relative endpoints, and these homotopies will represent the isomorphisms  $\sigma_1, \sigma_3$ .

The stated homotopy can be constructed by a formula of the form:

$$\sigma_{\eta,s,t} = \begin{cases} \sigma_{\eta s, g(\eta,t)} & \text{for } t \leq t_\eta, \\ \sigma_{f(\eta,s,t), 1} & \text{for } t \geq t_\eta, \end{cases}$$

where:

- $t_\eta$  satisfies  $t_1 = 1$  and  $t_0 = 1/2$
- $g(\eta, -)$  maps  $[0, t_\eta]$  onto  $[0, 1]$ ,  $f(\eta, s, -)$  maps  $[t_\eta, 1]$  onto  $[s\eta, 1]$ ,
- $g(0, t) = \beta(2t)$ ,  $f(0, s, t) = s\beta(2t - 1)$
- $g(1, t) = t$  and  $f(1, t) = 1$ .

This can be arranged by an explicit step-by-step construction. Finally, note:

$$\sigma_{s\beta(2t-1), 1} = \sigma_{s\beta(2t-1), 1} \circ \sigma_{0,1}^{-1} \circ \sigma_{0,1},$$

and so we set  $\psi_s = \sigma_{s,1} \circ \sigma_{0,1}^{-1}$  and  $\varphi_\tau = \sigma_{0,\tau}$  to satisfy Definition 4.8.  $\square$

<sup>45</sup>In the following  $\beta$  is a standard cut-off function; one can take, e.g.,  $\beta(x) = f(x)$  for  $x \in [0, 1]$  and  $\beta'(x) = 0$  for  $x \notin [0, 1]$ , where  $f$  is as in Figure 3

4.2.4. **Extender ansatz.** Suppose that  $\sigma_{s,t}$  is of type (M) for  $\varphi_\tau$  and  $\psi_s$  as in Definition 4.8. Then the generator of  $\sigma_{s,t}$  on the symplectization  $SY$  is:

$$\bullet \partial_t \sigma_{s,t} = X'_{s,t} \circ \sigma_{s,t} \text{ and } \lambda(X'_{s,t}) = H'_{s,t};$$

here the ‘‘prime’’ notation signifies that these are Liouville equivariant<sup>46</sup> for all  $s, t$ ; we will deform  $H'_{s,t}$  in the subsequent discussion to construct an extender  $H_{s,t}$  satisfying the axioms in §4.1. We observe:

$$H'_{s,t} = 2\beta'(2t)F_{\beta(2t)} + 2s\beta'(2t-1)S_{s\beta(2t-1)},$$

where  $S_s$  generates  $\psi_s$  and  $F_\tau$  generates  $\varphi_\tau$ .

Define the *extender ansatz* by the formula:

$$(41) \quad H_{s,t} = 2\beta'(2t)F_{\beta(2t)} + 2s\beta'(2t-1)\gamma(S_{s\beta(2t-1)})$$

where  $\gamma$  is the convex cut-off function illustrated in Figure 14, which is required to vanish in a neighborhood of 0.

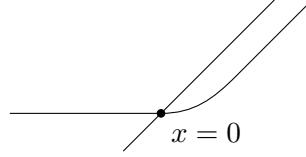


FIGURE 14. Convex cut-off function  $\gamma(x)$  used in the extender ansatz equals 0 in a neighborhood of  $x \leq 0$  and equals  $x - \epsilon$  for  $x \geq 1$ . Assume that  $\gamma''(x) > 0$  if  $\gamma'(x) \in (0, 1)$ .

**Lemma 4.10.** *The isotopy  $\Psi_{s,t}$  generated by (41) satisfies (E1) through (E5), but maybe not axiom (E6) (in other words, we do not show all the orbits of the extender ansatz have the same action).*

*Proof.* Recall that  $K_{s,t}$  is determined from  $H_{s,t}$  by the initial value problem:

$$(42) \quad \begin{cases} \partial_t K_{s,t} + \omega(X_{H_{s,t}}, X_{K_{s,t}}) = \partial_s H_{s,t}, \\ K_{s,0} = 0; \end{cases}$$

see, e.g., [Can23, pp. 16]. It follows that  $K_{s,t} = 0$  on the negative end where  $H_{s,t} = H_{0,t}$ . Thus (E1) holds. The two axioms (E2) and (E3) are clear.

We compute:

$$\partial_s H_{s,t} = 2\beta'(2t-1)[\gamma(S_{s\beta(2t-1)}) + s\gamma'(S_{s\beta(2t-1)})D_{s\beta(2t-1)}\beta(2t-1)],$$

where  $D_s = \partial_s S_s$ . After some manipulation, this yields:

$$\partial_s H_{s,t} = \partial_t(\beta(2t-1)\gamma(S_{s\beta(2t-1)})),$$

<sup>46</sup>Recall that here we mean these objects commute with the Liouville flow on the symplectization  $SY$ ; Liouville equivariant Hamiltonian vector fields are uniquely determined by the ideal restrictions to contact vector fields of  $Y$ .

and thus we can set:

$$K_{s,t} = \beta(2t - 1)\gamma(S_{s\beta(2t-1)})$$

to solve the IVP (42), as  $\omega(X_{K_{s,t}}, X_{H_{s,t}}) = 0$  (this is clear for  $t \leq 1/2$  as  $K_{s,t}$  vanishes there, and it holds for  $t \geq 1/2$  since then both vector fields are proportional to the vector field  $X_{S_s}$ ). Then, by inspection it follows that  $K_{s,1} = \gamma(S_s)$ , i.e., (E4) holds.

The remaining axiom (E5) is concerned with the actions of the orbits of  $X_{H_{1,t}}$ . If  $H_{s,t}$  is given by (41), then we compute:

$$\int_0^{1/2} H_{1,t}(y(t))dt - y^*\lambda = 0,$$

provided that  $y(t) = \Psi_{1,t}(y(0))$ . This is because  $H_{1,t}$  is 1-homogeneous with respect to the Liouville flow, for  $t \leq 1/2$ . Moreover:

$$(43) \quad \int_{1/2}^1 H_{1,t}(y(t))dt - y^*\lambda = \int_0^1 \gamma(S_s) - S_s\gamma'(S_s)ds < 0.$$

where the equality follows from the change of variables for integrals, and the inequality follows from the fact that  $\gamma(S_s) > 0$  must hold somewhere along the orbit  $y(t)$ , otherwise  $\varphi_t$  would lie on the discriminant; convexity then yields the inequality  $\gamma(S_s) - S_s\gamma'(S_s) < 0$ .

As a final remark, we note that compactness of the orbit set follows from the assumption that  $\varphi_1, \psi_1\varphi_1$  do not lie on the discriminant. This completes the proof.  $\square$

**4.2.5. The orbit set in the autonomous case.** Let  $H_{s,t}$  be as in (41). The final axiom (E6) concerns a global property of the orbit set of the system generated by  $H_{1,t}$ , namely, that all orbits have the same action. This property is not automatic, but there is a straightforward criterion in the case the input data was of type (AM), i.e., in the case  $\psi_\sigma$  autonomous.

**Lemma 4.11.** *Suppose that there is a unique  $\sigma_0 \in (0, 1)$  such that  $\psi_{\sigma_0}\varphi_1$  lies on the discriminant, and let  $H_{s,t}$  be the extender ansatz given in (41) for data  $\psi_\sigma, \varphi_\tau$  of type (AM). Then  $H_{s,t}$  satisfies the final axiom (E6).*

*Proof.* The argument is similar to many arguments in symplectic homology theory and its cousin Rabinowitz–Floer homology, in particular, the arguments used to analyze translated points. The key idea is that  $S$  is a positive one-homogeneous Hamiltonian and can be used as a coordinate on the symplectization. We observe:

$$H_{s,t} = 2\beta'(2t)F_{\beta(2t)} + 2s\beta'(2t-1)\gamma(S),$$

which can be solved explicitly:

$$\Psi_{s,t} = \psi_{s\beta(2t-1)\gamma'(S)} \circ \varphi_{\beta(2t)},$$

here  $\psi_{s\beta(2t-1)\gamma'(S)}$  is the diffeomorphism *preserving the level sets of  $S$*  obtained by flowing by  $2s\beta'(2t-1)\gamma'(S)X_S$  over the time interval  $t \in [0, 1]$ . Therefore  $\Psi_{1,1}$  has a fixed point at  $x$  if and only if:

$$(44) \quad \gamma'(S(x)) = \sigma_0.$$

Since  $\gamma$  is convex, and strictly convex when  $\gamma'(S) = \sigma_0$ , it holds that  $S(x)$  is uniquely determined by  $\sigma$  and (44). As in (43), the action at such a fixed point is equal to  $\gamma(S(x)) - S(x)\gamma'(S(x))$ . It follows there is a unique action value, i.e., (E6) follows.  $\square$

**4.2.6. Completing the proof of Theorem 4.2.** Let us describe our strategy for achieving (E6) in general. Fix a positive 1-simplex  $\sigma$ ; the first step is to use Lemma 4.9 to decompose  $\sigma$  into  $\sigma_1, \sigma_2, \sigma_3$  where  $\sigma_1, \sigma_3$  are isomorphisms in  $\mathcal{C}(Y)$  and  $\sigma_2$  is of type (M). For the purposes of proving Theorem 4.2, we can replace  $\sigma$  by  $\sigma_2$ , and just suppose that  $\sigma$  was (M) from the start.

Let us therefore suppose that  $\sigma$  is type (M) for inputs  $\psi_s, \varphi_\tau$ . The second step is the following statement concerning the generic intersections with the discriminant:

**Lemma 4.12.** *Suppose that  $\varphi_1$  and  $\psi_1\varphi_1$  do not have discriminant points. For a generic perturbation of  $\psi_s$  relative its endpoints, we can ensure that:*

$$D = \{(y, s) : \psi_s \circ \varphi_1 \text{ has } y \text{ as a discriminant point}\}$$

*is a finite set, and the projection map  $(y, s) \in D \mapsto s \in (0, 1)$  is injective.*

*Moreover, we can assume that:*

$$(45) \quad \text{im}(d\psi_s \circ d\varphi_1 - 1) \text{ is transverse to } T_s$$

*at all fixed points of  $\psi_s \circ \varphi_1$  in  $SY$ , where  $T_s$  is the vector field generating the positive isotopy  $\psi_s$  (the ideal restriction of  $T_s$  is an  $s$ -dependent Reeb flow on  $Y$ ).*

The proof is given in §4.3. Let us briefly comment on the conclusions:

- discriminant points, when they occur, are isolated in  $Y \times (0, 1)$ ;
- condition (45) says that the subspace  $\text{im}(d\psi_s \circ d\varphi_1 - 1)$  in  $TSY$  has codimension 1 at any lift to  $SY$  of a discriminant point in  $Y$ .

Because positivity of  $\psi_s$  is an open condition, we can assume that  $\psi_s$  and  $\varphi_1$  satisfy the conclusions of Lemma 4.12 by a small perturbation of  $\sigma$ .

The third step is to “chop up” the  $s$ -interval so as to isolate the discriminant points; this is illustrated in Figure 15. To be clear, “chopping up” refers to the process of decomposing  $[0, 1] = [s_0, s_1] \cup [s_1, s_2] \cup \dots [s_{n-1}, s_n]$  with  $0 = s_0 < \dots < s_n = 1$  so that:

- $\psi_{s_j}\varphi_1$  does not have discriminant points for  $j = 0, \dots, n$ .

Then one considers the 1-simplices  $\sigma_{s,t}^j$  given by:

$$(46) \quad \sigma_{j,s,t} = \sigma_{s_{j-1}+s(s_j-s_{j-1}),t} = \psi_{(s_{j-1}+s(s_j-s_{j-1}))\beta(2t-1)} \circ \varphi_{\beta(2t)}.$$

It is clear that  $\sigma$  can be decomposed into  $\sigma_1, \dots, \sigma_n$ . Observe that:

$$\psi_{(s(s_j-s_{j-1})\beta(2t-1)+s_{j-1}\beta(2t-\tau))} \circ \varphi_{\beta(2t)}.$$

is a homotopy relative the boundary of the square, as  $\tau \in [0, 1]$ , which agrees with (46) when  $\tau = 1$ , and which is of type (M) when  $\tau = 0$  with data:

$$\begin{aligned} (c1) \quad \varphi_t^j &= \psi_{s_{j-1}t} \varphi_t, \\ (c2) \quad \psi_s^j &= \psi_{s(s_j-s_{j-1})+s_{j-1}} \circ \psi_{s_{j-1}}^{-1}. \end{aligned}$$

Thus we can replace  $\sigma$  by the type (M) data given by (c1) and (c2). By choosing the partition fine enough, we can suppose that  $\sigma_{s,1}$  has at most one transverse intersection with the discriminant.

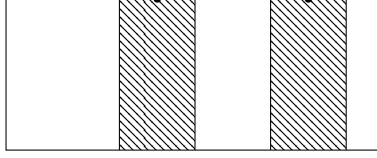


FIGURE 15. Chopping up a 1-simplex into a concatenation so that each piece has either 0 or 1 intersection with the discriminant; the shaded parts have 1 intersection with the discriminant, located at the midpoint.

The idea now is to pick the partition  $s_0 < \dots < s_n$  so that  $s_j - s_{j-1}$  becomes very small. Using this trick, we first determine what happens if the case  $\sigma_{s,1}$  has no intersections with the discriminant:

**Lemma 4.13.** *Suppose that  $\psi_s \varphi_1$  does not have discriminant points, for any value of  $s \in [0, 1]$ . For a sufficiently fine partition  $0 = s_0 < \dots < s_n = 1$ , the extender ansatz from §4.2.4 associated to the data given by (c1) and (c2) satisfies (E6); moreover it has no 1-periodic orbits at all.*

*Relatedly, if  $t \mapsto \psi_{st} \varphi_t$  has no discriminant orbits in  $\kappa \subset \pi_0(\Lambda Y)$ , then the extender ansatz for (c1) and (c2) has no 1-periodic orbits in  $\kappa$  for sufficiently fine partition.*

*Proof.* Unpacking the definitions, we need to show that, for  $s_j - s_{j-1}$  small enough, the system generated by:

$$(47) \quad H_{1,t} = 2\beta'(2t)F_{\beta(2t)}^j + 2\beta'(2t-1)\gamma(S_{\beta(2t-1)}^j)$$

has no 1-periodic orbits, where:

- $F_t^j$  generates  $\psi_{s_{j-1}t} \varphi_t$ , and,
- $S_s^j$  generates  $\psi_{s(s_j-s_{j-1})+s_{j-1}} \circ \psi_{s_{j-1}}^{-1}$ .

In particular,  $S_s^j$  is proportional to  $s_j - s_{j-1}$  and hence converges in the  $C^\infty$  topology to the zero function, as the partition gap tends to zero. The argument then follows from a standard compactness argument: since  $F_t^j$  generates  $\psi_{s_{j-1}t}\varphi_t$ , the system generated by (47) has a time-1 map which is close to  $\psi_{s_{j-1}}\varphi_1$ ; since the latter has no orbits (uniformly in  $s_{j-1} \in [0, 1]$ ), and this is an open condition on the generating Hamiltonian, we conclude that (47) defines a trivial extender, provided the partition is chosen small enough.

The same argument applies verbatim to the modified statement when the free homotopy class  $\kappa$  is included.  $\square$

**Lemma 4.14.** *Suppose  $\psi_s\varphi_1$  has exactly one discriminant point at  $s = s_0$ , and the intersection is transverse in the sense of (45). After localizing to a sufficiently small interval around  $s = s_0$ , the extender ansatz satisfies (E6).*

*Proof.* We will prove that, after a sufficiently small localization, the extender ansatz has a single 1-periodic orbit. Evidently, this implies there is a single action value.

Let us consider a sequence of localizations to the interval  $[s_0 - \epsilon_n, s_0 + \epsilon_n]$  of the extender ansatz:

$$H_{1,t}^n = 2\beta'(2t)F_{\beta(2t)}^n + 2\beta'(2t-1)\gamma(S_{\beta(2t-1)}^n),$$

where  $F_t^n$  generates  $\psi_{(s_0-\epsilon_n)t}\varphi_t$ , and  $S_s^n$  generates  $\psi_{s2\epsilon_n+s_0-\epsilon_n} \circ \psi_{s_0-\epsilon_n}^{-1}$ .

Following the same argument used in the proof of Lemma 4.13,  $H_{1,t}^n$  generates a system  $\xi_{n,t}$  whose time-1 map  $\xi_{n,1}$  is  $C^\infty$  close to  $\psi_{s_0}\varphi_1$ . In particular, if  $\xi_{n,1}$  has a fixed point  $z_n$ , then  $z_n$  must converge to some point on the Liouville flow line  $Z_{\mathbb{R}}(y)$  through the unique discriminant point  $y$  of  $\psi_{s_0}\varphi_1$ .

The time-1 map of  $\xi_{n,t}$  can be written as a composition of two maps:

- $\psi_{s_0-\epsilon_n}\varphi_1$ ,
- the isotopy generated by  $V_{n,t} = 2\epsilon_n\gamma'(S_t^n)T_{2\epsilon_nt+s_0-\epsilon_n}$ .

Since  $S_t^n \approx 2\epsilon_nr$ , where  $r$  is the Hamiltonian generating  $T_{s_0}$ , we conjugate with the Liouville flow  $Z_{-\log(2\epsilon_n)}$  so that  $2\epsilon_nr$  converges to  $r$ ; then the second part is approximately the flow by  $2\epsilon_n\gamma'(r)T_{s_0}$ , with error given by terms which are of order  $\epsilon_n^2$ .

Similarly, up to terms of order  $\epsilon_n^2$ , the first part can be replaced by  $\psi_{s_0}\varphi_1$  followed by the flow of  $-\epsilon_nT_{s_0}$  for time 1. Thus, in this analysis to first order, we have that  $\xi_{n,1}$  is approximately:

- $\psi_{s_0}\varphi_1$ , followed by,
- the time 1 flow of  $(2\gamma'(r) - 1)\epsilon_nT_{s_0}$ .

There is a unique fixed point located at  $\gamma'(r) = 1/2$  on the Liouville flow line  $Z_{\mathbb{R}}(y)$ . This is non-degenerate, by our transversality assumptions. This non-degeneracy ensures that the analysis up to first order in  $\epsilon_n$  is stable

under small perturbations of order  $\epsilon_n^2$ ; a more precise argument uses the Banach fixed point theorem to prove the existence of a unique fixed point. Thus  $\xi_{n,1}$  has a unique fixed point for  $n$  sufficiently large, as desired.  $\square$

**4.3. On the generic intersections with the discriminant.** The discriminant may be highly singular, but it has a canonical resolution as a smooth (Fréchet) manifold, as follows. Consider the graph map:

$$\Gamma : \text{Cont}(Y) \times SY \rightarrow SY \times SY$$

sending  $(\phi, y)$  to  $(y, \phi(y))$ ; this map is submersive (by the abundance of contact Hamiltonians), and the inverse image of the diagonal yields a Fréchet manifold  $\Gamma^{-1}(\Delta)$ . The projection (forgetting the  $SY$  factor) yields a continuous surjection:

$$(48) \quad \Gamma^{-1}(\Delta)/\mathbb{R} \rightarrow \text{the discriminant}$$

which is the aforementioned resolution; where the quotient corresponds to the  $\mathbb{R}$ -action by the Liouville flow on  $SY$ . This suggests the following characterization<sup>47</sup> of when a finite-dimensional family of contactomorphisms should be called transverse to the discriminant:

**Definition 4.15.** *Let  $P$  be a finite dimensional smooth manifold. We say that a smooth map  $f : P \rightarrow \text{Cont}(Y)$  is transverse to the discriminant if:*

$$\Gamma_f : (p, y) \in P \times SY \mapsto \Gamma(f(p), y) \in SY \times SY$$

*is transverse to the diagonal  $\Delta$ . One can also speak of maps which are transverse along some closed subset  $A \subset P$ .*

**Proposition 4.16.** *Let  $P$  be a smooth, finite dimensional, compact manifold, and  $f : P \rightarrow \text{Cont}(Y)$  a smooth map. Suppose  $f$  is transverse to the discriminant along some closed, possibly empty, subset  $A \subset P$ . Then  $f$  is homotopic relative  $A$  to a smooth map which is everywhere transverse to the discriminant and arbitrarily  $C^\infty$  close to  $f$ .*

*Proof.* This follows from the abundance of contact Hamiltonians and the Sard–Smale theorem. The details are left to the reader.  $\square$

By counting dimensions, one can also bound the number of discriminant points which appear at any given moment in generic 1-parameter families of contactomorphisms:

**Proposition 4.17.** *Suppose that  $P$  is a one-dimensional manifold and consider a smooth map  $f : P \rightarrow \text{Cont}(Y)$  which is transverse to the discriminant. After a generic perturbation, there is at most one discriminant point<sup>48</sup> which occurs at any given point  $p$ .*

<sup>47</sup>A toy model for discriminant points are critical points of a smooth function with critical value zero.

<sup>48</sup>In fact, if  $P$  has dimension  $d$ , then there are at most  $d - 1$  many discriminant points which occur above any given point  $p \in P$ , for generic  $f$ . We will not use this fact.

*Proof.* The subset  $D = \Gamma_f^{-1}(\Delta)/\mathbb{R}$  is a zero dimensional submanifold of the total space  $P \times Y$ . Thus, for any given  $p \in P$ , there are a finite number of discriminant points, say  $y_1, \dots, y_N$ . One perturbs  $f$  in a neighborhood of each point  $y_i$  in order to slightly move its basepoint  $p$ . The technical reason this is possible is that Sard Smale argument establishes the universal moduli space of all pairs  $(f, p, y)$  such that  $y$  is a discriminant point of  $f(p)$  has a submersive projection  $(f, p, y) \mapsto p$ , (again by the abundance of contact Hamiltonians).  $\square$

We relate this notion of transversality to Lemma 4.12.

**Proposition 4.18.** *Suppose that  $s \in [0, 1] \mapsto \psi_s \circ \varphi_1$  is transverse to the discriminant, where  $\psi_s$  is positive isotopy. Then, for each  $s$  such that  $\psi_s \circ \varphi_1$  lies on the discriminant, it holds that  $\text{im}(\text{d}\psi_s \circ \text{d}\varphi_1 - 1)$  is transverse to  $T_s$  in  $SY$ , where  $T_s$  is the generator of  $\psi_s$ .*

*Proof.* By the above discussion, weak transversality ensures:

$$(s, y) \in [0, 1] \times SY \mapsto (y, \psi_s(\varphi_1(y)))$$

is transverse to the diagonal. Differentiating with respect to  $y$  and  $s$  at a fixed point yields the subspace of tangent vectors of the form

$$v \oplus (\text{d}\psi_s \circ \text{d}\varphi_1(v) + T_s(y)).$$

If this is transverse to the diagonal, then we can solve:

$$u \oplus u + v \oplus (\text{d}\psi_s \circ \text{d}\varphi_1(v) + T_s(y)) = 0 \oplus w$$

for any  $w$ . The only option is  $u = -v$ , so:

$$-v + \text{d}\psi_s \circ \text{d}\varphi_1(v) + T_s(y) = w$$

can be solved for any  $w$ , for some  $v$ . This yields the desired result.  $\square$

Combining Propositions 4.16, 4.17, and 4.18 yields Lemma 4.12.  $\square$

**4.4. The local Floer cohomology of an extender.** In this section we develop the local Floer cohomology of an extender.

**4.4.1. Admissible perturbations.** Let  $\psi_{s,t}$  be an extender, as in §4.1. Fix a Liouville equivariant  $\omega$ -tame almost complex structure  $J$  on  $SY$ .

An  $\epsilon$ -admissible perturbation for  $\psi_{s,t}, J$  is a family  $\delta_{s,t}$  of Hamiltonian diffeomorphisms such that:

- $\delta_{s,0} = \delta_{0,t} = \text{id}$ , for all  $s, t$ ,
- $\delta_{s,t}$  is compactly supported, uniformly in  $s, t$ ,
- $\psi_{1,t}\delta_{1,t}$  is a non-degenerate Hamiltonian system, for which all Floer differential cylinders are cut transversally,
- the Hamiltonian generator of  $t \mapsto \delta_{s,t}$ , and its first derivative with respect to  $s$ , are bounded in absolute value by  $\epsilon$ .

As in the statement of Theorem 4.1, we let  $\kappa \subset \pi_0(\Lambda Y)$  be a collection of free homotopy classes of loops.

**Lemma 4.19.** *If  $\epsilon$  is small enough, then the Floer differential on:*

$$\text{CF}(\psi_{1,t}\delta_{1,t}, J, \kappa)$$

*squares to zero, and the chain homotopy type of the resulting complex is independent of the choice of  $J$  or the  $\epsilon$ -admissible perturbation  $\delta_{s,t}$ .*

*Proof.* The Floer differential is defined as usual (as a sum of the index 1 Floer differential cylinders, recording the input and output orbits, and the orientation line, as an endomorphism). To prove it squares to zero, we need only prove the required compactness result:

- *any sequence of Floer cylinders  $u_n$  remains in a compact subset of the symplectization  $SY$ .*

The other necessary a priori estimates on the derivatives of  $u_n$ , say, follow from the standard machinery once the above  $C^0$  bound is proved.

To analyze this compactness problem, we introduce the symplectization coordinate  $r$ . We assume that  $\psi_{1,t}$  is Liouville equivariant, and  $\delta_{s,t} = 0$ , on the ends  $r \leq 0$  and  $r \geq r_0$ , for some  $r_0 > 0$ , (positioning the left end at  $r \leq 0$  is without any loss of generality, as we can translate by the Liouville flow).

Then, using the mean-value property for energy density as in [RS01, Appendix B], we conclude that the  $C^0$  sizes of  $\partial_s u_n$  and  $\partial_s u_n - X_t(u_n)$  must be small; here  $X_t$  is the Hamiltonian vector field of  $\psi_{1,t}\delta_{1,t}$ . This is because the energy of Floer cylinders can be made arbitrarily small (by shrinking  $\epsilon$ ).

In particular, if  $u_n$  fails to remain bounded, then one can find numbers  $s_n$  so that subcylinders  $u_n([s_n - 1, s_n + 1] \times \mathbb{R}/\mathbb{Z})$  are mapped into the region  $r \leq 0$  (negative end), or the region  $r \geq r_0$  (positive end). In either case, we obtain a contradiction (for  $\epsilon$  small enough), because in these ends the Hamiltonian system  $\psi_{1,t}\delta_{1,t} = \psi_{1,t}$  has no orbits, so these subcylinders occupy a minimum quantum of energy  $\hbar$  (by a standard argument, e.g., [Sal97] or [BC24, Proposition 2.2.]). The contradiction is ensured by picking  $\epsilon$  small enough.

A similar argument works for proving a priori  $C^0$ -bounds on continuation cylinders relating two choices  $J, \delta_{s,t}$  and  $J', \delta'_{s,t}$ , if both are  $\epsilon$ -admissible and  $\epsilon$  is small enough. In this fashion we prove that the resulting chain complexes are chain homotopy equivalent (the quasi-isomorphisms are continuation maps, as in [HS95, Sal97]).  $\square$

By virtue of the preceding result, the following is well-defined:

**Definition 4.20.** *For any extender  $\psi_{s,t}$  and any  $\kappa \subset \pi_0(\Lambda Y)$ , we define  $\text{CF}_{\text{loc}}(\psi_{s,t}, \kappa)$  to be the chain homotopy type of  $\text{CF}(\psi_{1,t}\delta_{1,t}, J, \kappa)$  for any  $J$  and any  $\epsilon$ -admissible  $\delta_{s,t}$ , for  $\epsilon$  small enough.*

**4.4.2. Insertion of an extender into the filling.** Let us present the  $G$ -filling  $W$  of  $Y$  as a compact aspherical domain  $\Omega$  with contact type boundary, with the positive half of  $S\partial W$  attached as the convex end. This presentation specifies a radial coordinate  $r$  so that  $\partial\Omega = \{r = 1\}$ . We fix the geometric set-up as shown in Figure 16.

Pick an extender  $\psi_{s,t}$  whose ideal restriction is the 1-simplex  $\varphi_{s,t}$ , and pick an  $\epsilon$ -admissible perturbation  $\delta_{s,t}$ .

Fix regular Borel data  $\psi_{\eta,0,t}$  whose ideal restriction is  $\varphi_{0,t}$ . We can suppose that the  $t$ -generator  $X_{\eta,0,t}$  of  $\psi_{\eta,0,t}$  is Liouville equivariant outside of  $\Omega$ . We can also suppose (by appropriate choice of the initial radial coordinate) that all orbits of  $X_{v,0,t}$  remain entirely in  $\Omega$ , when  $v$  is any critical point on  $EG$ .

Introduce  $X_{s,t}$  as the  $t$ -generator of the perturbed extender  $\psi_{s,t}\delta_{s,t}$ ; we suppose that  $X_{s,t} = X_{0,t}$  holds on  $r \leq 1$ ,  $X_{s,t} = X_{1,t}$  holds on  $r \geq r_0$ , and that  $\delta_{s,t} = id$  holds for  $r$  outside  $[1, r_0]$ . This  $X_{s,t}$  lifts from  $SY$  to  $S\partial W$ , as a family of  $G$ -invariant vector fields.

This set-up ensures that the piecewise definition:

$$(49) \quad X_{\eta,s,t} = \begin{cases} X_{\eta,0,t} & \text{inside } \Omega, \\ dZ_L \circ X_{s,t} \circ Z_{-L} & \text{for } r \geq 1, \end{cases}$$

is smooth and integrates to a path of Borel data whose ideal restriction is  $\varphi_{s,t}$  (throughout we always use the same  $J$  in our Borel data). In other words, we conjugate the extender  $\psi_{s,t}$  and the  $\epsilon$ -admissible perturbation by Liouville flow. This has the effect of exponentially scaling the actions of the orbits arising from the extender; importantly, orbits with negative action will have a *very* negative action when we increase  $L$ .

**Claim 4.21.** *A generic choice of  $X_{\eta,0,t}$  and  $\delta_{s,t}$  ensures that (49) is a regular path of Borel data.*

*Proof.* The engine for ensuring regularity is the usual Sard-Smale argument of [FHS95, MS12].

Any continuation cylinder (or Floer differential cylinder) which intersects  $\Omega$  can be assumed to be regular by the generic choice of  $X_{\eta,0,t}$ . The other cylinders are contained entirely in the symplectization end; since the action is free on this part, the cylinders project to a cylinder in  $SY$  for data determined by  $X_{s,t}$ . One ensures regularity for the projected cylinders by generic variation of  $\delta_{s,t}$  (and the Sard-Smale theorem). The regularity of the original cylinders (before projection) follows since the projection map is a covering map (variations of the cylinder before projection correspond to variations of the cylinder after projection).  $\square$

To summarize, we have used the perturbed extender to build an explicit regular path of Borel data whose ideal restriction is the 1-simplex under consideration. The construction depends on the extender  $\psi_{s,t}$ , the initial

Borel datum  $X_{\eta,0,t}$ , the parameter  $L$ , and the choice of perturbation term  $\delta_{s,t}$ .

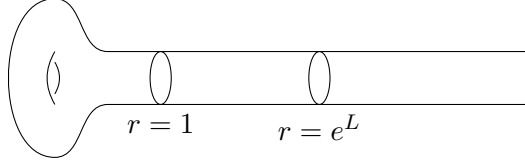


FIGURE 16. Regions in the  $G$ -filling  $W$  of  $Y$ . The length parameter  $L$  should be considered as quite large.

**4.4.3. Proof of Theorem 4.1.** In this section we prove that the cone of the morphism  $\text{CF}_{\text{eq}}(\varphi_{0,t}) \rightarrow \text{CF}_{\text{eq}}(\varphi_{1,t})$  associated to the 1-simplex  $\varphi_{s,t}$  arising as the ideal restriction of the extender  $\psi_{s,t}$  lies in the local Floer cohomology chain homotopy class  $\text{CF}_{\text{loc}}(\psi_{s,t}, \kappa)$  from Definition 4.20, where  $\kappa$  is the collection of  $W$ -contractible orbits.

We fix the geometric set-up from §4.4.2, yielding the regular path of Borel data  $X_{\eta,s,t}$ . We will now analyze the cone of the continuation map associated to this regular path of Borel data.

The equivariant chain complex for the Borel data determined by  $X_{\eta,1,t}$  is generated as a  $\mathbf{k}[[x]]$ -module by summands  $\mathfrak{o}(\gamma) \otimes \mathfrak{o}(v)$  where  $v$  is a critical point of the pseudogradient on  $BG$  lying above the pole  $[1 : 0 : \dots]$ , and  $\gamma$  is a contractible orbit of  $X_{v,1,t}$ , as we have explained already in §2.2.4, §2.3, §2.4.11. These summands come in two types:

- (a) summands where  $\gamma$  is contained in  $\Omega$ , and is an orbit of  $X_{0,1,t}$ ,
- (b) summands where  $\gamma$  passes through  $r \in [1, r_0]$ , and projects to a  $W$ -contractible orbit for the vector field  $X_{1,t}$  contributing to the local Floer cohomology of the extender  $\psi_{s,t}$  and the  $\epsilon$ -admissible perturbation  $\delta_{s,t}$ .

On the other hand, the equivariant chain complex for the Borel data determined by  $X_{\eta,0,t}$  is generated as a  $\mathbf{k}[[x]]$ -module by only those summands of type (a). The summands of type (a) for  $\text{CF}_{\text{eq}}(X_{\eta,0,t})$  and  $\text{CF}_{\text{eq}}(X_{\eta,1,t})$  are literally the same. In other words, there is a decomposition:

$$(50) \quad \text{CF}_{\text{eq}}(X_{\eta,1,t}) = \text{CF}_{\text{eq}}(X_{\eta,0,t}) \oplus E,$$

where  $E$  is the  $\mathbf{k}[[x]]$ -module generated by the summands of type (b).

**Claim 4.22.** *With respect to the direct sum decomposition (50), it holds that:*

$$d_{\text{eq}} = \begin{bmatrix} d_{\text{eq}} & \Delta \\ 0 & d_E \end{bmatrix} \quad \mathfrak{c} = \begin{bmatrix} id \\ 0 \end{bmatrix},$$

where  $d_{\text{eq}}$  is the differential on  $\text{CF}_{\text{eq}}(X_{\eta,1,t})$  and  $\mathfrak{c}$  is the continuation map. Moreover, the term  $d_E$  only counts cylinders which remain outside of  $\Omega$ .

*In the deduction, we allow the operations of increasing the length parameter  $L$  in §4.4.2, and picking the perturbation term  $\delta_{s,t}$  smaller, if necessary.*

*Proof.* By (E5), and the operation of increasing  $L$  which scales actions exponentially, we may assume that all of the orbits in summand (b) have actions which are much more negative than the actions of the orbits in summand (a). Even taking into account the curvature of Hamiltonian connections arising in the definition of the equivariant differential, we can then preclude the existence of cylinders with input in (a) and output in (b), in the definition of  $d_{eq}$ . This proves the first part.

Furthermore, by subsequently picking  $\delta_{s,t}$  small enough, and using (E6), we may suppose that all orbits contributing to summand (b) have nearby actions; then, by the same energy estimate used in Lemma 4.19 we conclude that all cylinders contributing to  $d_E$  remain in the complement of  $\Omega$ .

For the conclusion about the continuation map  $\mathfrak{c}$ , we need to be careful that this “increasing  $L$ ” operation does not simultaneously increase the curvature of the Hamiltonian connection used to define  $\mathfrak{c}$  — indeed, since the curvature is non-zero in the region where the extender is supported, there is a risk that we also scale the curvature exponentially when we conjugate by the Liouville flow. However, this is where we use assumption (E4); since  $K_{s,1}$  is non-negative, it follows, from the same arguments used in the cited references in Remark 2.13 (which we used to get the a priori energy estimates on the moduli spaces of §2.2.3), that the curvature of the Hamiltonian connection used to define  $\mathfrak{c}$  is non-positive everywhere on the region  $r \geq 1$ , and so exponential scaling will not introduce arbitrarily positive curvature.

To conclude that the matrix entry in  $\mathfrak{c}$  acting on the summands of type (a) is the identity, we use the standard fact (used in all Floer theoretic works concerning continuation maps) that continuation cylinders for  $s$ -independent continuation data which contribute to  $\mathfrak{c}$  are only the stationary cylinders.  $\square$

**Corollary 4.23.** *Assume the set-up, conclusion, and notation of Claim 4.22. The cone of the continuation map  $\mathfrak{c}$  is in the chain homotopy class of  $(E, d_E)$ .*

*Proof.* This is standard homological algebra; the projection map:

$$C \oplus E \oplus C[1] \rightarrow E$$

is the desired chain homotopy equivalence, when the domain is given the cone differential (with sign conventions due to supergradings left to the reader):

$$\begin{bmatrix} d_C & \Delta & id \\ 0 & d_E & 0 \\ 0 & 0 & d_C \end{bmatrix}.$$

The inverse map sends  $e \in E$  to  $0 \oplus e \oplus \Delta e$ .  $\square$

To complete the proof of Theorem 4.1, we prove:

**Claim 4.24.** *Assume the set-up, conclusion, and notation of Claim 4.22. The complex  $(E, d_E)$  is in the chain homotopy class of  $\text{CF}_{\text{loc}}(\psi_{s,t}, \kappa)$ .*

*Proof.* This claim should be understood as the well-known ‘‘Cartan isomorphism’’ in equivariant cohomology, which asserts that the  $G$ -equivariant cohomology of a space with a free  $G$ -action is isomorphic to the ordinary cohomology of the quotient space. Versions of the Cartan isomorphism in equivariant Floer theory have appeared before, see, e.g., [SC25].

For the Borel equivariant cohomology of spaces, the idea is the following, if a space  $M$  admits a free action, then:

$$M \times_G EG \rightarrow M/G,$$

sending  $[(m, \eta)]$  to  $[m]$  is a fibration with contractible fibers, and hence is a homotopy equivalence. The task of the present claim is to translate this into Floer theoretic language.

Let us identify  $E$  with the direct sum of generators:

$$(51) \quad E = \bigoplus \mathfrak{o}(\zeta) \otimes \mathfrak{o}(\eta) \otimes \mathbf{k} \simeq \text{CF}(X_{1,t}) \otimes \text{CM}(EG)$$

where  $\eta$  is a critical point of the pseudogradient on  $EG$  and  $\zeta$  is a  $W$ -contractible orbit in  $SY$ . This is a small sleight of hand: this complex is identified with the usual complex by the identification:

$$\mathfrak{o}(\zeta) \otimes \mathfrak{o}(x^k v_{i,g}) = x^k \mathfrak{o}(g\gamma_\zeta) \otimes \mathfrak{o}(v_i),$$

where we pick some distinguished lift  $\gamma_\zeta$  to  $S\partial W$  of each  $W$ -contractible orbit  $\zeta$ ; here  $v_{i,g}$  is the lift of  $v_i$  as indicated in Figure 7. That this identification is well-defined uses the freeness of the action on  $S\partial W$ .

Because the data is equivariant in the region  $r \geq 1$ , the trajectories  $(\pi, u)$  contributing to  $d_E$  project to Floer cylinders in  $SY$  and come in two types:

- $\pi$  has index 1 and  $u$  has index 0,
- $\pi$  has index 0 and  $u$  has index 1.

It follows that, with respect to the decomposition (51), the differential is a tensor product differential on  $\text{CF}(X_{1,t}) \otimes \text{CM}(EG)$ .

Let  $1$  denote the sum of the local minima in  $\text{CM}(EG)$  (in the notation of Figure 7, this element  $1$  is the sum  $v_{0,g}$  as  $g$  ranges over all elements of  $G$ ), and consider the chain map:

$$a \in \text{CF}(X_{1,t}) \mapsto a \otimes 1 \in \text{CF}(X_{1,t}) \otimes \text{CM}(EG).$$

The contractibility of  $EG$  and standard Morse theoretic arguments imply that this is a chain homotopy equivalence, as desired.  $\square$

**4.4.4. Proof of Theorem 1.18.** The first part of the theorem follows from Lemma 4.13 and Theorem 4.1; briefly, one decomposes the 1-simplex into a composition of 1-simplices associated to extenders which have zero local Floer cohomology in the free homotopy class of  $W$ -contractible orbits.

The second part follows from a soft argument involving approximating paths by zig-zags of positive and negative paths; if we assume the original path remains in the complement of the  $W$ -contractible discriminant, then we can pick the approximation so that each segment of the zig-zag is also in the complement of the  $W$ -contractible discriminant.  $\square$

## 5. Tying up loose ends

In this section, we prove various technical lemmas needed to complete the proofs of the main applications stated in §1.2. Throughout we assume that  $Y$  admits a  $G$ -filling  $W$ , with  $G = \mathbb{Z}/p\mathbb{Z}$ , as in §1.5, and that the  $G$ -action has at least one fixed point  $q_0$ . At this stage, we have completed the proof of our main structural theorems Theorems 1.14, 1.16, and 1.18.

**5.1. The unit element.** We prove a few lemmas about unit elements. Let us briefly recall the construction; in  $\text{CM}_{\text{eq}}(X_\eta)$ , for any Morse–Borel datum  $X_\eta$  on  $W$ , there is a well-defined cycle  $1$  by summing up all local minima. Then, for any 1-simplex  $\varphi_{s,t}$  in the PSS category  $\mathcal{P}(Y)$  starting at  $id$  and ending at  $\varphi_{1,t} \in \mathcal{C}(Y)$ , one has a chain map:

$$\text{PSS} : \text{CM}_{\text{eq}}(X_\eta) \rightarrow \text{CF}_{\text{eq}}(\varphi_{1,t}).$$

This gives a class  $1(\varphi_{s,t}) \in \text{HF}_{\text{eq}}(\varphi_{1,t})$ , and pushing forward by the colimit map  $\text{HF}_{\text{eq}}(\varphi_{1,t}) \rightarrow \text{SH}_{\text{eq}}(W)$  gives a class  $1(\varphi_{s,t}) \in \text{SH}_{\text{eq}}$ .

**5.1.1. The unit element is well-defined.** We prove Lemma 1.17 stating that the unit element  $1(\varphi_{s,t}) \in \text{SH}_{\text{eq}}(W)$  is well-defined independently of the 1-simplex  $\varphi_{s,t}$  chosen.

*Proof of Lemma 1.17.* The key idea is the following: for any 1-simplex  $\varphi_{s,t}$  in  $\mathcal{P}(Y)$  starting at  $id$ , there is a 2-simplex whose 01 face is  $\varphi_{s,t}$  and whose 02 face is a Reeb flow  $R_{ast}$  for all sufficiently large speeds  $a > 0$ .

Indeed, if  $R_{ast}\varphi_{s,t}^{-1}\varphi_{1,t}$  is positive (which it certainly is for  $a$  large enough), then we can simply use Lemma 4.5.

The result then follows, since the axioms of an  $\infty$ -functor:

$$\mathcal{P}(Y) \rightarrow \text{N}_{\text{dg}}\text{Ch}(\mathbf{k}[[x]])$$

imply  $1(R_{ast}) = \mathbf{c}(1(\varphi_{s,t}))$ , where  $\mathbf{c}$  is the chain map associated to the 12 face. Passing to the colimit, we conclude  $1(\varphi_{s,t}) = 1(R_{ast})$ , for all sufficiently large  $a$ , for any 1-simplex  $\varphi_{s,t}$ , and the desired result follows.  $\square$

5.1.2. **The unit element is not eternal.** We prove the assertion:

**Lemma 5.1.** *If  $\mathrm{SH}_{\mathrm{eq}}(W) \neq 0$ , the unit 1 does not lie in the image of:*

$$\mathrm{HF}_{\mathrm{eq}}(R_{-ct}) \rightarrow \mathrm{SH}_{\mathrm{eq}}(W)$$

for any negative Reeb flow  $R_{-ct}$ .

*Proof.* This is related to the results of [Rit13, Rit14, CHK23, Can24b, DUZ25] on non-existence of eternal classes in the symplectic cohomology of Liouville manifolds.

Let us fix an equivariant Morse–Borel datum  $X$  on  $W$  as in §3.1.6, and consider the chain complexes:

- $\mathrm{CM}_{\mathrm{eq}}(X)$ ,
- $\mathrm{CM}_{\mathrm{neq}}(X) = \mathrm{CM}_{\mathrm{eq}}(X) \otimes_{\mathbf{k}[[x]]} \mathbf{k}$  (the non-equivariant quotient).

Since  $\mathrm{HM}_{\mathrm{eq}}(X)$  is a finitely generated  $\mathbf{k}[[x]]$ -module, it decomposes into a free summand and a torsion summand (by classification of modules over a principal ideal domain). Thus it makes sense to speak of the *free part* of the unit element 1.

**Claim 5.2.** *The free part of the unit element cannot be written as  $xa$  for any element  $a$  in the free summand.*

*Proof of Claim 5.2.* This follows from a similar argument to the one used in the localization statement §3.2.1; one can reduce the general case to the case  $W = \mathrm{pt}$ , by “localizing” at a local minimum  $q_0$  on the fixed point submanifold. Here we need to again appeal to the fact that the characteristic of the coefficient field  $\mathbf{k}$  matches the order of the prime cyclic group  $G$ , to conclude that the free part of the unit cannot be written as  $xa$  for any element  $a$  in the free summand of  $\mathrm{HM}_{\mathrm{eq}}^*(\mathrm{pt})$ , since in this case:

$$(52) \quad \mathrm{HM}_{\mathrm{eq}}^*(\mathrm{pt}) \simeq \mathbf{k}[[x]]1 \oplus \mathbf{k}[[x]]\theta,$$

where  $\theta$  is an element of degree 1 (here we assume that  $p \geq 3$ , for simplicity, the result is similar and easier in the case  $p = 2$ ).  $\square$

Now, following the same argument as the one given in [CHK23, §2.3.3], we conclude that, if 1 lies in the image of  $\mathrm{HF}_{\mathrm{eq}}(R_{-ct}^\alpha) \rightarrow \mathrm{SH}_{\mathrm{eq}}$ , then there would be some element  $b \in \mathrm{HM}_{\mathrm{eq}}(W)$  so that:

- (a)  $1 - b$  is mapped (via PSS and continuation) to zero in  $\mathrm{SH}_{\mathrm{eq}}$ ,
- (b)  $b$  is represented by a cycle lying in the image of the Morse continuation map  $\mathrm{CM}_{\mathrm{eq}}(-X) \rightarrow \mathrm{CM}_{\mathrm{eq}}(X)$ ;

here  $-X$  points inwards, but otherwise the construction of the chain complex is exactly the same as §3.1; see [CHK23, Theorem 2.6] for details.

Observe that  $(1 - b)^{kp} = 1 - b^{kp}$  for any number  $k$  (where  $p$  is the prime characteristic), and so that (a) and (b) still hold, with  $b$  replaced by  $b^{kp}$ .

Here we use the product structure on the equivariant cohomology  $\mathrm{HM}_{\mathrm{eq}}(W)$ ; we do not need to assume that this product structure respects the  $\mathbf{k}[[x]]$ -module structure, and so we can use the soft construction in §5.2 (below).

By taking  $k$  large enough, we can then assume  $b^{kp}$  is mapped to zero in the non-equivariant group  $\mathrm{HM}_{\mathrm{neq}}(W)$  (the homology of the complex  $\mathrm{CM}_{\mathrm{neq}}(X)$ ). This is because every cycle lying in the image of  $\mathrm{CM}_{\mathrm{neq}}(-X)$  is nilpotent<sup>49</sup>, for degree reasons; this is same argument as [CHK23, §2.3.3].

In particular, it follows by the long-exact sequence:

$$\mathrm{HM}_{\mathrm{eq}}(W) \rightarrow \mathrm{HM}_{\mathrm{eq}}(W) \rightarrow \mathrm{HM}_{\mathrm{neq}}(W),$$

that  $b^{kp} = xa$  for some  $a \in \mathrm{HM}_{\mathrm{eq}}(W)$ .

Thus  $1 - xa$  is mapped to zero in  $\mathrm{SH}_{\mathrm{eq}}(W)$ , via PSS and continuation. Taking the free parts, and using the fact that the free part of  $\mathrm{HM}_{\mathrm{eq}}(W) \simeq \mathrm{HF}_{\mathrm{eq}}(R_{et})$  is mapped injectively into  $\mathrm{SH}_{\mathrm{eq}}(W)$  (since the cones of any continuation map are torsion, by Theorem 1.14), we conclude that  $1 - xa$  is zero in  $\mathrm{HM}_{\mathrm{eq}}(W)$ , contradicting Claim 5.2. This completes the proof of the lemma.  $\square$

**5.2. Pair of pants product.** In this section, we briefly outline the construction of the pair-of-pants product on equivariant Floer cohomology, which is required to establish the sub-additivity property (R5) in §5.3.5. Due to the technical nature of this construction, the similarity with the main results of [Can24b], and the existence of similar equivariant product structures in [GMP23], we do not give a detailed construction.

The structural theorem we claim is:

**Theorem 5.3.** *Assume the setup and conclusion of Theorem 1.14. Then there exists a natural transformation  $\Pi$  between the two functors:*

- $\varphi_{0,t}, \varphi_{1,t} \in \mathrm{hC} \times \mathrm{hC} \mapsto \mathrm{HF}_{\mathrm{eq}}(\varphi_{0,t}) \otimes \mathrm{HF}_{\mathrm{eq}}(\varphi_{1,t})$ ,
- *constant functor*  $\mathrm{hC} \times \mathrm{hC} \rightarrow \mathrm{SH}_{\mathrm{eq}}$ ,

and  $\Pi_{\varphi_{0,t}, \varphi_{1,t}}$  factors through the natural map  $\mathrm{HF}_{\mathrm{eq}}(\varphi_{0,t}\varphi_{1,t}) \rightarrow \mathrm{SH}_{\mathrm{eq}}$ .

Furthermore, the induced product:

$$\Pi : \mathrm{SH}_{\mathrm{eq}} \otimes \mathrm{SH}_{\mathrm{eq}} \rightarrow \mathrm{SH}_{\mathrm{eq}}$$

is unital with 1 acting as the unit element.

The construction is similar to the non-equivariant approach detailed in [Can24b, §3], adapted to the equivariant setting, and with coefficients over a field  $\mathbf{k}$  with arbitrary characteristic.

The generalization from “non-equivariant” to “equivariant” is akin to the generalization of “non-family Floer cohomology” to “family Floer cohomology” in the sense of [Hut08]. In this context, we refer the reader to the

<sup>49</sup>It is important here that  $\omega$  vanishes on holomorphic spheres; see [Can24b, §1.7]. Otherwise quantum corrections can ruin this argument.

discussion in [BC25, §3.6.3] which constructs (in an unsurprising manner) a product structure on some version of family Floer cohomology. See [GMP23] for an approach in this vein.

Let  $\Sigma = \mathbb{C} \setminus \{0, 1\}$  be the pair-of-pants surface equipped with two positive cylindrical inputs  $C_0, C_1$  and one negative output  $C_\infty$ . For a triple of contact isotopies  $\varphi_{0,t}, \varphi_{1,t}, \varphi_{\infty,t}$  such that  $\varphi_{0,t} \circ \varphi_{1,t} \leq \varphi_{\infty,t}$ , we will consider a suitable space of Hamiltonian connections on  $SY \times \Sigma$  whose ideal restrictions match the respective isotopies at the cylindrical ends. These Hamiltonian connections are constructed exactly as in [Can24b].

These lift to  $G$ -invariant Hamiltonian connections on  $S\partial W \times \Sigma$ . The subtle part is how to extend these to the compact part of  $W$  in a way which is compatible with Borel data  $\varphi_{i,\eta,t}$  extending  $\varphi_{i,t}$ .

We do not really want to get into a technical discussion of Hamiltonian connections on the pair of pants surface (for this, we refer to the cited references) but let us just say that:

- one can speak of families of Hamiltonian connections  $\mathfrak{H}_\eta$  on  $\Sigma \times W$ , parametrized by points  $\eta \in EG$ , which agree with the connections determined by Borel data  $\varphi_{i,\eta,t}$  in the ends, and whose ideal restriction are the  $G$ -invariant connections described above;
- given a function  $f : \Sigma \mapsto EG$ , it makes sense to speak of the Hamiltonian connection obtained by setting  $\eta = f(z)$  (in a similar manner to how we set  $\eta = \pi(s)$  when defining the equivariant operations on cylinders).

Just as the equivariant operations in §2.2.4, etc, were defined using flow lines of a pseudogradient on  $BG$ , the product we are describing will be defined in terms of *Morse flow trees* on  $BG$ . There are various approaches to this problem. For our purposes, let us just suppose there is a well-defined space  $\mathcal{T}$  of trees  $\tau$  which have evaluation maps  $\tau : T \rightarrow BG$  and which agree with flow lines of our pseudogradients on the legs of the flow tree. Let us denote by  $\bar{\tau}$  the lift of a flow tree to  $EG$ . Here  $T$  is the underlying space of a flow tree (i.e., three copies of  $[0, \infty)$  connected at the vertex).

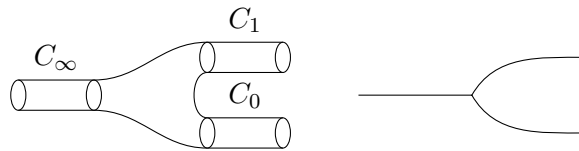


FIGURE 17. Pair of pants surface and the flow tree  $T$ .

The moduli space used to define the product consists of rigid pairs  $(\bar{\tau}, u)$ , where  $\tau \in \mathcal{T}$ , and  $u : \Sigma \rightarrow W$  is a solution to the Floer equation associated to the Hamiltonian connection  $\mathfrak{H}_\eta$  obtained by setting  $\eta = \bar{\tau}(p(z))$  where  $p : \Sigma \rightarrow T$  is an appropriate map sending the pair of pants surface onto

the trivalent graph. This moduli space carries a  $G$ -action, and we count the rigid  $G$ -orbits. The asymptotics are interpreted using the same language of “distinguished lifts” common to all of our equivariant operations.

To make a long story short, this induces a chain map which descends to a product on homology:

$$* : \mathrm{HF}_{\mathrm{eq}}(\varphi_{0,t}) \otimes \mathrm{HF}_{\mathrm{eq}}(\varphi_{1,t}) \rightarrow \mathrm{HF}_{\mathrm{eq}}(\varphi_{\infty,t}).$$

This a priori depends on the choice of Hamiltonian connections  $\mathfrak{H}_\eta$  used (unlike the setting of Borel data, it is not clear whether the space of Hamiltonian connections on  $SY$  with non-positive curvature behaves as a contractible space). However, the same arguments used in [Can24b] show that the map obtained by post-composing  $*$  with the colimit map yield a well-defined product  $\Pi_{\varphi_{0,t}, \varphi_{1,t}}$  as in Theorem 5.3.

By standard arguments, this operation satisfies two properties:

- (1) The product commutes with the continuation maps.
- (2) The element  $1 \in \mathrm{SH}_{\mathrm{eq}}(W)$  from in §5.1 is the unit for this product.

These ensure that if the colimit maps for  $\varphi_{0,t}, \varphi_{1,t}$  hit the unit, then the colimit map for  $\varphi_{\infty,t}$  also hits the unit. This, as in [Can24b], is the mechanism used to verify the sub-additivity of the spectral invariants (R5).

**Remark 5.4.** What we *do not* verify is that the resulting product structure respect the  $\mathbf{k}[[x]]$ -module structure. This seems to be a subtle point. We do believe the product can be made to be  $\mathbf{k}[[x]]$ -linear on the level of homology groups. Whether this can be done on chain level seems far less certain. We leave this question for future research. If one proves the product structure is  $\mathbf{k}[[x]]$ -linear, then the invariants of type  $\mu$  will be super-additive.

**5.3. The axioms for the spectral invariants.** We establish the axioms (R1) through (R5) for the spectral invariants  $c_R$  defined in (7) in §1.5.

**5.3.1. Spectrality.** Property (R1) follows from Theorem 1.18 and the fact the unit is not eternal §5.1.2. Indeed, the unit not being eternal implies  $c_R(\varphi_t) \in \mathbb{R}$ . Having established this finiteness, if  $s$  is not in the spectrum, then  $s$  cannot be the infimal value for which  $\mathrm{HF}(\varphi_t^{-1} \circ R_{st}) \rightarrow \mathrm{SH}_{\mathrm{eq}}(W)$  hits the unit element, since the continuation maps for slightly lower values of  $s$  are isomorphisms by Theorem 1.18.

Moreover, since the statement of Theorem 1.18 includes the refinement by the  $W$ -contractible discriminant, we conclude that

**5.3.2. Monotonicity.** Property (R2) follows from the fact that, if  $\varphi_{0,t} \leq \varphi_{1,t}$ , then there is a 1-simplex  $R_{st} \circ \varphi_{1,t}^{-1} \rightarrow R_{st} \circ \varphi_{0,t}^{-1}$ , for any value of  $s$ .

**5.3.3. Continuity from above.** Property (R3) follows by definition when computing  $c_R(\varphi_t)$  when  $\varphi_t$  lies on the discriminant. Otherwise it follows from Theorem 1.18.

**5.3.4. Normalization.** Property (R4) follows from Lemma 5.1 in §5.1.2.

**5.3.5. Sub-additivity.** Property (R5) follows from the existence of the pair of pants product outlined in §5.2; once the product structure is set-up, the argument is exactly the same as one used to prove [Can24b, Theorem 4] and [DUZ25, Theorem 2.11].

**5.4. The axioms for the integer-valued measurements.** In this section, we establish the axioms (G1) through (G5) for the measurement  $\mu$  in §1.5. We also prove the implication (9) concerning the vanishing of  $\text{SH}_{\text{neq}}(W)$ , and Proposition 1.12 relating the values of  $\mu$  on linear symplectic isotopies with the Conley-Zehnder index.

**5.4.1. On the non-equivariant symplectic cohomology.** In this section we prove implication (9). Observe that for a Borel data  $(\psi_{\eta,t}, J)$  it holds that:

$$(53) \quad \text{CF}_{\text{neq}}(\psi_{\eta,t}, J) = \begin{cases} \text{CF}(\psi_{v_0,e}) \otimes \sigma(v_0) \oplus \text{CF}(\psi_{v_1,e}) \otimes \sigma(v_1) & \text{if } p \geq 3, \\ \text{CF}(\psi_{v_0,+}) & \text{if } p = 2, \end{cases}$$

where the differential only counts the flow lines which lie above the pole  $[1 : 0 : \dots]$  in projective space (either  $\mathbb{C}P^\infty$  or  $\mathbb{R}P^\infty$ , depending on  $p \geq 3$ ). See §2.4.11 and §3.1.3 for details on the chain complex.

It follows fairly tautologically that  $\text{SH}_{\text{neq}}(W) = \text{SH}(W; \mathbf{k})$  in the case  $p = 2$ , so we henceforth assume  $p \geq 3$ .

Recall  $\text{SH}_{\text{neq}}(W)$  is the colimit of  $\text{HF}_{\text{neq}}(\psi_{\eta,t}, J)$ , and so we need to prove this colimit vanishes under the assumption that the “ordinary” symplectic cohomology  $\text{SH}(W; \mathbf{k})$  vanishes. We will use a spectral sequence argument.

With respect to the direct sum decomposition in (53) we can write the differential on  $\text{CF}_{\text{neq}}(\psi_{\eta,t}, J)$  as a lower triangular matrix, where the diagonal entries are the ordinary Floer differentials.

Now, for any element  $e \in \text{SH}_{\text{neq}}(W)$ , we can find some  $\psi_{\eta,t}$  so that  $e$  lies in the image of the continuation map from  $\text{HF}_{\text{neq}}(\psi_{\eta,t}, J)$ . This element  $e$  then splits into a summand  $e_0 + e_1$ . Since the element  $e_1$  is a cycle, and the ordinary symplectic cohomology vanishes, we may assume (by replacing  $\psi_{\eta,t}$  by something closer to the colimit) that  $e_0$  is an exact cycle, say  $e_0 = df_0$ . Then  $e$  is cohomologous to  $e'_1 = e_1 + \Delta f_0$ , where  $\Delta$  is the off-diagonal term in the differential. But now  $e'_1$  is a cycle. By passing further into the colimit *and using the fact that the continuation maps are also lower triangular*, we may suppose  $e'_1$  is an exact cycle. But thus the entire cycle  $e$  is exact (up to passing closer to the colimit). Since  $e$  was an arbitrary element of the colimit, we conclude  $\text{SH}_{\text{neq}}(W) = 0$ , as desired.

**5.4.2. Integer valuedness.** Since  $\mu(\varphi_t)$  is defined as a supremum of the set of integers  $d$  for which  $x^{-d}1$  lies in the image of the colimit map, to prove  $\mu(\varphi_t)$  is finite, it is sufficient (to obtain the integer valuedness of  $\mu$ ) to prove that this set of integers is non-empty and bounded from above.

That the set of integers is bounded from above follows from the structure theorem for finitely generated modules over the ring  $\mathbf{k}[[x]]$ . We argue by contradiction, and suppose  $\mathrm{HF}_{\mathrm{eq}}(\varphi_t) \rightarrow \mathrm{SH}_{\mathrm{eq}}(W)$  can hit  $x^{-d}1$  for arbitrarily large values of  $d$ . Since  $x$  acts invertibly on  $\mathrm{SH}_{\mathrm{eq}}(W)$ , the  $x$ -torsion part of  $\mathrm{HF}_{\mathrm{eq}}(\varphi_t)$  is mapped to zero under the colimit map. On the other hand, since the cones of continuation maps are  $x$ -torsion, the free part of  $\mathrm{HF}_{\mathrm{eq}}(\varphi_t)$  is mapped injectively into  $\mathrm{SH}_{\mathrm{eq}}(\varphi_t)$ . By our assumption, there exists  $a_0, a_d$  in the free part such that:

- $a_0 \mapsto 1$
- $a_d \mapsto x^{-d}1$ .

But then  $a_0 = x^d a_d$  by the aforementioned injectivity. In particular,  $a_0$  can be divided by  $x^d$  for arbitrarily large  $d$ . In a finitely generated  $\mathbf{k}[[x]]$ -module, this can only happen if  $a_0 = 0$ , which implies  $1 \in \mathrm{SH}_{\mathrm{eq}}(W)$  vanishes, contradicting Lemma 5.1 that the unit was not eternal.

To prove the set of integers is non-empty (i.e.,  $\mu$  is not  $-\infty$ ), we need to prove that *some* power of the unit is hit. However, since the cones of continuation maps are torsion, the map  $\mathrm{HF}_{\mathrm{eq}}(R_{-at}) \rightarrow \mathrm{HF}_{\mathrm{eq}}(R_{\epsilon t})$  hits  $x^k 1$  for large enough  $k$ , for any  $a > 0$ . Thus  $\mu(R_{-at})$  is at least  $-k$ . By monotonicity (see §5.4.3) it follows that  $\mu(\varphi_t)$  is at least  $-k$  if  $\varphi_t$  is greater than  $R_{-at}$ . Since  $a > 0$  was arbitrary, we conclude the desired result.

**5.4.3. Monotonicity.** Property (G1) follows from the fact that, if  $\varphi_{0,t} \leq \varphi_{1,t}$ , then image of the map  $\mathrm{HF}_{\mathrm{eq}}(\varphi_{0,t}) \rightarrow \mathrm{SH}_{\mathrm{eq}}(W)$  is contained in the image of the map  $\mathrm{HF}_{\mathrm{eq}}(\varphi_{1,t}) \rightarrow \mathrm{SH}_{\mathrm{eq}}(W)$ .

In particular, if  $\mathrm{HF}_{\mathrm{eq}}(\varphi_{0,t}) \rightarrow \mathrm{SH}_{\mathrm{eq}}(W)$  hits  $x^{-d}1$ , then the same holds for  $\varphi_{1,t}$ ; thus  $\mu(\varphi_{1,t}) \geq \mu(\varphi_{0,t})$ .

**5.4.4. Continuity from above.** For (G2) we use the same argument as 5.3.3.

**5.4.5. Normalization.** (G3) follows from Claim 5.2 and the injectivity of  $\mathrm{HF}_{\mathrm{eq}}(R_{\epsilon t}) \rightarrow \mathrm{SH}_{\mathrm{eq}}(W)$  on the free part recently explained in §5.4.2.

**5.4.6. Non-triviality.** Every element of the colimit, including  $x^{-d}1$ , lies in the image of  $\mathrm{HF}_{\mathrm{eq}}(\varphi_t) \rightarrow \mathrm{SH}_{\mathrm{eq}}(W)$  for some zero-simplex  $\varphi_t$ . Thus (G4) holds, i.e.,  $\mu$  attains arbitrarily large values.

**5.4.7. Discriminant.** Property (G5) follows from Theorem 1.18. Moreover, we have the refinement from Theorem 1.11 where the discriminant is replaced by the  $W$ -contractible discriminant (Definition 1.9), since the statement of Theorem 1.18 includes this refinement.

**5.4.8. Agreement with the Conley-Zehnder index.** In this section we prove Proposition 1.12 that  $\mu(\varphi_t) = \text{CZ}(\varphi_t) - n$  whenever  $\varphi_t$  is a linear symplectic isotopy whose time-1 map does not have 1 as an eigenvalue. In this setting, we focus only on the case  $G = \mathbf{k} = \mathbb{Z}/2\mathbb{Z}$ .

The result follows easily from the fact that  $\varphi_t$  admits a canonical extension to  $\mathbb{C}^n$  as equivariant Borel data with a single generator  $\gamma(\varphi_t)$  (the constant orbit located at the origin). The underlying chain complex is:

$$\mathbf{k}[[x]]\gamma(\varphi_t),$$

and all differentials vanish.

Now consider  $s$  large enough that the  $\varphi_t$  admits a continuation to  $R_{st}$ , and let  $\mathbf{c}$  be the continuation map. Similarly let  $\mathbf{c}$  denote the continuation map from  $R_{\epsilon t}$  to  $R_{st}$  for  $\epsilon \in (0, 1)$ . It follows that:

- $\mathbf{c}(\gamma(\varphi_t)) = x^{k(\varphi_t)}\gamma(R_{st})$ ,
- $\mathbf{c}(\gamma(R_{\epsilon t})) = x^{k(R_{\epsilon t})}\gamma(R_{st}) = 1$ .

Note that we exclude the case that  $\mathbf{c} = 0$  by the structural theorems we have proved above (e.g., otherwise  $\mu$  would not be a finite integer).

The integers  $k(\varphi_t)$  and  $k(R_{\epsilon t})$  can be computed using the Fredholm index formula for Cauchy-Riemann operators, and we have:

- $\text{CZ}(\varphi_t) - \text{CZ}(R_{st}) + k(\varphi_t) = 0$ ,
- $\text{CZ}(R_{\epsilon t}) - \text{CZ}(R_{st}) + k(R_{\epsilon t}) = 0$ .

Subtracting, we conclude:

$$\text{CZ}(\varphi_t) - n = k(R_{\epsilon t}) - k(\varphi_t).$$

Because  $x^{k(\varphi_t)}\gamma(R_{st}) = x^{k(\varphi_t)-k(R_{\epsilon t})}1$ , we conclude  $\mu(\varphi_t) = k(R_{\epsilon t}) - k(\varphi_t)$ , and the desired result follows.  $\square$

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