EQUIVARIANT LAGRANGIAN DISPLACEMENTS

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Abstract. This paper proves that certain monotone Lagrangians in the standard symplectic vector space cannot be displaced by a Hamiltonian isotopy which commutes with the antipodal map. The method of proof is to develop a Borel equivariant version of the quantum cohomology of Biran and Cornea, and prove it is sensitive to equivariant displacements. The Floer–Euler class of Biran and Khanevsky appears as a term in the equivariant differential in certain cases.

1. *Introduction*

Suppose that L is a Lagrangian submanifold of a symplectic manifold W admitting a symplectic involution a, and suppose L is preserved by the involution as a set. The question we are interested in is the following:

Does there exist an equivariant Hamiltonian isotopy φ_t which displaces L?

Here *equivariant* is in the sense $a\varphi_t = \varphi_t a$ holds for all t. If the answer is "yes" then we say L is *equivariantly displaceable*.

The prototypical example is $W = \mathbb{C}^n$, $a(z) = -z$; here one can consider Lagrangians L of the following form: if $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$ is the standard Hopf fibration, then $L = \pi^{-1}(\bar{L})$ is invariant under a for any Lagrangian submanifold $\bar{L} \subset \mathbb{C}P^{n-1}$. As an application of our methods, we will prove:

Theorem 1.1. *The Lagrangian $L = \pi^{-1}(\bar{L})$ in $\mathbb{C}^n$ is not equivariantly displaceable, with respect to $a(z) = -z$, provided both:*

1. *$\bar{L}$ is monotone with minimal Maslov number at least 2,*
2. *the quantum cohomology of $\bar{L}$ with coefficient field $\mathbb{Z}/2\mathbb{Z}$, in the sense of* [BC07, BC09]*, is non-vanishing.*

Remark 1.2. In the case when $n = 1$, the Lagrangian L is simply the unit circle, and an elementary argument can be used to prove Theorem 1.1. We give such an argument in §B, although we are confident readers can furnish a proof without much difficulty.

Remark 1.3. It is known that the quantum cohomology of $\bar{L}$ is non-vanishing in various cases, e.g., if $\bar{L}$ is the real projective space or the Clifford torus (in this case L is just the product of n circles); see [BC07, BC09] and [Cho04].

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1.1. *Quantum cohomology for monotone Lagrangians.* Throughout we work in the context of a single monotone Lagrangian $L \subset W$, with minimal Maslov number $N \geq 2$; here W should be tame at infinity as in [Gro85].

Recall that the *quantum cohomology* for a monotone Lagrangian L (with minimal Maslov number at least 2) is a deformation of the Morse cohomology of L. See [BC09] for an overview of the construction. To define the quantum cohomology, one requires:

- a Morse-Smale pseudogradient P on L,
- an ω-tame almost complex structure J on W.

One defines a differential on the vector space:

$$\mathrm{CM}(P) \otimes \mathbf{k}[q^{-1}, q]]$$

where $\mathrm{CM}(P)$ is the free $\mathbf{k}$-vector space generated by the zeros of P (graded by the Morse index), and q is a formal variable of degree N. Throughout this paper, we work with the coefficient field $\mathbf{k} = \mathbb{Z}/2\mathbb{Z}$.

The differential is a polynomial expression:

$$d = d_0 + qd_1 + q^2 d_2 + \cdots,$$

where $d_i : \mathrm{CM}(P) \to \mathrm{CM}(P)$ is a linear endomorphism of degree $1 - Ni$. Moreover, d_0 is the classical Morse differential associated to the pseudogradient P. The quantum terms d_i, $i > 0$, are defined by counting *pearl trajectories*, and are described in detail in §2.1. As we will explain later, these terms depend on a choice of almost complex structure J.

As established by Biran and Cornea, provided J is generic, the total differential $d = d_0 + qd_1 + \cdots$ squares to zero; the resulting homology vector space is denoted by $\mathrm{QH}^*(L; P, J)$. The significance of this invariant is:

Proposition 1.4. *If* $\mathrm{QH}^*(L; P, J) \neq 0$*, then* L *cannot be displaced by a Hamiltonian isotopy.* □

The quantum cohomology is known to be isomorphic to the Lagrangian Floer cohomology of L with itself; see [BC09, Theorem A(vi)]. As in [BC09], in this paper we do not make any use of the fact that the Lagrangian quantum homology can be identified with the Floer homology.

1.2. *Equivariant quantum cohomology.* Suppose now that $a : W \to W$ is a symplectic involution and L is preserved by a. Our approach to $\mathbb{Z}/2\mathbb{Z}$-equivariant quantum cohomology is a "pearl trajectory" version of the equivariant cohomology of [SS10, Sei15], which is itself a Morse/Floer version of the Borel equivariant cohomology. As in the ordinary quantum cohomology, the definition depends on some auxiliary choices. As part of the data, one picks P, J exactly as in §1.1. The underlying equivariant complex is:

$$\mathrm{CM}(P) \otimes \mathbf{k}[q^{-1}, q]] \otimes \mathbf{k}[e],$$

and the differential d_{eq} is a polynomial expression:

$$d_{eq} := \sum q^i e^j d_{i,j},$$

where $d_{i,j} : \mathrm{CM}(P) \to \mathrm{CM}(P)$ is a graded endomorphism of degree $1-Ni-j$. The definition will be such that the specialization $e = 0$ is the ordinary quantum cohomology differential; the terms appearing with coefficient e^j form a graded endomorphism of $\mathrm{CM}(P) \otimes \mathbf{k}[q^{-1}, q]]$ of degree $1 - j$.

The terms $d_{i,j}$ are constructed by following the recipe of [SS10, Sei15]. For now, we will just say that $d_{i,j}$ depends on the choice of sufficiently regular data (P_η, J_η) parametrized by $\eta \in S^\infty$. These data are required to restrict to (P, J) at various points on the sphere. All of this is explained properly in §2.3. For such regular data (P_η, J_η), we define $\mathrm{QH}^*_{eq}(L; P_\eta, J_\eta)$ to be the homology of this complex.

Remark 1.5. It is important to note that QH^*_{eq} is an *uncompleted* equivariant cohomology, in the sense that it is a module over $\mathbf{k}[e]$ rather than over the ring $\mathbf{k}[[e]]$ as in [SS10, Sei15]. This algebraic curiosity plays an important role in our proof of Theorem 1.1.

Remark 1.6 (Invariance statement)**.** Well-established continuation arguments then prove the map sending regular Borel data (P_η, J_η) to $\mathrm{QH}^*_{eq}(L; P_\eta, J_\eta)$ can be equipped with the structure of a *connected simple system*[1] [Sal85]. In other words, if one considers the set of regular data as an indiscrete groupoid (a category with a unique morphism between any two elements) then:

$$(P_\eta, J_\eta) \mapsto \mathrm{QH}^*_{eq}(L; P_\eta, J_\eta)$$

extends to a functor; we explain the definition of the morphisms and prove this invariance in §2.4. The limit[2] of this connected simple system is denoted $\mathrm{QH}^*_{eq}(L)$, and is the key player in our paper.

1.3. *Equivariant displacements.* Equivariant quantum cohomology is sensitive to equivariant displacements:

Theorem 1.7. *If ψ_t is an equivariant Hamiltonian isotopy displacing L then the equivariant quantum cohomology vanishes $\mathrm{QH}^*_{eq}(L) = 0$.*

The proof of the theorem is given in §2.5 and goes as follows: first one shows:

$$\mathrm{id} : \mathrm{QH}^*_{eq}(L; P_\eta, J_\eta) \to \mathrm{QH}^*_{eq}(L; P_\eta, J_\eta)$$

is represented as a certain chain level continuation map. The continuation data can be deformed using the equivariant isotopy ψ_t, and, under the hypothesis that $\psi_1(L) \cap L = \emptyset$, one can prove that the deformed continuation map is zero. Thus the identity map is zero, as desired.

[1] A *connected simple system* is a functor from an indiscrete groupoid.

[2] In the category theory sense, see [Mac71]

1.4. *The Biran-Khanevsky Floer-Gysin sequence.* The paper [BK13] considers the quantum cohomology of Lagrangians $L \in \partial\Omega$ where Ω is a Liouville domain and $\partial\Omega$ has a Reeb flow inducing a free $\mathbb{R}/\mathbb{Z}$-action. Throughout this section we assume $\pi_2(\Omega, \partial\Omega) = 0$, in addition to the usual assumptions that L is monotone with minimal Maslov number at least 2.

In this setting the *reduced Lagrangian* $\bar{L} = L/(\mathbb{R}/\mathbb{Z})$ is a monotone Lagrangian in the symplectic reduction $M = \partial\Omega/(\mathbb{R}/\mathbb{Z})$; moreover $\bar{L}$ and L have the same minimal Maslov number. The paper [BK13] constructs an element $F \in \mathrm{QH}^2(\bar{L})$ called the *Floer-Euler class*, and proves an exact triangle (the *Floer-Gysin sequence*):

$$\cdots \longrightarrow \mathrm{QH}(\bar{L}) \xrightarrow{F} \mathrm{QH}(\bar{L}) \longrightarrow \mathrm{QH}(L) \longrightarrow \cdots$$

where F as a map denotes the product with F with respect to Biran-Cornea's quantum multiplication [BC07, BC09]. An example of such domain Ω is:

$$\Omega = B(1) = \left\{ z \in \mathbb{C}^n : \pi |z|^2 \leq 1 \right\},$$

and this has reduction $M = \mathbb{C}P^{n-1}$; this is relevant to Theorem 1.1.

1.5. *Main technical result.* We prove the following:

Theorem 1.8. *In the context of §1.4, if the $\mathbb{Z}/2\mathbb{Z}$-action on $\partial\Omega$ inherited from its $\mathbb{R}/\mathbb{Z}$-action extends to an exact symplectic involution of Ω, then:*

$$\mathrm{QH}^*_{eq}(L) \simeq (\mathrm{QH}^*(\bar{L}) \otimes \mathbf{k}[e])/(e^2 + F) \simeq \mathrm{QH}^*(\bar{L}) \oplus \mathrm{QH}^{*-1}(\bar{L}), \tag{1}$$

Here $(e^2 + F)$ is the image of the morphism $e^2 + F$.[3]

Remark 1.9. If $\mathrm{QH}^*(L) = 0$, e.g., if $\mathrm{SH}(\Omega) = 0$, then F must be a unit by the Floer-Gysin sequence. Then, if we extend coefficients $\mathbf{k}[e] \to \mathbf{k}[[e]]$ it holds:

$$\mathrm{QH}^*_{eq}(L) \otimes_{\mathbf{k}[e]} \mathbf{k}[[e]] \simeq (\mathrm{QH}^*(\bar{L}) \otimes \mathbf{k}[[e]])/(e^2 + F) = 0,$$

since $e^2 + F$ is a unit in $\mathrm{QH}^*(\bar{L}) \otimes \mathbf{k}[[e]]$. However, the same conclusion does not hold in our uncompleted version of equivariant quantum cohomology; one has instead (1) which is often non-zero even when $\mathrm{QH}^*(L) = 0$.

Remark 1.10. This final result can be interpreted as an suitable quantum analogue of a result in algebraic topology relating $\mathbb{Z}/2\mathbb{Z}$- and S^1-equivariant cohomology, see, e.g., [Sei15, pp. 957] (in the same manner that [BK13] is a quantum analogue of the Gysin sequence in algebraic topology).

Proof of Theorem 1.1 using Theorem 1.8. Assume $n \geq 2$, as the case $n = 1$ can be handled by the elementary argument given in §B. Then Theorem 1.8 applied with $L = \mathrm{pr}^{-1}(\bar{L})$ implies:

$$\mathrm{QH}^*_{eq}(L) \simeq \mathrm{QH}^*(\bar{L}) \oplus \mathrm{QH}^{*-1}(\bar{L})$$

[3] In classical algebraic topology, this would be an ideal in the corresponding commutative algebra.

Then Theorem 1.7 implies L is not equivariantly displacable (since the hypothesis of Theorem 1.1 is that $\mathrm{QH}^*(\bar{L}) \neq 0$) $\square$

1.6. *The case of free actions.* Suppose the context for Theorem 1.8, i.e., Ω is a Liouville domain, $\partial\Omega$ has a one-periodic Reeb flow, $\pi_2(\Omega, \partial\Omega) = 0$, and the time-1/2 flow extends to Ω as an exact involution a. In this section, we are concerned with the case when a acts freely on Ω, and so one has a smooth quotient domain Ω'. Then we can deduce:

Theorem 1.11. *If a symplectic involution a acts freely on a Liouville domain Ω satisfying $\pi_2(\Omega, \partial\Omega) = 0$, with smooth quotient domain Ω', and if:*

$$\bar{L} \subset M = \partial\Omega/(\mathbb{R}/\mathbb{Z})$$

is a monotone Lagrangian with minimal Maslov number at least 2, then:

$$\mathrm{QH}^*(L') \simeq \mathrm{QH}^*(\bar{L}) \oplus \mathrm{QH}^{*-1}(\bar{L}),$$

where L' is the lift of $\bar{L}$ to $\partial\Omega'$. Equivalently, the Floer-Euler class associated to the covering $L' \to \bar{L}$ vanishes.

What is interesting is the implication:

$$\mathrm{QH}^*(\bar{L}) \neq 0 \implies \mathrm{QH}^*(L') \neq 0,$$

as it enables one to bootstrap non-vanishing quantum cohomology in a lower dimension to a higher dimension. This result can be applied to, e.g., T^*S^n, $n \geq 3$, to produce monotone Lagrangians in $T^*\mathbb{R}P^n$ which are disjoint from the zero section and have non-vanishing quantum cohomology (by pulling back Lagrangians in the reduction of ST^*S^n, which is known to be the quadric hypersurface in $\mathbb{C}P^n$).

The proof of Theorem 1.11 is based on a variation of a result of [SC25]:

Theorem 1.12. *If W is a tame-at-infinity symplectic manifold with a symplectic involution a which acts freely, and $L \subset W$ is a monotone Lagrangian with minimal Maslov number at least 2, there is an isomorphism:*

$$\mathrm{QH}^*_{eq}(L) \simeq \mathrm{QH}^*(L'),$$

where $L' \subset W'$ is the quotient of $L \subset W$ by the involution.

Proof of Theorem 1.11. Using Theorem 1.8 and Theorem 1.12 we have:

$$\mathrm{QH}^*(L') \simeq \mathrm{QH}^*_{eq}(L) \simeq \mathrm{QH}^*(\bar{L}) \oplus \mathrm{QH}^{*-1}(\bar{L}),$$

which gives the isomorphism. To see that the Floer-Euler class vanishes, we count dimensions using the Floer-Gysin sequence for $L' \to \bar{L}$ from §1.4; the relevant segment of the sequence is:

$$\mathrm{QH}^{*+1}(\bar{L}) \longrightarrow \mathrm{QH}^{*+1}(L') \xrightarrow{A} \mathrm{QH}^*(\bar{L}) \xrightarrow{F} \mathrm{QH}^{*+2}(\bar{L}) \longrightarrow \cdots.$$

We conclude from exactness and a dimension counting argument that $\dim \ker A = \dim \mathrm{QH}^{*+1}(\bar{L})$, which implies, by exactness, that:

$$\dim \ker F = \dim \mathrm{QH}^*(\bar{L}) \implies \ker F = \mathrm{QH}^*(\bar{L}) \implies F = 0,$$

as desired. □

1.7. *Further questions.* Does an analogous story hold with when $\mathbb{Z}/2\mathbb{Z}$ is replaced[4] by another prime cyclic group $\mathbb{Z}/p\mathbb{Z}$? Surely the construction of the invariant and its main formal properties does not require $\mathbb{Z}/2\mathbb{Z}$, (see, e.g., [SZ21] for a generalization of [Sei15] to other prime cyclic groups). However, it seems our computation of the equivariant cohomology vis-a-vis the circle bundle construction of [BK13] requires further analysis in the case of $\mathbb{Z}/p\mathbb{Z}$ equivariant cohomology if $p > 2$.

Another question: can one replace the assumption that L is monotone with the weaker assumption that $\mu(A) > 0$ for each class A represented by a non-constant J-holomorphic disk? We use the stronger assumption of monotonicity in a seemingly crucial way in the proof that our (uncompleted) equivariant cohomology QH^*_{eq} is sensitive to equivariant displacements.

Another question: in the case of $\mathbb{C}^n$, can one prove similar equivariant non-displaceability of Lagrangians results using generating functions à la [Vit92]?

1.8. *Other approaches and non-displaceability of Legendrians leaves.* We assume the context of §1.4. If:

- $L \subset \partial\Omega$ is the lift of a *Lagrangian stem* $\bar{L} \subset M$, in the sense there is an involutive map $M \to \mathbb{R}^n$ such that $\bar{L}$ is the only non-displaceable fibre (as in [EP09]), and
- an appropriate version of the symplectic cohomology of SY is non-vanishing (e.g., as in [BKK24, SUV25]), where we consider the contact manifold $Y = \partial\Omega/(\mathbb{Z}/2\mathbb{Z})$,

then the projection of L to SY is non-displaceable by the *contact big fiber theorem* of [SUV25]. This in turn gives equivariant non-displaceability of L in the symplectization of $\partial\Omega$. This applies to the Clifford torus in $\partial B(1)$, as remarked in the proof of [SUV25, Corollary 1.16].

Let us also note that the trick of [SUV25, Theorem 1.15] implies that, whenever the contact structure on $\partial\Omega$ foliates L by closed Legendrian leaves Λ, these leaves Λ are not equivariantly displaceable from L by contact isotopies of $\partial\Omega$, provided that L is not equivariantly displaceable. The proof that non-displaceability of L implies non-displaceability of Λ from L uses elementary arguments in symplectic and contact geometry. The conclusion is similar to those obtained in [EHS95].

[4]Let us note that the linear symplectic group acts by $\mathbb{Z}/2\mathbb{Z}$-equivariant Hamiltonian isotopies of $\mathbb{C}^n$, but the action of a linear symplectic matrix is rarely $\mathbb{Z}/p\mathbb{Z}$-equivariant.

1.9. *Outline of the rest of the paper.* It remains to construct QH^*_{eq}, and prove it is sensitive to equivariant displacements (Theorem 1.7); this is done in §2. We also prove Theorem 1.8 in §3 and Theorem 1.12 in §4.

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2. *Equivariant quantum cohomology and equivariant displacements*

In §2.1 we recall the non-equivariant theory in a bit more detail, and we discuss the special role of disks u with $\mu(u) = 2$ in §2.2. In §2.3 we construct the equivariant differential d_{eq} for regular data and prove that it squares to zero. In §2.4 we explain that the resulting invariant is independent of the choice of regular data as described in Remark 1.6. In §2.5, we show the vanishing of QH^*_{eq} in the presence of an equivariant Hamiltonian displacement.

2.1. *Recollection of Lagrangian quantum cohomology.* We continue the discussion started in §1.1. Let $\mathcal{G}_k$ be the (open) parameter space of ordered tuples $b = (b_1, \dots, b_k)$ satisfying $-\infty < b_1 < \cdots < b_k < \infty$, so $\mathcal{G}_k$ is a smooth open manifold of dimension k.

A *pearl trajectory* is a tuple $w = (b; u_1, \dots, u_k)$ where $b \in \mathcal{G}_k$ and:

$$\begin{cases} u_\nu : \mathbb{R} \times [0,1] \to (W, L), \\ \partial_s u_\nu + J(u_\nu)\partial_t u_\nu = 0, \\ 0 < \int \omega(\partial_s u_\nu, \partial_t u_\nu) ds dt < \infty, \end{cases}$$

for $\nu = 1, \dots, k$, such that the flow line of P of time $b_{\nu+1} - b_\nu$ joins the removable singularities $u_\nu(+\infty)$ to $u_{\nu+1}(-\infty)$, $\nu = 1, \dots, k-1$. Each pearl trajectory has asymptotic zeros x_-, x_+ of P obtained by flowing $u_1(-\infty)$ backwards by P and $u_k(+\infty)$ forwards by P. In other words, $u_1(-\infty)$ lies in the stable manifold S_{x_-}, and $u_k(+\infty)$ lies in the unstable manifold U_{x_+}.

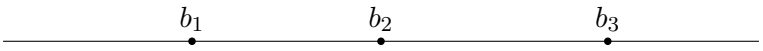


FIGURE 1. $\mathcal{G}_k$ is the parameter space of strictly ordered k-tuples (shown with $k = 3$).

For generic choice of J, the space of pearl trajectories w with k pearls and $\mu(w) = \sum \mu(u_\nu) = Ni$ is a smooth manifold of dimension:

$$Ni + \mathrm{ind}(x_+) - \mathrm{ind}(x_-) + k; \tag{2}$$

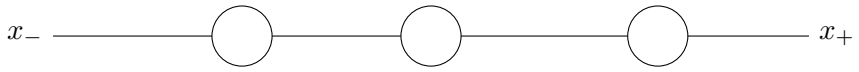


FIGURE 2. Illustration of a pearl trajectory w with $k = 3$ and asymptotics x_-, x_+.

moreover, the space of pearl trajectories carries a free $\mathbb{R} \times \mathbb{R}^k$ action, where the $\mathbb{R}$ action translates g, and the $\mathbb{R}^k$ action translates the "pearls" $u_1, \dots, u_k$, that is, $(s_1, \dots, s_k) \cdot (u_1, \dots, u_k) = (u_1(s + s_1, t), \dots, u_k(s + s_k, t))$.

Still assuming a generic J, the space of rigid orbits of this $\mathbb{R}^{k+1}$-action is a compact zero dimensional manifold. The count of these rigid orbits with asymptotics x_-, x_+ produces a coefficient $N_{i,k}(x_-, x_+) \in \mathbf{k}$, and one defines:

$$d_i(x_-) := \sum_{k=0}^{\infty} N_{i,k}(x_-, x_+)x_+.$$

Finally, we observe the rigid orbits satisfy $\mathrm{ind}(x_+) - \mathrm{ind}(x_-) = 1 - Ni$; thus d_i defines an endomorphism of degree $1 - Ni$.

Theorem 2.1 (Biran-Cornea)**.** *For regular data (P, J), the differential:*

$$d = \sum q^i d_i$$

squares to zero. (See Remark 2.9 for the definition of regular).

This result is a special case of our result that the equivariant differential squares to zero in §2.3. One subtlety v concerns disks with $\mu(v) = 2$, discussed next.

2.2. *Disks with Maslov number 2.* The definition of Lagrangian quantum cohomology depends on a choice of (P, J). It will be important to consider:

- the moduli space $\mathcal{M}_2(L, J)$ of J-holomorphic disks v with $\mu(v) = 2$,
- the moduli space $\mathcal{M}_1(J)$ of J-holomorphic spheres with $c_1(u) = 1$.

Such special treatment of disks with Maslov number 2 appears throughout the literature on enumerating holomorphic disks with boundary on a Lagrangian; we refer the reader to [BC12, §3.3], and [Oh95] for further discussion.

Each $v \in \mathcal{M}_2(L, J)$ or $v \in \mathcal{M}_1(J)$ determines a linearized operator:

$$D_v : W^{1,p}(v^*TW) \to L^p(v^*TW)$$

which is obtained by differentiating the J-holomorphic curve equation.

Lemma 2.2. *Fix a precompact neighborhood U of L and an ω-tame almost complex structure J_0. Let $\mathcal{J}(U)$ be the set of ω-tame almost complex structures which agree with J_0 outside of U, topologized with the C^∞ topology. There is an open dense set $\mathcal{J}^{\mathrm{reg}}(U) \subset \mathcal{J}(U)$ of complex structures J which*

render D_v surjective for each $v \in \mathcal{M}_2(L,J)$ and for each $v \in \mathcal{M}_1(J)$ whose image intersects U.

Proof. It is straightforward to adapt the arguments of [MS12] to handle those v which have "injective points" mapped into U. Each sphere v with Chern number 1 cannot be a multiple cover, and so v must simple[5] by the work of [MW95]. It follows that if v intersects U then v has injective points mapped into U. In the case of holomorphic disks with Maslov number 2, we follow [BC07, §3.2] to explain why we can assume such injective points exist. The key result [BC07, Theorem 3.2.1] is due to [Laz00, Laz11] which proves that any holomorphic disk v can be decomposed into multiple covers of simple disks. If v has the minimal Maslov number 2, it must therefore have injective points passing through U. This completes the proof. □

To each $J \in \mathcal{J}^{\mathrm{reg}}(U)$ we can then define:

$$\Pi_2 = (\mathcal{M}_2(L,J) \times \partial D(1) \times \partial D(1))/\mathrm{Aut}(D(1)),$$
$$\Pi_1 = \mathrm{ev}_1^{-1}(U) \text{ where } \mathrm{ev}_1 : (\mathcal{M}_1(J) \times S^2)/\mathrm{Aut}(S^2) \to W,$$

with evaluation maps $\mathrm{ev}_2 : \Pi_2 \to L \times L$ and $\mathrm{ev}_1 : \Pi_1 \to W$. By the regularity ensured by Lemma 2.2, Π_2 is a smooth manifold of dimension $n+1$ and Π_1 is a smooth manifold of dimension $2n-2$.

Lemma 2.3. *Given a Morse-Smale pseudogradient P, there is an open and dense subset $\mathcal{J}^{\mathrm{reg}}(P,U) \subset \mathcal{J}^{\mathrm{reg}}(U)$ such that:*

(1) $\mathrm{ev}_1^{-1}(x) = \emptyset$ *for each zero x of P,*
(2) ev_2 *is transverse to $S_x \times U_y$ for each pair of zeros x, y of P, where S_x and U_y are the stable and unstable manifolds, respectively.*

Proof. The conditions are open conditions, so only needs to prove their density; this in turn follows from standard application of the Sard-Smale theorem as in [MS12], together with the injectivity considerations from Lemma 2.2. The first condition is equivalent to ev_1 being transverse to each singleton $\{x\}$, since $\dim \Pi_1 < \mathrm{codim}\,\{x\}$. □

For $J \in \mathcal{J}^{\mathrm{reg}}(P,U)$ it then follows that:

$$\Pi_2^x := \mathrm{ev}_2^{-1}(S_x \times U_x)$$

is an open 1-manifold whose non-compact ends are in bijection with the zero manifold:

$$\partial \Pi_2^x = \mathrm{ev}_2^{-1}(\partial S_x \times U_x) \cup \mathrm{ev}_2^{-1}(S_x \times \partial U_x),$$

where ∂S_x and ∂U_x are the codimension 1 boundaries. This observation is used to prove that $d^2 = 0$ in the case $N = 2$.

[5] *simple*, in the sense that v is an injective holomorphic immersion on the complement of a finite set of critical points and self-intersections.

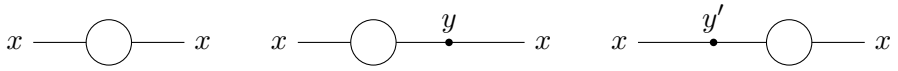


FIGURE 3. The 1-manifold Π_2^x and example boundary configurations. Here y is in the unstable manifold of x and y' is in the stable manifold of x.

2.3. *The equivariant differential.* We continue from §1.2. The goal is to define the moduli spaces used in the definition of the equivariant differential, and define a notion of *regular data*; such data will yield a well-defined equivariant differential which squares to zero.

2.3.1. *Borel data.* The terms in the equivariant differential are defined using data parametrized by:

$$S^\infty = \left\{(\eta_0, \eta_1, \dots) : \sum \eta_i^2 = 1 \text{ and } \eta_i = 0 \text{ for } i \text{ sufficiently large}\right\}.$$

The required data is:

(1) a family P_η of vector fields on L, $\eta \in S^\infty$,
(2) a family J_η of ω-tame almost complex structures, $\eta \in S^\infty$,

and this so-called *Borel data* is required to satisfy the following properties:

(3) $(P_{-\eta}, J_{-\eta}) = (a^* P_\eta, a^* J_\eta)$,
(4) $(P_\eta, J_\eta) = (P, J)$ if $\eta_j \geq 4/5$ for some j,
(5) $(P_{\tau(\eta)}, J_{\tau(\eta)}) = (P_\eta, J_\eta)$,

where $\tau : S^\infty \to S^\infty$ is the self-similarity shift map:

$$\tau(\eta_0, \eta_1, \dots) = (0, \eta_0, \eta_1, \dots).$$

It is important to note that the neighborhoods $\{\eta_j \geq 4/5\}$ and $\{\eta_j \leq -4/5\}$ are mutually disjoint neighborhoods of the "poles:"

$$v_{j,\pm} := (0, \dots, 0, \pm 1, 0, \dots) \text{ where the } \pm 1 \text{ is in the } j\text{th position.}$$

As explained in [SS10] and [Sei15, §3], there is a vector field $\mathfrak{S}$ on S^∞ which behaves as a Morse-Smale pseudogradient for the function $\sum j\eta_j^2$. To be precise, we will take $\mathfrak{S}$ as the generator of the flow:

$$[\eta_0 : \eta_1 : \eta_2 : \dots] \mapsto [\eta_0 : e^t \eta_1 : e^{2t} \eta_2 : \dots],$$

defined on RP^∞ using real projective coordinates (this lifts uniquely to a pseudogradient on S^∞). Let us note that:

- the poles $v_j^\pm$ are the zeros of $\mathfrak{S}$,
- $\mathfrak{S}(-z) = -\mathfrak{S}(z)$, using the identification of TS^∞_z and TS^∞_{-z},
- $\mathfrak{S}$ is tangent to $S^n \subset S^\infty$,
- $\mathfrak{S}(\tau(z)) = \tau_* \mathfrak{S}(z)$.

Associated to this pseudogradient is the space of parametrized flow lines $\mathcal{P}^{j,\pm}$ asymptotic to $v_{0,+}$ and $v_{j,\pm}$, which forms an (open) manifold of dimension j, and carries an $\mathbb{R}$-action by translation.

2.3.2. *Definition of the differential.* The terms $d_{i,j}$ are of the form:

$$d_{i,j} = \sum_{k=0}^{\infty} \left(d^+_{i,j,k} + d^-_{i,j,k} \right),$$

where $d^{\pm}_{i,j,k}$ is defined by counting pearl trajectories in the parametric moduli space of solutions $w = (b, \pi, u_1, \dots, u_k)$ where:

$$\begin{cases} b = (b_1, \dots, b_k) \in \mathcal{G}_k \text{ and } \pi \in \mathcal{P}^{j,+}, \\ \partial_s u_\nu + J_{\pi(b_\nu)} \partial_t u_\nu = 0 \text{ and } 0 < \omega(u_\nu) < \infty, \text{ for } \nu = 1, \dots, k, \\ \mu(u_1) + \cdots + \mu(u_k) = Ni, \end{cases} \tag{3}$$

such that the flow line of $P_{\pi(s)}$, $s \in [b_\nu, b_{\nu+1}]$ joins $u_\nu(+\infty)$ to $u_{\nu+1}(-\infty)$.

Let us denote this moduli space by $\mathcal{M}^{\pm}_{i,j,k}$. As above, this has an $\mathbb{R}^{k+1}$ action, and we are interested in the rigid orbits.

In the "+" case, such a solution has asymptotic zeros $x_-(w), x_+(w)$ of P at its ends, by flowing $b_1(-\infty)$ backwards by $P_{\pi(s)}$, $s \in (-\infty, b_1]$, and by flowing $b_k(+\infty)$ forwards by $P_{\pi(s)}$, $s \in [b_k, \infty)$. Define:

$$N^+_{i,j,k}(x_-, x_+) = \text{count of rigid orbits of } \mathcal{M}^+_{i,j,k} \text{ with asymptotics } x_-, x_+.$$

In the "−" case, one can still associate zeros x_-, x_+ of P, where x_- is as above, but now $a(x_+)$ (instead of x_+) is the asymptotic of the trajectory of $P_{\pi(s)}$, $s \in [b_k, \infty)$, starting at $u_k(+\infty)$. In words, the "−" case twists the geometric asymptotic by the involution a. With this convention, each solution w has asymptotic zeroes $x_\pm(w)$ of P. Define:

$$N^-_{i,j,k}(x_-, x_+) := \text{count of rigid orbits of } \mathcal{M}^-_{i,j,k} \text{ with asymptotics } x_-, x_+.$$

This twisting of the asymptotics is the same as in [SS10, Sei15].

Our main structural result is the following:

Theorem 2.4. *If (P, J) are regular for defining the quantum cohomology, then they can be extended to Borel data (P_η, J_η) satisfying 1–5 which is regular in the sense of Definition 2.5 below. In this case, the endomorphisms of* $\mathrm{CM}(P)$ *given by:*

$$d^{\pm}_{i,j,k}(x_-) = \sum_{x_+} N^{\pm}_{i,j,k}(x_-, x_+) x_+$$

have degree $1 - j - Ni$ and $d_{eq} = \sum e^j q^i d^{\pm}_{i,j,k}$ squares to zero; the specialization to $e = 0$ (i.e., the $j = 0$ term) agrees with the non-equivariant quantum differential.

The precise notion of regularity is given in Definition 2.6, and Theorem 2.4 is proved in §2.3.6.

2.3.3. *Linearized operators.* In this section we consider classes of Borel data which achieve transversality for various moduli spaces.

Let $w = (b, \pi, u_1, \dots, u_k)$ be a pearl trajectory with k pearls, total Maslov number Ni, and with $\pi \in \mathcal{P} = \mathcal{P}^{j,\pm}$, using Borel data (P_η, J_η). Here we assume that the data (P, J) near the positive poles is generic in the sense of §2.2 if the minimal Maslov number is 2.

We define the *ambient tangent space* of w to be:

$$AT_w = W^{1,p,\delta}(u_1^*TW) \oplus \cdots \oplus W^{1,p,\delta}(u_k^*TW) \oplus (T\mathcal{G}_k)_b \oplus (T\mathcal{P})_\pi \oplus TL^{\oplus 2k};$$

where we abbreviate $W^{1,p,\delta}(u_\nu^*TW) = W^{1,p,\delta}(u_\nu^*TW, u_\nu^*TL)$ and:

$$TL^{\oplus 2k} = (TL)_{p_{1,-}} \oplus (TL)_{p_{1,+\infty}} \oplus \cdots \oplus (TL)_{p_{k,-}} \oplus (TL)_{p_{k,+}},$$

with $p_{i,\pm} = u_i(\pm\infty)$. This is the tangent space of all variations of w, including those which do not respect the incidence conditions.

The usual Floer theory linearization process produces a linear operator:

$$AD_w : AT_w \to L^{p,\delta}(u_1^*TW) \oplus \cdots \oplus L^{p,\delta}(u_k^*TW);$$

which maps $W^{1,p,\delta}(u_i^*TW)$ into $L^{p,\delta}(u_i^*TW)$ as an operator of Cauchy-Riemann type (in particular, as a Fredholm operator). For related details, we refer the reader to [Can22, §4], [BC24, §4.1.2], and [BC07, §4.4].

To incorporate the incidence conditions, we define:

$$\mathcal{I} \subset (T\mathcal{G}_k)_b \oplus (T\mathcal{P})_\pi \oplus TL^{\oplus 2k}$$

to be the subspace of *coincident variations*: to state the definition, let us denote by $\phi^\pi_{b_1,b_2}$ the flow by $P_{\pi(s)}$ on the interval $[b_1, b_2]$, where $b_1 < b_2$. For each $i = 1, \dots, k-1$, differentiating:

$$\phi^\pi_{b_i,b_{i+1}}(p_{i,+}) = p_{i+1,-},$$

gives a linear map:

$$C_i : (T\mathcal{G})_b \oplus (T\mathcal{P})_\pi \oplus (TL)_{p_{i,+}} \to (TL)_{p_{i+1,-}}.$$

Similarly one can differentiate the relation that $\phi^\pi_{-\infty,b_1}(p_{1,-})$ lies in the stable manifold of x_- and the relation that $\phi^\pi_{b_k,+\infty}(p_{k,+})$ lies in the unstable manifold of x_+ as follows: flowing for a sufficiently long time and then projecting onto unstable and stable spaces produces maps:

$$C_0 : (T\mathcal{G}_k)_b \oplus (T\mathcal{P})_\pi \oplus (TL)_{p_{1,-}} \to TU_{x_-},$$
$$C_k : (T\mathcal{G}_k)_b \oplus (T\mathcal{P})_\pi \oplus (TL)_{p_{k,+}} \to TS_{x_+}.$$

In order to lie in the stable/unstable manifold, C_0 and C_k should vanish. We explain these maps with a bit more detail: by definition of Morse-type, P has distinguished linear charts around its critical points (where the flow

is diagonal and linear with no zero eigenvalues). This linear chart gives a decomposition:

$$\mathrm{pr}_U \times \mathrm{pr}_S : \mathrm{Nbd}(x) \to U_x \times S_x,$$

where U_x is the direct sum of the negative eigenspaces and S_x is the direct sum of the positive eigenvalues. By differentiating:

$$\mathrm{pr}_S \circ \phi^{\pi}_{b_k,\tau}(p_{k,+}) \in S_{x_+},$$

for τ sufficiently large, one obtains a map C_k which is well-defined up to composition with an isomorphism. In particular, the kernel of C_k is well-defined independently of τ. The kernel of C_0 is defined similarly.

The variation $(\delta b, \delta\pi, \delta p)$ lies in $\mathcal{I}$ provided:

(1) $\delta p_{i+1,-} = C_i(\delta b \oplus \delta\pi \oplus \delta p_{i,+})$, and,
(2) $C_0(\delta b \oplus \delta\pi \oplus \delta p_{1,-}) = 0$ and $C_k(\delta b \oplus \delta\pi \oplus \delta p_{k,+}) = 0$.

The subspace $\mathcal{I}$ is cut out transversally and has codimension:

$$n(k-1) + \dim TU_{x_-} + \dim TS_{x_+}.$$

Indeed, we are imposing $k-1$ many equations of type 1, each of which imposes conditions that increase the codimension by n, as well as the two equations in 2 which impose conditions that increase the codimension by $\dim TU_{x_-} + \dim TS_{x_+}$. To see that the equations are cut transversally, we observe that, for equations of type 1, transversality can be achieved by variations of $\delta p_{i+1,-}$, and for variations of type 2, one achieves transversality by variations of $\delta p_{k,+}$ and $\delta p_{1,-}$. It follows that one achieves transversality for the whole system of equations by some variation of the form $(\delta p_{1,-}, \delta p_{2,-}, \ldots, \delta p_{k,-1}, \delta p_{k,+})$.

It then follows that the restriction $D_w = AD_w|_{T_w}$ where:

$$T_w = W^{1,p,\delta}(u_1^* TW) \oplus \cdots \oplus W^{1,p,\delta}(u_k^* TW) \oplus \mathcal{I}$$

is a Fredholm operator of index:

$$\mathrm{index}(D_w) = \mu(w) + \dim TU_{x_+} - \dim TU_{x_-} + k + \dim(T\mathcal{P})_\pi.$$

This leads to the definition of a regular solution:

Definition 2.5. *A solution w is regular if* $\dim\ker(D_w) = \mathrm{index}(D_w)$.

It is convenient to introduce $\mathrm{ind}(x) = \dim TU_x$ and the *virtual dimension*:

$$\mathrm{vdim}(w) = \mu(w) + \mathrm{ind}(x_+) - \mathrm{ind}(x_-) + \dim(T\mathcal{P})_\pi - 1. \tag{4}$$

This equals $\mathrm{index}(D_w) - k - 1$, and is the expected dimension of $[w]$ in the quotient by the $\mathbb{R}^{k+1}$ action which acts by translations on the pearls (this gives an $\mathbb{R}^k$ action) and by simultaneous translation of $b \in \mathcal{G}$ and $\pi \in \mathcal{P}$ (this gives an additional $\mathbb{R}$ action).

2.3.4. *Nodal curves.* Unfortunately, to prove finiteness of the counts appearing in d_{eq}, and to prove $d_{eq}^2 = 0$ in the standard way, it will be necessary to enforce additional transversality requirements for certain nodal configuations. These curves can be considered as lying above the compactification of $\mathcal{G}_k$.

Given a surjective non-decreasing map $f : \{1, \dots, k\} \to \{1, \dots, \ell\}$, consider solutions $w = (b, \pi, u_1, \dots, u_k)$ of the *nodal equation*:

$$\begin{cases} u_i : \mathbb{R} \times [0,1] \to (W, L), \\ \partial_s u_i + J_{\pi(b_{f(i)})} \partial_t u_i = 0, \\ 0 < \omega(u_i) < \infty, \end{cases}$$

subject to the incidence equation:

$$\phi^{\pi}_{b_{f(i)}, b_{f(i+1)}}(u_i(+\infty)) = u_{i+1}(-\infty),$$

where the flow ϕ is defined above. This incidence forces $u_i(+\infty) = u_{i+1}(-\infty)$ whenever $f(i) = f(i+1)$. See Figure 4 for an illustration.

Similarly to §2.3.3, such curves also have a linearized operator D_w defined on a space $T_w = W^{1,p,\delta} \oplus \mathcal{I}$ of coincident variations with index:

$$\operatorname{index}(w) = \mu(w) + \operatorname{ind}(x_+) - \operatorname{ind}(x_-) + \ell + \dim \mathcal{P}.$$

This linearized operator has a finite dimensional kernel (consisting of smooth coincident variations). One can quotient the kernel by an $\mathbb{R}^{k+1}$ action; the expected dimension of this quotient space equals the *virtual dimension*:

$$\operatorname{vdim}(w) = \mu(w) + \operatorname{ind}(x_+) - \operatorname{ind}(x_-) + \dim \mathcal{P} - 1 - \mathfrak{n}, \tag{5}$$

(where $\mathfrak{n} = k - \ell$ is the number of nodes). Then:

Definition 2.6. *Borel data (P_η, J_η) is called regular if D_w is surjective for each (nodal) solution w satisfying* $\operatorname{vdim}(w) \leq -\mathfrak{n}$*, and if (P, J) is regular in the sense of §2.2. This includes the case of non-nodal solutions with $\mathfrak{n} = 0$.*

For regular data, the moduli spaces of *free* orbits of the $\mathbb{R}^{k+1}$-action with $\operatorname{vdim} = i$, have the structure of a smooth i-dimensional manifold, for $i \leq 1$; in particular, *non-constant* solutions with negative virtual dimension do not exist. However, the constant solutions $k = 0$, $j = 0$, $i = 0$ do exist in the $\operatorname{vdim} = -1$ component (these are not free orbits of the $\mathbb{R}^{k+1} = \mathbb{R}^1$ action).

As explained in §2.3.2, the solutions with $\operatorname{vdim} = \mathfrak{n} = 0$ contribute to the definition of the Floer differential. For regular Borel data, the counts converge: by monotonicity $\mu(w)$ is non-negative, which combined with the relation given by (5) imply a uniform energy bound on all trajectories. Standard compactness arguments apply and hence the index 0 component is finite. Observe that the $\pi \in \mathcal{P}$ factor varies in a compact space (in the usual Morse-theoretic sense) because $\dim \mathcal{P}$ is a priori bounded above; indeed, along any sequence with $\operatorname{vdim} = \mathfrak{n} = 0$, all the terms in (5) are bounded from below (by $-\dim L$), hence $\dim \mathcal{P}$ must be bounded from above.

Definition 2.7. *We say that (P_η, J_η) is very regular Borel data if it is regular and if additionally all solutions w with $\mathrm{vdim}(w) \leq 1 - \mathfrak{n}$ are regular. This includes the nodal and non-nodal solutions.*

Theorem 2.8. *Given any set-up (W, L, a) as above, there exists very regular Borel data (P_η, J_η) in the sense of Definition 2.7, and such data can be chosen to be arbitrarily close to any chosen Borel data (i.e., very regular data is generic). If the involution a acts freely on W, then one can take (P_η, J_η) to be independent of η. In particular, (very) regular data for the non-equivariant quantum cohomology (see Remark 2.9) also exists.*

We postpone the proof to §A; the argument is based on the one in [BC07].

Remark 2.9. It will be important to also mention when data (P, J) for defining the *non-equivariant* quantum cohomology is (very) regular. As a shortcut, we adopt the following definition: we consider (P, J) as Borel data on the space $W \sqcup W$ where the involution interchanges the two factors (in this case (P_η, J_η) is equal to (P, J) on each factor and is independent of η). If this trivial extension is (very) regular Borel data, then we say (P, J) is *(very) regular for defining the non-equivariant quantum homology.*

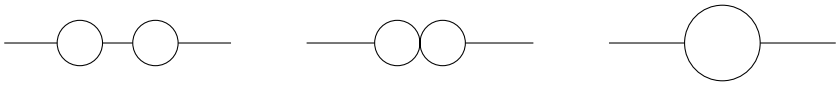


FIGURE 4. Nodal curve lying at the interface of two moduli spaces of non-nodal curves.

We continue to discuss the role of nodal curves in the proof that $d_{eq}^2 = 0$. The ideas are essentially the same as those used in [CL05, BC07, BC09]. The set of curves solving the nodal equation can be considered as compactifying the one-dimensional components in the moduli spaces of non-nodal curves, in a loose sense: a sequence of curves w_n lying above $\mathcal{G}_k$ can fail to converge because b_n diverges in $\mathcal{G}_k$ because some markers collide, say, $\lim b_{n,i} = b_{f(i)}$ for some map f satisfying the requirements of the nodal equation. Ignoring for the moment other possible degenerations, w_n would then converge to a nodal curve with $k > \ell$. By inspection, $\mathrm{vdim}(w) = \mathrm{vdim}(w_n) - (k - \ell)$. Thus the virtual dimension drops by the number of nodal interfaces created.

There is another way to create nodal interfaces: one can have a sequence of curves w_n which fails to converge because some of the $u_{n,i}$ break into a chain of holomorphic disks. Again ignoring the other failures of compactness, one concludes that w_n converges to a nodal curve w with $k > \ell$, and we still have $\mathrm{vdim}(w) = \mathrm{vdim}(w_n) - (k - \ell)$.

As we explained above, the equivariant differential d_{eq} is defined by counting non-nodal solutions with $\mathrm{vdim}(w) = 0$; all of these solutions are non-nodal. To prove d_{eq} squares to zero one needs to consider:

- $\mathcal{M}_1$ the non-nodal pearl trajectories with $\mathrm{vdim}(w) = 1$,

- $\mathcal{M}_{0,\text{node}}$ nodal curves with $k = \ell + 1$ (i.e., only one node) and $\text{vdim}(w) = 0$.

The idea is to glue together different components of $\mathcal{M}_1$ along $\mathcal{M}_{0,\text{node}}$.

2.3.5. *Fibre product curves.* To complete the proof that $d_{eq}^2 = 0$, we consider one final configuration of pearl trajectories. A *fibre product* curve is a tuple of nodal solutions $w = (w_1, \ldots, w_m)$, subject to the following conditions:

(1) $x_+(w_i) = x_-(w_{i+1})$, *(in the twisted sense defined above)*,
(2) $\text{vdim}(w_i) \geq 0$.

Set $x_+(w) = x_+(w_m)$ and $x_-(w) = x_-(w_1)$. We define the virtual dimension of a fibre product curve as the sum of $\text{vdim}(w_i)$. Let us call the interface between w_i and w_{i+1} a *Morse breaking*.

Denote by $\mathcal{M}_{\text{break}}(x, y)$ the collection of fibre product curves with $x_-(w) = x$ and $x_+(w) = y$, only one Morse breaking, and virtual dimension 0. These appear as ends of the 1-dimensional component $\mathcal{M}_1(x, y)$.

2.3.6. *Proof of Theorem 2.4.* First we prove it for very regular data following the ideas of [BC07], and the ideas of [SS10, Sei15] for the $\pi \in \mathcal{P}$ component; one shows:

$$\#\partial\mathcal{M}_1(x, y) + 2\#\mathcal{M}_{\text{node}}(x, y) + \#\mathcal{M}_{\text{break}}(x, y) = 0 \bmod 2$$

because each element of $\mathcal{M}_{\text{node}}(x, y)$ can be glued into two different ways to produce ends of moduli spaces in $\mathcal{M}_1$, while each Morse breaking can only be glued in one way. Moreover, the count of $\mathcal{M}_{\text{break}}$ represents the coefficients appearing in d_{eq}^2, and one thereby concludes that $d_{eq}^2 = 0$ modulo two.

The trick to upgrade from very regular (Definition 2.7) to just regular (Definition 2.5) is the following: one uses that regular data can be approximated by a converging sequence of very regular data (by Theorem 2.8). Then, because we begin with regular data, the counts used in the definition of d_{eq} stabilize in this approximating sequence. Thus one can replace the regular data by very regular data without changing d_{eq}, and the desired result follows. For a similar argument, see [BK13, §6.3]. □

In the remaining sections §2.3.7, §2.3.8, §2.3.9, and §2.3.10 we comment on some of the technical points of the compactness and gluing results implicitly used in the above proof. The reader who is comfortable with the above proof that $d_{eq}^2 = 0$ is encouraged to continue to §2.4.

2.3.7. *Gradient bound.* Consider a sequence of solutions w_n for regular Borel data satisfying $\text{vdim}(w_n) \leq 1$. Then:

Lemma 2.10 (A priori C^1 bound)**.** *If $x_-(w_n) \neq x_+(w_n)$ and $\text{vdim}(w_n) \leq 1$, then the first derivatives of each pearl $u_{n,i}$ are uniformly bounded.*

Proof. If not, then a disk bubble or sphere bubble forms. By standard compactness arguments, one concludes that w_n converges to a (nodal, fibre product) underlying solution after removing small neighborhoods of the bubbling regions.

The limiting underlying solution w_∞ has virtual dimension:

$$\mathrm{vdim}(w_\infty) = \mathrm{vdim}(w_n) - \mu(\text{bubbles}) - \#\{\text{additional nodes/breakings}\}.$$

Because the minimal Maslov number is at least 2, we have $\mathrm{vdim}(w_\infty) \leq -1$. If $\mathrm{vdim}(w_\infty) \leq -1$, then $x_- = x_+$, which we exclude from the lemma. Thus there could not have been bubbling. □

2.3.8. *The exceptional case.* Establishing compactness when $x_- = x_+$ is more subtle, but it only occurs in the moduli space with $\mu(w) = 2$, $k = 1$, and $\dim \mathcal{P} = 0$ (and is therefore present in the non-equivariant differential). For this, we appeal to the regularity of (P, J) in the sense of §2.2. The component of curves with $\mathrm{vdim}(w) = 1$ joining x to x is:

$$\mathrm{ev}_2^{-1}(S_x \times U_x) \setminus \{z_1 = z_2\},$$

where ev_2 is the evaluation map of §2.2. Thus we work instead with the compactification:

$$\Pi_2^x = \mathrm{ev}^{-1}(S_x \times U_x),$$

which is the 1-manifold studied in §2.2. In this manner, we see that the bubbling off of disks with $\mu = 2$ in this exceptional component behaves as a sort of removable singularity; see [Oh95].

Let us note that $\partial\Pi_2^x$ can be described explicitly and consists of fibre product curves; see §2.2. The count of these boundary points yields the relation:

$$\left\langle x, d_{eq}^2(x) \right\rangle = 0.$$

The gluing result needed to characterize $\partial\Pi_2^x$ follows from standard transversality methods: the evaluation map ev is sufficiently submersive that one can make it transverse to $S_x \times U_x$, and its codimension 1 boundary strata, and taking preimage of a submanifold with boundary yields a submanifold with boundary.

For the rest of the compactness argument, we will assume that $x_+ \neq x_-$ in the 1-dimensional components.

2.3.9. *Gromov-Floer compactness.* We are concerned with the compactness of the moduli space quotiented by the $\mathbb{R}^{k+1}$ action. Because of the gradient bound, and because we work in a convex-at-infinity symplectic manifold, we can apply the Arzelà-Ascoli theorem to conclude Floer-type compactness along sequences w_n in the component with $\mathrm{vdim}(w_n) \leq 1$. Because the pearls are defined on strips, each pearl can break into a chain of holomorphic strips (this is one way the nodal solutions of §2.3.4 are formed); see Figure 4.

One important subtlety introduced by the Borel construction is that the flow line $[\pi_n] \in \mathcal{P}/\mathbb{R}$ may diverge (and break in the Morse theory sense). In this case, w_n will converge to a fibre product curve; $[\pi_n]$ converges in the usual Morse theoretic sense to a configuration of flow lines, but with the caveat that the flow lines appearing in the limit are all transported by self-similarity map τ and the involution so as to start at $v_{0,+}$ (this is merely a question of interpretation of the limit). This part of the story is treated carefully in [Sei15, pp. 970].

Similarly one can have divergence in $\mathcal{G}_k$ which corresponds to nodal degenerations (when two points collide), or fibre product curves (when points drift off to $\pm\infty$). Finally, one can always have breakings of Morse flow lines for P at either end, even if $[(\pi_n, b_n)] \in (\mathcal{P} \times \mathcal{G}_k)/\mathbb{R}$ converges. The analysis needed to "capture" all the components of the limit is fairly standard by now; we refer the reader to [BC07, §5.1], [Sch16, §4 and 9], [Sei15, pp. 972] for further details.

2.3.10. *Gluing.* There are three types of "interfaces" one needs to consider:

(1) node removal by gluing disks,
(2) node removal by adding flow segments,
(3) fibre product gluing.

Each type of result is essentially a gluing result: one uses regularity of the solution w to build "pre-glued" approximate solutions $\bar{w}_R$ depending on a parameter R. For R sufficiently large, one proves the pre-glued solutions $\bar{w}_R$ is close to a unique one-parameter family of solutions w_R diverging to w.

The overall strategy and techniques of gluing are by now standard; see, e.g., [BC07, §4], [Sch16, §7] and also [BC24, §4], [BC25, §4.4], for details.

2.4. *Continuation isomorphisms.* In this section, we explain the invariance of the equivariant quantum cohomology with respect to the choice of Borel data (P_η, J_η). The main results are the following. First, one defines a category $\mathcal{D}$ whose objects are choices of regular Borel data (P_η, J_η) and whose morphisms are homotopy classes of paths of Borel data with fixed endpoints. Then one shows $\mathcal{D}$ is an indiscrete groupoid (the choice of data is contractible). Finally one shows the assignment $(P_\eta, J_\eta) \mapsto \mathrm{QH}^*_{eq}(P_\eta, J_\eta)$ extends to a functor defined on $\mathcal{D}$, taking values in the category of modules over the algebra $\mathbf{k}[q^{-1}, q]] \otimes \mathbf{k}[e]$. It follows that all of the quantum cohomologies for different choics of regular data are canonically isomorphic.

2.4.1. *Continuation data.* We define *Borel continuation data* to be a one-parameter family $P_{\eta,\sigma}, J_{\eta,\sigma}$ of Borel data where $\sigma \in \mathbb{R}$. The data should be σ-independent outside of a finite interval $(-\sigma_0, \sigma_0)$. We allow one relaxation: when $\sigma \in (-\sigma_0, \sigma_0)$, $P_{\eta,\sigma}$ does not need to be a Morse pseudogradient when η is at a pole, (it can be an arbitrary vector field). However, for $\sigma \notin (-\sigma_0, \sigma_0)$, the Borel continuation data is required to be regular in the sense of Definition

2.3.3. Let us denote by $P_{\eta,\pm}, J_{\eta,\pm}$ the limiting Borel data as $\sigma \to \pm\infty$, and by $P_\pm, J_\pm$ the restrictions of the Borel data to the pole $v_{0,+}$.

Homotopy classes of Borel continuation data with fixed endpoints are defined in the obvious way, and such homotopy classes can be concatenated, endowing $\mathcal{D}$ with the structure of a category. It is readily verified that for any two choices of regular Borel data, there is a unique homotopy class of continuation data between them, (since the space of almost complex structures and vector fields on L is contractible); thus $\mathcal{D}$ is an indiscrete groupoid.

2.4.2. *Moduli space of continuation trajectories.* Consider the moduli space of solutions $w = (b, \pi, u_1, \dots, u_k)$ to the problem:

$$\begin{cases} b = (b_1, \dots, b_k) \in \mathcal{G}_k \text{ and } \pi \in \mathcal{P}^{j,\pm}, \\ \partial_s u_\nu + J_{\pi(b_\nu),b_\nu}(u_\nu)\partial_t u_\nu = 0, \\ 0 < \omega(u_\nu) < \infty. \end{cases}$$

subject to the incidence condition that the flow line of $P_{\pi(s),s}$ for $s \in [b_\nu, b_{\nu+1}]$ joins $u_\nu(+\infty)$ to $u_{\nu+1}(-\infty)$; the asymptotics $x_-(w)$ and $x_+(w)$ are as in the definition of the equivariant differential. In the case $\pi^{j,-}$, the $x_+(w)$ asymptotic is the twist by a of the underlying geometric asymptotic zero; this is necessary so that $x_\pm(w)$ are zeros of $P_\pm$.

We denote by $\mathcal{M}_{\mathrm{cont}}(x, y)$ the moduli space of such continuation trajectories with $x_-(w) = x$ and $x_+(w) = y$. Similarly to §2.3.3, each w determines a certain linearized operator D_w defined on:

$$W^{1,p,\delta}(u_1^*TW, u_1^*TL) \oplus \cdots \oplus W^{1,p,\delta}(u_k^*TW, u_k^*TL) \oplus \mathcal{I}_w,$$

where $\mathcal{I}_w$ is a certain subspace of $TL^{2k} \oplus T\mathcal{P} \oplus T\mathcal{G}_k$, and D_w is a Fredholm operator of index:

$$\operatorname{index}(D_w) = \mu(u_1) + \cdots + \mu(u_k) + \dim \mathcal{P} + \operatorname{ind}(x_+) - \operatorname{ind}(x_-) + k.$$

We say that w is *regular* provided that D_w is surjective. Due to the free action by $\mathbb{R}^k$, it is convenient to the introduce the virtual dimension as $\operatorname{vdim}(w) = \operatorname{index}(D_w) - k$.

One defines nodal continuation trajectories in the obvious way, combining the definition in §2.3.4 with the above definition of continuation trajectories. Nodal solutions also have a virtual dimension as in §2.3.4, the general formula for the virtual dimension is:

$$\operatorname{vdim}(w) = \mu(w) + \dim \mathcal{P} + \operatorname{ind}(x_+) - \operatorname{ind}(x_-) - \mathfrak{n},$$

where $\mathfrak{n}$ is the number of nodes. Emulating the definition of regular and very regular used in §2.3.2, we have:

Definition 2.11. *Continuation data $(P_{\eta,\sigma}, J_{\eta,\sigma})$ is regular provided all (nodal) continuation trajectories with $\operatorname{vdim}(w) \leq -\mathfrak{n}$ are regular, and $P_{\eta,\pm}, J_{\eta,\pm}$ are regular in the sense of Definition 2.6. Similarly, continuation data is called*

very regular provided it is regular and additionally all (nodal) continuation trajectories with $\mathrm{vdim}(w) \leq 1 - \mathfrak{n}$ *are regular.*

In §A.1 we explain why very regular continuation data is generic, in a manner analogous to Theorem 2.8.

2.4.3. *Continuation map.* The continuation map associated to regular continuation data is defined by appropriately counting the non-nodal continuation trajectories with $\mathrm{vdim}(w) = 0$. To be precise, we define:

$$\mathfrak{c}(x_-) := \sum q^{\mu(w)} e^{\dim \mathcal{P}} x_+(w);$$

the sum is over $\mathbb{R}^k$-orbits of continuation solutions $w = (b; \pi; u_1, \ldots, u_k)$ with $\mathrm{vdim}(w) = 0$, $\mathfrak{n} = 0$, and $x_-(w) = x_-$. Here $\mu(w)$ is the total Maslov number of the pearls, and $\mathcal{P}$ is the space of flow lines containing π.

Modulo issues of convergence of sums, this defines a map:

$$\mathfrak{c} : \mathrm{CM}(P_-) \to \mathrm{CM}(P_+) \otimes \mathbf{k}[q] \otimes \mathbf{k}[e], \tag{6}$$

which is extended to a map of $\mathbf{k}[q^{-1}, q]] \otimes \mathbf{k}[e]$-modules.

Lemma 2.12. *The map written as* (6) *is well-defined (the sum converges).*

Proof. Because the sum is only over solutions w with $\mathrm{vdim}(w) = 0$, there is a uniform a priori upper bound on k (the number of pearls), $\dim \mathcal{P}$, and $\mu(w)$, similarly to the argument in §2.3.4. Here it is important that the pearls appearing in w are holomorphic (for some ω-tame almost complex structure) so that (because of monotonicity) the Maslov number of each pearl is strictly positive. Thus, in order to prove convergence, it suffices to prove the number of $\mathbb{R}^k$-orbits of solutions with fixed k, $\mathcal{P}$, and $\mathrm{vdim}(w) = 0$ is finite. This is then a standard Gromov-Floer compactness argument; if there was a diverging sequence, one would necessarily conclude the existence of a continuation trajectory with negative virtual dimension, contradicting regularity.

□

2.4.4. *Chain map property.* The fact that the continuation map is a chain map with respect to d_{eq} is typically proved by analyzing the moduli space of continuation solutions with $\mathrm{vdim}(w) = 1$ and $\mathfrak{n} = 0$. Similarly to the proof of $d_{eq}^2 = 0$, one considers the nodal solutions with $\mathrm{vdim}(w) = 0$ and $\mathfrak{n} = 1$ as an interface between different components.

The standard arguments (in the spirit of the proof that $d_{eq}^2 = 0$) show that, for very regular continuation data, it holds that $d_{eq} \circ \mathfrak{c} = \mathfrak{c} \circ d_{eq}$. then:

Theorem 2.13. *For regular continuation data, it holds that* $d_{eq} \circ \mathfrak{c} = \mathfrak{c} \circ d_{eq}$.

Proof. First one proves it for very regular continuation data as explained above. Then one follows the same approximation argument used in §2.3.6 to deduce it for regular data. □

2.4.5. *Homotopies of continuation data give chain homotopic maps.* The argument follows more or less verbatim the argument of [HS95] (in Hamiltonian Floer theory). See also [BC07, §5.1.2] for a related argument in the setting of pearl complexes.

Let $(P_{\eta,\sigma,\tau}, J_{\eta,\sigma,\tau})$ be a homotopy of continuation data, defined for $\tau \in [0,1]$, with fixed endpoints:

- $P_{\eta,\pm\sigma,\tau} = P_{\eta,\pm,0}$ and $J_{\eta,\pm\sigma,\tau} = J_{\eta,\pm,\tau}$ for $\sigma \geq \sigma_0$.

Then one can consider the parametric moduli space of pairs (τ, w) where w solves the (nodal) continuation equation for the data at parameter τ. After quotienting by the $\mathbb{R}^k$ action on by translation of the pearls, such solutions have virtual dimension:

$$\mathrm{vdim}(\tau, w) = \mu(w) + \dim \mathcal{P} + \mathrm{ind}(x_+) - \mathrm{ind}(x_-) + 1 - \mathfrak{n};$$

provided the solution space is regular (in the parametric sense) the dimension of the quotiented moduli space near (τ, w) agrees with the above virtual dimension. If the endpoints of the path are regular continuation data, then one can ensure the parametric moduli space is regular by a generic perturbation of $(P_{\eta,\sigma,\tau}, J_{\eta,\sigma,\tau})$ for $\tau \in (0,1)$.

The component with $\mathrm{vdim}(w,\tau) = 1$ and $\mathfrak{n} = 0$, is used to prove the continuation maps defined using the data at endpoints $\tau = 0, 1$ are chain homotopic (as before, one needs to compactify this component by gluing along the solutions with $\mathrm{vdim}(w,\tau) = 0$ and $\mathfrak{n} = 1$ considered as interfaces between moduli spaces with different numbers of pearls). The chain homotopy term is defined by counting the component with $\mathrm{vdim}(w,\tau) = 0$ and $\mathfrak{n} = 0$.

Consequently $\mathfrak{c} : \mathrm{QH}^*_{eq}(L; P_{\eta,-}, J_{\eta,-}) \to \mathrm{QH}^*_{eq}(L; P_{\eta,+}, J_{\eta,+})$ *is independent of the choice of regular continuation* $(P_{\eta,\sigma}, J_{\eta,\sigma})$.

2.4.6. *Concatenations yield compositions of continuation maps.* This relies on §2.4.5 and a gluing result similar to those invoked already in §2.3.10. The outline of the argument is still the same as the one in [HS95], and we omit the verification that the standard argument works as expected. The outcome is: *the composition of two continuation maps is a continuation map associated to the concatenation.*

2.4.7. *The continuation endomorphism equals the identity map.* Suppose that $(P_{\eta,\sigma}, J_{\eta,\sigma}) = (P_\eta, J_\eta)$ is regular Borel data (for defining d_{eq}), considered as σ-independent continuation data. Then the only solutions which contribute to the continuation map are the constant solutions localized to zeros of P, in the component $k = 0$ and $\dim \mathcal{P} = 0$. It follows that this continuation data is regular for defining $\mathfrak{c}$, and $\mathfrak{c}$ is the identity map. With this established, the functoriality of $(P, J) \mapsto \mathrm{QH}^*(L; P, J)$ is proved, and the invariance of $\mathrm{QH}^*(L; P, J)$ follows. See [BC07, §5.1.2] and [Sch16, §10.3] for related discussion of invariance.

2.5. *Equivariant displacements.* In this section we prove that $\mathrm{QH}^*_{eq}(L) = 0$ if L is equivariantly displaceable (Theorem 1.7). The idea is to deform the continuation trajectories from §2.4 using a Hamiltonian perturbation term.

The deformation trajectories we consider will involve holomorphic pearls, except that one pearl is allowed to be non-holomorphic, and is instead a solution of Floer's continuation equation.

2.5.1. *Deformation pearls.* Let J be an ω-tame almost complex structure, and let $X_{s,t}$ be a family of Hamiltonian vector fields whose generator $H_{s,t}$ satisfies $H_{s,t} = 0$ for $|s| \geq s_0$.

A *deformation pearl* for this data is simply a map:

$$\begin{cases} u : \mathbb{R} \times [0,1] \to (W, L), \\ \partial_s u + J(u)(\partial_t u - X_{s,t}(u)) = 0, \\ \int \omega(\partial_s u, \partial_t u - X_{s,t}(u)) < \infty. \end{cases}$$

We note that deformation pearls have well-defined asymptotics $u(\pm\infty) \in L$. Unlike the pearls which appear in the quantum differential or continuation isomorphism, deformation pearls can have negative Maslov number. However, the non-negative energy $E(u)$ (the above integral) is equal to:

$$E(u) = \omega(u) + \int \partial_s H_{s,t}(u) ds dt,$$

which implies $\omega(u)$ is bounded from below by minus the curvature:

$$\text{curvature} = \sup \left\{ \int \partial_s H_{s,t}(v) ds dt \right\},$$

where the supremum is over all smooth maps v. The relevant Floer theory constructions we will appeal to naturally yield Hamiltonians $H_{s,t}$ with a priori bounds on their curvature. This curvature bound and monotonicity imply that the Maslov index of deformation pearls cannot be too negative; we return to this point in §2.5.3.

2.5.2. *Deformation trajectories.* Define *deformation data* as:

- a family of Hamiltonians $H_{\eta,s,t}$ as in §2.5.1 parametrized by $\eta \in S^\infty$,
- Borel data (P_η, J_η) for defining d_{eq} as in §2.3.1,

where $H_{\eta,s,t}$ satisfies the properties:

(H1) $H_{-\eta,s,t} = a^* H_{\eta,s,t}$,
(H2) $H_{\tau(\eta),s,t} = H_{\eta,s,t}$,
(H3) $H_{\eta,s,t} = 0$ for $|s| \geq s_0$,
(H4) the curvature of $H_{\eta,s,t}$ is bounded from above by a constant C_0.

Different choices of deformation Borel data are allowed different choices for the constants C_0 and s_0, but these constants should be uniform in the variable $\eta \in S^\infty$.

Given such data, consider the solutions $(b, \pi, u_1, \dots, u_k)$ to the *deformation equation*:

$$\begin{cases} b \in \mathcal{G}_k^{\times} \text{ and } \pi \in \mathcal{P}, \\ u_\nu : \mathbb{R} \times [0,1] \to (W, L), \\ \partial_s u_\nu + J_{\pi(b_\nu)}(u_\nu)\partial_t u_\nu = 0 \text{ except for } \nu = m \\ \mu(u_\nu) > 0 \text{ except for } \nu = m. \end{cases}$$

The missing pearl u_m instead solves the deformation pearl equation:

$$\partial_s u_m + J_{\pi(b_m)}(u_m)(\partial_t u_m - X_{\pi(b_m),s,t}(u_m)) = 0,$$

and is allowed to have non-positive Maslov number. Each deformation trajectory has exactly one deformation pearl. The deformation pearl may be a constant map. Solutions must satisfy the following incidence condition:

- the flow line of $P_{\pi(s)}$ on the interval $s \in [b_i, b_{i+1}]$ takes $u_i(+\infty)$ to $u_{i+1}(-\infty)$ for $i = 1, \dots, k-1$.

The asymptotics $x_\pm$ of such a solution are obtained by flowing the initial and terminal points and twisting by a depending on whether $\pi \in \mathcal{P}^-$ or $\pi \in \mathcal{P}^+$, as in the pearl differential.

As in §2.3.3, this equation determines a linearized operator acting on:

$$W^{1,p,\delta}(u_1^* TW) \oplus \cdots \oplus W^{1,p,\delta}(u_k^* TW) \oplus \mathcal{I}_{b,\pi,p}$$

where $\mathcal{I}_{b,\pi,p} \subset T\mathcal{G}_{k,b} \oplus T\mathcal{P}_\pi \oplus TL_p^{\oplus 2k}$ forms the coincident variations: here $p = (u_1(-\infty), u_1(+\infty), \dots, u_k(-\infty), u_k(+\infty))$.

By a similar argument to the one given in §2.3.3, this linearized operator is a Fredholm operator with index:

$$\operatorname{index}(w) = \mu(w) + \operatorname{ind}(x_+) - \operatorname{ind}(x_-) + k + \dim \mathcal{P}.$$

Now there is only an $\mathbb{R}^k$-action on the moduli space rather than a $\mathbb{R}^{k+1}$-action — this is because the deformation pearl should not be translated.

Similarly to §2.3.4, one also considers nodal solutions, where nodal chains of holomorphic disks located at the same position b_i are allowed (this includes chains of holomorphic pearls at either end of the deformation pearl). The virtual dimension formula for a configuration with k pearls and $\mathfrak{n}$ nodes is:

$$\operatorname{vdim}(w) = \operatorname{index}(w) - k = \mu(w) + \operatorname{ind}(x_+) - \operatorname{ind}(x_-) + \dim \mathcal{P} - \mathfrak{n}.$$

This leads to the definition of regular deformation data:

Definition 2.14. *Data $(P_\eta, J_\eta, H_{\eta,s,t})$ is said to be regular deformation data provided that:*

(1) P_η, J_η is regular Borel data for defining the quantum differential,
(2) each (nodal) deformation trajectory with $\operatorname{vdim}(w) \leq -\mathfrak{n}$ has a surjective linearized operator.

If instead we replace the second item by:

(3) each (nodal) deformation trajectory with $\mathrm{vdim}(w) \leq 1 - \mathfrak{n}$ *has a surjective linearized operator,*

then the data is said to be very regular.

Let us note that that the constant data $H_{\eta,s,t} = 0$ is very regular deformation data provided (P_η, J_η) is regular data for the quantum differential; the argument is the same one given in §2.4.7. In §A.1 we claim that very regular deformation data is generic, in a manner analogous to Theorem 2.8.

2.5.3. *The deformed continuation map.* Let $(P_\eta, J_\eta, H_{\eta,s,t})$ be regular deformation data. Then one can count the moduli space of orbits of the $\mathbb{R}^k$ action on the component with $\mathrm{vdim}(w) = 0$ and $\mathfrak{n} = 0$. In this count, one considers all the moduli spaces, allowing any possible choice for:

- the number of pearls k,
- the flow line space $\mathcal{P}$ on S^∞ starting at $v_{0,+}$,
- the total Maslov number, and,
- the position $\ell \in \{1, \ldots, k\}$ of the deformation pearl.

The count takes a generator $x_- \in \mathrm{CM}(P)$ as input, and returns an element of $\mathrm{CM}(P) \otimes \mathbf{k}[q^{-1}, q]] \otimes \mathbf{k}[e]$ as output:

$$\mathfrak{h}(x_-) := \sum e^{\dim \mathcal{P}} q^{\mu(w)} x_+(w).$$

We recall that, as always, the asymptotics are appropriately twisted by the involution a, as explained in §2.3.2.

Lemma 2.15. *The sum over the* $\mathrm{vdim}(w) = \mathfrak{n} = 0$ *component is finite for regular deformation data.*

Proof. The crux of the matter is bounding $\dim \mathcal{P}$ and μ from above. Once these bounds are established, it is easy to prove compactness following a standard argument we outline at the end of the proof.

The vanishing of the virtual dimension, and the positivity of $\dim \mathcal{P}$ and $\mu(u_i)$ for holomorphic pearls show that it is sufficient to bound $\mu(u_m)$ from below for the deformation pearl.

We have $\mu(u_m) = N\omega(u_m)$ after rescaling ω if necessary. The energy identity from Floer theory §2.5.1 gives:

$$\omega(u_m) > -(\text{curvature of } H_{\pi(b_m),s,t}) \geq -C_0$$

where C_0 is from the hypotheses of deformation data §2.5.2. It therefore follows that $\mu(u_m)$ is bounded from below by $-NC_0$. Then we have:

$$\dim \mathcal{P} \leq \dim L + NC_0.$$

Similarly the total Maslov number of the holomorphic pearls is bounded from above by $\dim L + NC_0$. This also implies that the number of pearls k is bounded from above.

Now suppose one has a failure of compactness represented as a sequence w_n. One can pass to a subsequence so that $\mathcal{P}$, k, and the choice $m \in \{1, \ldots, k\}$ are fixed along w_n.

Then, this failure of compactness must produce:

- a limiting (nodal) solution with $\mathrm{vdim}(w_\infty) < 0$ which is regular;

note that the limiting solution may be non-nodal if it appears as one component of a Morse-type breaking (fibre product curve in §2.3.5) or by some bubbling off of holomorphic spheres or disks. Such a limiting solution cannot exist by regularity. If a nodal solution develops it is straightforward to verify that $\mathrm{vdim}(w_\infty) \leq -\mathfrak{n}$, and thus the limiting solution is automatically regular. □

2.5.4. *Chain map property.* It follows from the preceding Lemma that $\mathfrak{h}$ is a well-defined endomorphism of $\mathrm{CM}(P) \otimes \mathbf{k}[q^{-1}, q]] \otimes \mathbf{k}[e]$ when it defined using regular data. We show:

Lemma 2.16. *It holds that* $\mathfrak{h} \circ d_{eq} = d_{eq} \circ \mathfrak{h}$ *for regular data.*

Proof. As explained in §2.3.6, it suffices to prove it for very regular data (see Definition 2.14). We count the ends of the moduli space of deformation trajectories in the component $\mathrm{vdim}(w) = 1$ and $\mathfrak{n} = 0$. These ends come in two types:

(1) fibre-product configurations contributing to $\mathfrak{h} \circ d_{\mathrm{eq}}$ or $d_{\mathrm{eq}} \circ \mathfrak{h}$, (which case is determined by where the deformation pearl ends up),
(2) nodal configurations with $\mathrm{vdim}(w) = 0$ and $\mathfrak{n} = 1$.

Other failures of compactness can be precluded using the arguments in the proof of Lemma 2.15. By the aforementioned gluing results §2.3.10, each end of the second type appears exactly twice, and each end of the first type appears exactly once. One concludes the desired relation. □

2.5.5. *Chain homotopy property.* It follows that $\mathfrak{h}$ induces a well-defined endomorphism of the homology $\mathrm{QH}^*_{eq}(L; P_\eta, J_\eta)$. We show:

Lemma 2.17. *It holds that* $\mathfrak{h} = \mathrm{id}$ *as maps on* $\mathrm{QH}^*_{eq}(L; P_\eta, J_\eta)$*, provided that* $(P_\eta, J_\eta, H_{\eta,s,t})$ *is regular deformation data.*

Proof. First we claim that the chain homotopy class of $\mathfrak{h}$ depends only on the homotopy class of $H_{\eta,s,t}$ in the space of data satisfying (H1) through (H4). The argument establishing this is an application of parametric moduli spaces; see §2.4.5. Then we claim that $\tau \mapsto (P_\eta, J_\eta, \tau H_{\eta,s,t})$ is a homotopy between $(P_\eta, J_\eta, 0)$ and $(P_\eta, J_\eta, H_{\eta,s,t})$, satisfying (H1) through (H4) throughout the homotopy. By the same argument in §2.4.7, it follows that the map $\mathfrak{h}$ defined using $(P_\eta, J_\eta, 0)$ is the identity map on chain level; the desired result follows. □

2.5.6. *Vanishing property.* We now come to the crucial property, that $\mathfrak{h}$ vanishes in the presence of an equivariant displacement of L. This section will complete the proof of Theorem 1.7. In fact, we will prove a seemingly stronger statement. Suppose that $H_{\eta,t}$ is a family satisfying:

- $H_{-\eta,t} = a^* H_{\eta,t}$,
- $H_{\tau(\eta),t} = H_{\eta,t}$,
- $\int \max H_{\eta,t} - \min H_{\eta,t} dt$ is uniformly bounded from above.

Then, if we define, for a standard cut-off function[6] β,

$$H_{\eta,s,t} = \beta(s_1 - s)\beta(s + s_1)H_{\eta,t}, \tag{7}$$

we obtain a system satisfying (H1) through (H4), for any choice of s_1. Moreover, the curvature bound is independent of s_1. One obvious choice of $H_{\eta,t}$ is $H_{\eta,t} = H_t$ where H_t is equivariant. To ensure regularity, one allows a small η-dependent perturbation which can be made without increasing the curvature by more than any small amount ϵ; these perturbations should be supported outside $[1 - s_1, s_1 - 1]$. Then:

Lemma 2.18. *If the isotopy generated by $H_{\eta,t}$ displaces L for each η, then the deformation map $\mathfrak{h}$ defined using* (7) *vanishes for s_1 sufficiently large.*

Proof. The energy of the deformation pearl u_m is bounded from above by $\omega(u_m)$ plus the curvature of $H_{\pi(b_m),s,t}$; by our assumption, The curvature term is uniformly bounded from above. Consequently, the symplectic area cannot be too negative. It follows (by monotonicity) that the Maslov class $\mu(u_m)$ also cannot be too negative, and hence $\dim \mathcal{P}$ cannot be too large (each rigid element appearing in the count corresponds to some underlying flow line $\pi \in \mathcal{P}$, and the spaces which appear in rigid elements cannot have a too high dimension if $\mu(u_m)$ is bounded below).

On the interval $[1 - s_1, s_1 - 1]$, u_m solves Floer's equation for $H_{\pi(b_m),t}$. Let

$$\nu(s_1) = \lfloor 2s_1 - 2 \rfloor,$$

and observe that $[1 - s_1, s_1 - 1]$ fits $\nu(s_1)$ disjoint intervals of length 1 in its interior.

In particular, inside the domain of u_m, there are $\nu(s_1)$ disjoint strips of side length 1 solving Floer's equation for $H_{\pi(b_m),t}$. We claim that: *such strips carry a minimum quantum of energy $\hbar > 0$.* One proves this by a compactness argument[7] and the fact that zero energy solutions to Floer's equation imply intersection points between L and its image under the limiting Hamiltonian isotopy (one passes to a limit so $\pi(b_m)$ converges). Since

[6] That is, $\beta(s) = 0$ for $s \leq 0$, $\beta(s) = 1$ for $s \geq 1$ and β is smooth and non-decreasing.

[7] Here we use that uniformly small energy on the larger region $[0, 1] \times [0, 1]$ imply a priori C^∞ bounds on the interior domain $[\epsilon, 1 - \epsilon] \times [0, 1]$, for any fixed $\epsilon > 0$, and therefore the restrictions to the shrunken domains converge, up to a subsequence, by the theorem of Arzelà and Ascoli.

the Hamiltonian isotopies displaces L, by assumption, no such zero energy strip exists. Consequently, the energy of u_m on $[-s_1+1, s_1-1]$ is at least $\nu(s_1)\hbar$.

Thus the (total) energy of u_m must go to infinity as $s_1 \to \infty$; the only way this can happen is if the symplectic area $\omega(u_m)$ goes to infinity, and the only way *this* can happen is if $\mu(u_m)$ goes to infinity (again, by monotonicity). But if $\mu(u_m)$ is too large, then u_m will not appear in a rigid element, by the dimension formula for the relevant moduli spaces.

Therefore, for s_1 large enough, no deformation pearl can exist, and so the map $\mathfrak{h}$ must vanish identically on chain level. This completes the proof of Theorem 1.7. □

3. *The Biran-Khanevsky construction and the proof of Theorem 1.8*

To prove Theorem 1.8, we introduce a special class of Borel data (P_η, J_η) which we call *Biran-Khanevsky type*, inspired by the paper [BK13]. To construct such data, we fix once and for all data (Q, J) for defining the quantum cohomology complex of $\bar{L}$ inside of the reduction $M = \partial\Omega/(\mathbb{R}/\mathbb{Z})$. The circle bundle $L \to \bar{L}$ is endowed with a flat connection coming from the contact form on $\partial\Omega$ (as in [BK13, pp. 900]). Consequently, there exists a unique horizontal lift of Q to TL, which we denote by P_{hor}.

Unfortunately P_{hor} is not a Morse pseudogradient vector field. We explain how to modify this. Pick disjoint coordinate disks around each zero z of Q, on which the flow is diagonal and linear, with radial coordinate ρ, and trivialize the circle bundle over these neighborhoods, obtaining a new fibre coordinate $x_0 \in \mathbb{R}/\mathbb{Z}$. Given $\epsilon > 0$ and $\theta \in \mathbb{R}$, define:

$$V_{z,\epsilon,\theta} = \beta(1 - \epsilon^{-1}\rho)\sin(2\pi(x_0 - \theta))\partial_{x_0}, \tag{8}$$

where β is a standard cut-off function. For $\theta : \{\text{zeros of } Q\} \to \mathbb{R}$, define the pseudogradient:

$$P_{\epsilon,\theta} = P_{\text{hor}} + \sum_z V_{z,\epsilon,\theta(z)},$$

where the sum is over zeros z of Q. Since $\rho \circ a = \rho$ and $x_0 \circ a = x_0 - 0.5$, it holds that:

$$a^* P_{\epsilon,\theta} = P_{\epsilon,\theta+0.5}.$$

This vector field $P_{\epsilon,\theta}$ is a Morse pseudogradient, is related[8] to Q under the projection $L \to \bar{L}$, and there are two critical points $x(z), y(z)$ in the fibre over each zero z of Q, where $\text{ind}(y(z)) = \text{ind}(x(z)) + 1$.

For the remainder of this section, we will consider $\epsilon \in (0,1)$ fixed and θ as a generic choice, and abbreviate $P = P_{\epsilon,\theta}$. The following lemma is in §3.1.

[8]Here "related" means that $d\text{pr}_x P(x) = Q(\text{pr}(x))$, where $\text{pr} : L \to \bar{L}$ is the projection

Lemma 3.1. *Fix $\epsilon \in (0,1)$. There is a dense open set of choices of θ which render $P_{\epsilon,\theta}$ a Morse-Smale pseudogradient.*

Then the genericity results in §2.3.1 imply P can be extended to some regular Borel data (P_η, J_η). However, we will need to pick our extension (P_η, J_η) from a highly restricted class. To state the condition, we introduce the *symplectization end* $SY \subset W$. Recall that $SY \subset W$ is formally defined to be the set of points $z \in W$ whose future Liouville flowline escapes to infinity. The Liouville flow on SY induces a free $\mathbb{R}$-action, and the quotient space $Y = SY/\mathbb{R}$ carries a canonical contact structure. Reeb flows on Y are in 1-to-1 correspondence with sections $Y \to SY$; the images of these sections in W are hypersurfaces which bound compact domains. Since W is constructed as the completion of a compact Liouville domain Ω, there is a preferred such hypersurface, namely $\partial\Omega$. We also consider the smooth 1-homogeneous function $r : SY \to \mathbb{R}$ such that $\{r = 1\} = \partial\Omega$, and note that it extends continuously to the skeleton $W \setminus SY$. It will be important to consider the following class of almost structures defined on SY.

Definition 3.2. *Given J on the reduction Q, define J_S on SY to be the unique almost complex structure satisfying:*

- *J_S satisfies $J_S Z = R$ where Z is the Liouville vector field,*
- *the projection map $SY \to Q$ is holomorphic from J_S to J;*

This latter projection map is the quotient by the Liouville and Reeb flows.

Definition 3.3. *A Borel data (P_η, J_η) satisfying $P_{v_{0,+}} = P$ is said to be of Biran-Khanevsky type provided that:*

- *$J_\eta = J_S$ (SFT type) on the region $r \geq 1$,*
- *P_η is related to Q under the projection $L \to \bar{L}$, for each η.*

Working in a Liouville manifold opens the possibility of neck stretching: let us say that J'_η is obtained from J_η by *neck-stretching* with parameter s provided that:

(9) $J'_\eta(z) = (\mathrm{d}\rho_{s,z})^{-1} J_\eta(\rho_s(z))\mathrm{d}\rho_{s,z}$ where ρ_s is the time s Liouville flow.

There are two key lemmas concerning such data:

Lemma 3.4. *Let (P_η, J_η) be of Biran-Khanevsky type. After sufficiently large neck-stretching of J_η, all solutions w of the equivariant differential equation* (3) *from §2.3.2 with* $\mathrm{vdim}(w) = 0$ *have the property that all pearls u_i remain a minimum positive distance away from the skeleton,*[9] *and lie in the region where $J_\eta = J_S$, and therefore can be projected down to J-pearls on $\bar{L}$.*

Proof. See [BK13, Proposition 5.1]. We recall the argument in §3.2. □

[9]The skeleton is the complement of the symplectization end.

Henceforth we assume our data is sufficiently neck stretched so that Lemma 3.4 applies. The next two important results we will require from [BK13] are proved in §3.3 and §3.4; see [BK13, §7].

Lemma 3.5 (Unique lifting of disks)**.** *Let J_S be the SFT type almost complex structure on SY, and let:*

$$v : (D, \partial D) \to (M, \bar{L})$$

be a J-holomorphic disk. Then there is a unique lift $u : (D, \partial D) \to (SY, L)$ up to the Reeb flow (i.e., once the lift of any point $z \in \partial D$ is chosen in the circular fibre above z in the level $r = 1$).

Lemma 3.6. *For generic θ, and sufficiently neck stretched J_η, the data (P_η, J_η) is regular in the sense of Definition 2.6.*

Turning now to the equivariant differential, let us set up some notation: if $P = P_{\epsilon,\theta}$, then the chain complex is:

$$\mathrm{CM}(P) = \mathrm{CM}(Q) \oplus \mathrm{CM}(Q)[-1];$$

and above each generator $z \in \mathrm{CM}(Q)$, recall there are two generators:

- $y(z)$ with $\mathrm{ind}(y(z)) = \mathrm{ind}(z) + 1$,
- $x(z)$ with $\mathrm{ind}(x(z)) = \mathrm{ind}(z)$.

This induces a splitting of the total complex:

$$\mathrm{CM}(P) \otimes \mathbf{k}[x^{-1}, x]] \otimes \mathbf{k}[e] \simeq (\mathrm{CM}(Q) \oplus \mathrm{CM}(Q)) \otimes \mathbf{k}[q^{-1}, q]] \otimes \mathbf{k}[e] \tag{10}$$

Our main computational theorem in this section is proved in §3.5.

Theorem 3.7. *For generic θ and generic extension P_η the equivariant differential computed using sufficiently neck stretched (P_η, J_η) takes the form:*

$$d_{eq} = \begin{bmatrix} d & e^2 + F \\ 0 & d \end{bmatrix} \tag{11}$$

where d is the quantum differential on $\mathrm{CM}(Q) \otimes \mathbf{k}[q^{-1}, q]]$, and F is an endomorphism of $\mathrm{CM}(Q) \otimes \mathbf{k}[q^{-1}, q]]$ that agrees with multiplication with the Floer-Euler class on quantum cohomology (see §1.4). Both d and F are extended by linearity over $\mathbf{k}[e]$.

In the final subsection §3.6 we use Theorem 3.7 to prove Theorem 1.8.

3.1. *The Morse-Smale property.* We prove Lemma 3.1. Each flow line $\bar{\gamma}$ of the underlying pseudogradient Q lifts to a $\mathbb{R}/\mathbb{Z}$-family of flow lines γ for the pseudogradient $P_{\epsilon,\theta}$; one simply lifts the initial point $\bar{\gamma}(0)$. Suppose first that $z_- \neq z_+$. The space of lifts lying over $\mathcal{P}(z_-, z_+)$ is a circle bundle $\mathcal{L}(z_-, z_+) \to \mathcal{P}(z_-, z_+)$, defined by pulling back $L \to \bar{L}$ via $\bar{\gamma} \mapsto \bar{\gamma}(s_0)$ where s_0 is the first time when $\bar{\gamma}(s_0)$ exits the disk $D(z_-)$.

The local model (8) implies:

- there is a unique lift γ_y of $\bar{\gamma}$ starting at $y(z_-)$,
- there is a unique lift γ_x of $\bar{\gamma}$ ending at $x(z_+)$.

These can be interpreted as smooth sections of $\mathcal{L}(z_-, z_+) \to \mathcal{P}(z_-, z_+)$. In particular, we can compare the two lifts and obtain an $\mathbb{R}/\mathbb{Z}$-valued function t_0; prosaically, $\gamma_y(s) = R_{t_0}\gamma_x(s)$ for some $t_0 \in \mathbb{R}/\mathbb{Z}$, where R acts by the rotation in the fiber. This produces a smooth map:

$$f_{z_-,z_+} : \{\bar{\gamma} \text{ joining } z_- \text{ to } z_+\} \to \mathbb{R}/\mathbb{Z} \text{ given by } f_{z_-,z_+}(\bar{\gamma}) = t_0.$$

Claim A. *For a dense and open set of θ, it holds that f_{z_-,z_+} has zero as a regular value for each pair $z_- \neq z_+$.*

This follows immediately from Sard's theorem [Mil65], as rotating θ at z_- changes f_{z_-,z_+} by addition of a constant (here we assume that the disks D_{z_-} is sufficiently small so that any flow line starting at z_- never re-enters D_{z_-} after it leaves). Next:

Claim B. *If f_{z_-,z_+} has zero as a regular value for each pair $z_- \neq z_+$, then $P_{\epsilon,\theta}$ is Morse-Smale.*

The technical definition of Morse-Smale requires consideration of the linearized operators, which we do at the end of the proof. First we show that the moduli spaces are manifolds of the correct dimension.

Inside of $\mathcal{P}(z_-, z_+)$ there is the submanifold $\mathcal{R}(z_-, z_+) \subset \mathcal{P}(z_-, z_+)$ cut out by $f^{-1}_{z_-,z_+}(0)$. In fact, we have constructed two sections γ_x and γ_y of the circle bundle $\mathcal{L}(z_-, z_+)$, and $\mathcal{R}(z_-, z_+)$ is identified with their transverse intersection $\gamma_x \cap \gamma_y$. By inspection, there are isomorphisms:

$$\begin{aligned}
&\gamma_x : \mathcal{P}(z_-, z_+) - \mathcal{R}(z_-, z_+) \to \mathcal{P}(x(z_-), x(z_+)) \\
&\gamma_y : \mathcal{P}(z_-, z_+) - \mathcal{R}(z_-, z_+) \to \mathcal{P}(y(z_-), y(z_+)) \\
&\gamma_x = \gamma_y : \mathcal{R}(z_-, z_+) \to \mathcal{P}(y(z_-), x(z_+)) \\
&\mathrm{id} : (\mathcal{L}(z_-, z_+) - (\gamma_x \cup \gamma_y)) \to \mathcal{P}(x(z_-), y(z_+));
\end{aligned} \tag{12}$$

in the final line, the domain is the total space of a circle bundle with the images of two sections removed. Each space of trajectories is identified with a manifold of the correct dimension. To complete the proof of the claim, we linearize the problem and compute the kernel of the linearized operator.

Let γ be a flow line lifting $\bar{\gamma}$. There is a canonical splitting using the horizontal distribution $\xi \cap TL$ as a (flat) Ehresmann connection:

$$TW^{1,p}(\mathbb{R}, L)_\gamma \simeq W^{1,p}(\mathbb{R}, \mathbb{R}) \oplus TW^{1,p}(\mathbb{R}, \bar{L})_{\bar{\gamma}}.$$

Claim C. *With respect to this splitting, the linearized operator takes the form:*

$$D_\gamma = \begin{bmatrix} D_{\mathbb{R}} & A \\ 0 & D_{\bar{\gamma}} \end{bmatrix}$$

where $D_{\mathbb{R}}$ is a Fredholm operator of index $0, 1, -1$ depending on whether the asymptotics are of x type or y type. To be precise, the index is:

(i) 1 *if* $\gamma \neq \gamma_x$ *and* $\gamma \neq \gamma_y$,
(ii) zero if $\gamma = \gamma_x$ *or* $\gamma = \gamma_y$, *but not both, and,*
(iii) -1 *if* $\gamma = \gamma_x = \gamma_y$.

Let us note that the claim about the Fredholm index of $D_{\mathbb{R}}$ follows from the general properties of Morse theory, once we prove that the linearization D_Γ splits in the stated fashion. To prove the splitting, we briefly recall how the linearization is defined, following [Can22, §4].

Pick a coordinate chart $x_1, \ldots, x_{n-1}$ on the reduced space $\bar{L}$, and lift this to a leaf of $\xi \cap TL$, which is a flat distribution. Then we define an additional $\mathbb{R}/\mathbb{Z}$-valued coordinate x_0 by rotating the point on the chosen leaf. By our construction, the vector field $P_{\epsilon,\theta}$ appears in the form:

$$V(x_0, x_1, \ldots, x_{n-1})\frac{\partial}{\partial x_0} + \sum Q_i(x_1, \ldots, x_{n-1})\frac{\partial}{\partial x_i}.$$

Then, if η points in the $\partial/\partial x_0$ direction, it holds that:

$$D^x_\gamma(\eta) = \eta'(s) - \frac{\partial V}{\partial x_0}\eta(s),$$

where D^x_γ is the local coordinate linearization.[10] The genuine linearized operator is related to the local linearization by $\mathrm{d}x \circ D_\gamma \circ \mathrm{d}x^{-1} = D^x_\gamma$. Then, since $\mathrm{d}x$ respects the splitting, we conclude the claim.

From this splitting, we see that the reduction map takes elements in the kernel of D_γ into the kernel of $D_{\bar{\gamma}}$. It is clear from basic ODE theory that $\dim\ker D_R \leq 1$. Thus in case (i) we conclude:

$$\dim\ker D_\gamma \leq \dim\ker D_{\bar{\gamma}} + \dim\ker D_{\mathbb{R}} \leq \dim\ker D_{\bar{\gamma}} + 1 = \mathrm{index}(D_\gamma),$$

proving regularity in this case. In case (ii) and (iii) we differentiate the inverses of the lifting maps in (12) and thereby prove:

$$\dim\ker D_\gamma - \mathrm{index}(D_\gamma) \leq \dim\ker D_{\mathbb{R}}.$$

However, in both cases $\dim\ker D_{\mathbb{R}} = 0$, by analysis of linear ODEs on $W^{1,p}(\mathbb{R},\mathbb{R}) \to L^p(\mathbb{R},\mathbb{R})$ with non-degenerate asymptotics.[11] Thus we have regularity in the cases $z_- \neq z_+$.

---

[10]The local coordinate linearization is computed in the obvious way:

$$D^x_\gamma(\eta) = \eta' - \lim_{\epsilon\to 0}\epsilon^{-1}\left(\mathrm{d}xPx^{-1}(x\gamma(s) + \epsilon\eta(s)) - \mathrm{d}xP(\gamma(s))\right);$$

see, e.g., [Can22, §4.2]. This is only defined for η compactly supported in the open set $\gamma^{-1}(U)$ where U is the domain of the coordinates x. These local linearized operators uniquely determine the global linearized operator D_γ by the relation $\mathrm{d}x \circ D_\gamma \circ \mathrm{d}x^{-1} = D^x_\gamma$.

[11]Here we implicitly appeal to structural results in Morse theory. In particular, we know that $D_{\mathbb{R}}$ is represented by $\partial_s + S(s)$ where $S(s)$ converges to non-zero numbers at both ends, and the signs of these non-zero numbers can be determined by computing Morse indices. Such ODE can be solved explicitly; see, e.g., [Can22, §2.4].

It remains only to treat the case $z_- = z_+$. In this case the same arguments (considering an upper triangular linearization) work in the same way to prove D_γ is transverse. This completes the proof of the lemma. □

3.2. *Neck stretching.* This section recalls the proof of Lemma 3.4. We first show that solutions appearing in the moduli space with $\mathrm{vdim}(w) = 0$ must map all of their pearls into the symplectization end SY, into the region where $J_\eta = J_S$, provided that J_η is sufficiently neck stretched.

To be precise, let us pick a convergent family of pseudogradients $P_{\eta,n}$, and complex structures $J_{\eta,n}$ which are getting more and more neck-stretched, and suppose (in search of a contradiction) that there is a sequence of solutions w_n which have pearls intersecting the neighborhood of the skeleton where $J_{\eta,n} \neq J_S$. One shows w_n converges to a pearl trajectory where disks are replaced by pseudoholomorphic SFT type buildings. The argument which derives the contradiction is well-explained in [BK13, §5], so we only highlight the salient points:

- the "top-level" building projects to a genuine solution $\bar{w}$ in the reduction M, because being asymptotic to a Reeb orbit in the negative end implies the pearls in $\bar{w}$ are punctured disks with removable singularities,
- the "middle components" of the building are non-constant rational holomorphic curves in SY which project to non-constant holomorphic spheres in M,
- the "deepest components" of the building are non-constant rational curves in W using the original (non-neckstretched) almost complex structure J_η.

Each deepest component is a non-constant map $v : \mathbb{C} \to W$.[12]

Without loss of generality, let us suppose v is transverse to $r = 1$ (otherwise just perturb slightly $r = 1 + \epsilon$). Then $v^{-1}(\{r = 1\})$ is a collection of disjoint circles in $\mathbb{C}$, which bound disjoint disks $D_1, \ldots, D_k$.[13]

[12]The explanation of this fact uses the rather deep compactness theory of [BEH$^+$03] (with suitable modifications due to the boundary conditions on the compact Lagrangian $L \subset \partial\Omega$). First of all, the domain of w_n will split into a tree of disks with boundary on L, with interior punctures, and punctured spheres (rational curves). Each punctured disk has only negative interior punctures asymptotic to Reeb orbits in the negative end. Each punctured sphere can have either positive or negative punctures (in the SFT sense). These spheres need to have at least one positive puncture, because the ambient spaces (either SY or W) have exact symplectic structures. As explained in, e.g., [Bou03, pp. 8], the maximum principle for SFT-type almost complex structures implies each sphere cannot have *more* than one positive puncture — this is the same reason why the contact homology algebra introduced in [EGH00, §2] is well-defined. Thus the deepest components are spheres with one positive puncture, i.e., planes $v : \mathbb{C} \to W$.

[13]Let us note that the case $D_i \subset D_j$ implies u restricted to the complement $D_j \setminus D_i$ attains a local maximum in $W - \{r \leq 1\}$, which cannot happen (one can, e.g., appeal to the argument of [AS10, §7]).

We can homotope v on the union of the disks $D = D_1 \cup \cdots \cup D_k$ so that it takes values in SY, by our assumption that $\pi_2(\Omega, \partial\Omega) = 0$. Denote this deformation by v_1. Set $\alpha = \lambda/r$ and estimate:

$$\begin{aligned} \int v_1^* \mathrm{d}\alpha &= \int_{D^c} v^* \mathrm{d}\alpha + \int_D v_1^* \mathrm{d}\alpha \\ &= \int_{D^c} v^* \mathrm{d}\alpha + \int_{\partial D} v^* \lambda = \int_{D^c} v^* \mathrm{d}\alpha + \int_D v^* \mathrm{d}\lambda > 0. \end{aligned} \tag{13}$$

The equalities use Stokes' theorem. The inequality is straightforward (although is standard in SFT, see, e.g., [BEH$^+$03, §6.1]); briefly:

- $\int_{D^c} v^* \mathrm{d}\alpha > 0$ follows by the assumption that the projection from J_S on SY to J on Q is holomorphic;
- $\int_D v^* \mathrm{d}\lambda > 0$ follows from non-negativity of symplectic area for J-holomorphic curves (since J is presumed ω-tame).

By definition of symplectic reduction, the ω_M-area of the projection of v_1, say $\bar{v}_1$, is given by:

$$\int \bar{v}_1^* \omega_M = \int v_1^* \mathrm{d}\alpha.$$

Indeed, $\bar{v}_1^* \omega_M$ is computed by projecting $\tilde{v}$ onto the level $r = 1$, and then integrating $\mathrm{d}\lambda$ over this projection of $\tilde{v}$. But $\mathrm{d}\lambda = \mathrm{d}\alpha$ on the level $r = 1$. It therefore follows from (13) that $c_1(\bar{v}_1) > 0$, since the reduction $\bar{v}_1$ has positive ω_M-area (here we use the monotonicity of M in a seemingly crucial way rather than the weaker assumption that μ is positive on J-holomorphic curves).

In this manner, we observe that $\mu(\bar{w})$ must be strictly lower than $\mu(w_n)$, because the deepest components or middle components always consume a positive amount of intersections with the first Chern class. One then concludes from the fact that w_n was in a moduli space with $\mathrm{vdim}(w_n) = 0$ (and the fact the minimal Maslov number is at least 2) that $\bar{w}$ lives in a moduli space with virtual dimension -1 or less, which is impossible since $\bar{w}$ needs to have at least one pearl. □

3.3. *Unique lifting of holomorphic disks.* We prove Lemma 3.5. As in the setting of the previous section, after sufficient neck stretching we may assume that all the relevant pearls lie in the symplectization end $SY \subset W$ (i.e., we use a sufficiently neck-stretched almost complex, so that all pearls lie in the region where the almost complex structures agrees with J_S). The Liouville and Reeb flows induce actions on SY, which in turn induce the following $\mathbb{C}^\times$-action on SY:

$$e^{a+ib} \cdot p = \rho_a \circ \psi_b(p),$$

where ρ_a is the time a flow of the Liouville vector field Z and ψ_b is the time b flow of the vector field R (the flows commute, so the order is immaterial).

Set $\pi : SY \to M = SY/\mathbb{C}^\times$ to be the reduction map. Then $\pi : SY \to M$ is a principal bundle with structure group $\mathbb{C}^\times$, and the projection map is (J_S, J)-holomorphic.

Let $\bar{u} : (D, \partial D) \to (M, \bar{L})$ be a holomorphic disk in M. The pullback bundle $\bar{u}^* SY$ admits a section v (probably non-holomorphic) taking values in $\bar{u}^* L$ along ∂D. To construct a holomorphic lift u of $\bar{u}$ we consider an ansatz of the form:

$$u = e^f v \text{ where } f : (D, \partial D) \to (\mathbb{C}, \mathbb{R}) \text{ is a smooth function,} \tag{14}$$

and where $e^f \in \mathbb{C}^\times$ acts on v as explained above.

Lemma 3.8. *There is unique solution f which renders u holomorphic, up to the additions of imaginary constants.*

Proof. Set $f = a + ib : (D, \partial D) \to (\mathbb{C}, i\mathbb{R})$ and consider the map $u = e^f v$ which coincides with $\rho_a \circ \psi_b(v)$. Compute

$$\begin{cases} \partial_s u = d\rho_a \circ d\psi_b(\partial_s v) + \partial_s a \cdot Z(u) + \partial_s b \cdot R(u) \\ \partial_t u = d\rho_a \circ d\psi_b(\partial_t v) + \partial_t a \cdot Z(u) + \partial_t b \cdot R(u) \end{cases}$$

where $d\rho_a$ and $d\psi_b$ denote the spatial derivatives. Then the desired equation $\partial_s u + J_S(u)\partial_t u = 0$ is equivalent to:

$$d\rho_a d\psi_b(\partial_s v + J\partial_t v) + (\partial_s a - \partial_t b) \cdot Z(u) + (\partial_s b + \partial_t a) \cdot R(u) = 0. \tag{15}$$

Define the connection ξ on the bundle $\pi : SY \to M$ to be the symplectic orthogonal to the fibres. The flows ρ and ψ preserve this connection and J_S is preserved under the splitting by virtue of its compatibility with ω. Decompose $\partial_s v$ and $\partial_t v$ according to $TSY = \xi \oplus \mathbb{R}Z \oplus \mathbb{R}R$. After some manipulation, and because we assume that $\bar{u}$ was holomorphic to begin with, (15) reduces to a system of two equations:

$$\begin{cases} \partial_s a - \partial_t b = h_1, \\ \partial_s b + \partial_t a = h_2. \end{cases}$$

This is an inhomogenous Cauchy-Riemann problem:

$$\bar{\partial} f = h_1 + i h_2.$$

Such a solution exists, one simply recalls that the Cauchy-Riemann operator on maps $f : (D, \partial D) \to (\mathbb{C}, i\mathbb{R})$ is surjective.[14] The boundary conditions easily prove that $\ker \partial f = i\mathbb{R}$, and hence any other solution is of the form $f + ic$ where $c \in \mathbb{R}$ is a constant, as desired. □

[14]It is a Fredholm operator of index 1, and $\dim \ker \bar{\partial} = 1$, thus its cokernel vanishes.

3.4. *Regularity of pearl trajectories.* We prove Lemma 3.6. The proof is similar in spirit to the proof in §3.1. Let us denote by $\mathcal{M}_d(z_-, z_+)$ the moduli space of pearl trajectories $\bar{w}$ from z_- to z_+ appearing in moduli spaces with $\mathrm{vdim}(\bar{w}) = d$ and $\mathfrak{n} = 0$. We only consider $d \leq 1$. Because we assume (Q, J) is very regular in the sense of Remark 2.9, these moduli spaces are manifolds of dimension d (after quotient by an $\mathbb{R}^{k+1}$-action), except in the case $d = -1$ and $k = 0$ when the moduli space consists only of the constant trajectories.

We consider pearl trajectories $w = (b, \pi, u_1, \ldots, u_k)$ for (P_η, J_S) for the pair (SY, L) with $\mathrm{vdim}(w) = 0$ and $\dim \mathcal{P} = j$ which project to pearl trajectories $\bar{w} = (b, \bar{u}_1, \ldots, \bar{u}_k)$ in $\mathcal{M}_d(z_-, z_+)$. By the neck-stretching argument in §3.2, such trajectories w compute the quantum cohomology differential for sufficiently neck stretched data.

We first enumerate the curves which can appear. We treat the inverse images of $\mathcal{M}_d$ for $d = -1, 0, 1$. The inverse image over $\mathcal{M}_{-1}(z, z)$ can be analyzed directly using Morse theory on the circle. The only trajectories in this inverse image (with $\mathrm{vdim}(w) = 0$) are those without pearls and are in components:

(a) $y(z)$ to $x(z)$ with $\dim \mathcal{P} = 2$,
(b) $x(z)$ to $x(z)$ and $y(z)$ to $y(z)$ with $\dim \mathcal{P} = 1$.

Next the inverse image over $\mathcal{M}_0(z_-, z_+)$ consists of three types of trajectories:

(c) $x(z_-)$ to $x(z_+)$ with $\dim \mathcal{P} = 0$,
(d) $y(z_-)$ to $y(z_+)$ with $\dim \mathcal{P} = 0$,
(e) $y(z_-)$ to $x(z_+)$ with $\dim \mathcal{P} = 1$.

Finally we consider the inverse image over $\mathcal{M}_1(z_-, z_+)$:

(f) $y(z_-)$ to $x(z_+)$ with $\dim \mathcal{P} = 0$.

The counts of these trajectories are determined in §3.5. In this section, we provide the arguments explaining why we can assume these are regular.

3.4.1. *Regularity in cases (a) and (b).* These cases follow easily from, e.g., the usual Sard-Smale argument as in [MS12, §3.2]. The fact that the vector field V_η appearing in $P_\eta = Q + V_\eta$ is arbitrary when η is far from poles implies that (provided π is not constant) the universal operator obtained by varying V_η will be surjective. Here we appeal to similar analysis to §3.1.

3.4.2. *Regularity in cases (c), (d), (f).* This occurs solely in the $\dim \mathcal{P} = 0$ component, and so is a "non-equivariant" phenomenon, and is therefore covered by the work of [BK13]. The argument we give below in §3.4.3 can be adapted to show the present case, and so we move directly onto the next case for the sake of saving space.

3.4.3. *Regularity in case (e).* The same argument given in §3.4.1 works to establish the $k = 0$ case, so we assume $k > 0$ in this subsection.

The logic is similar to §3.1. The choice of solution w projecting to a solution $(\bar{w}, \pi) \in \mathcal{M}_0(z_-, z_+) \times \mathcal{P}$ forms a sort of circle bundle. Since $k > 0$, there is always a first pearl $\bar{u}_1$ and we trivialize this bundle by evaluation at $\bar{u}_1(-\infty)$ and pulling back $\bar{L} \to L$. The fiber over $(\bar{w}, \pi)$ is identified with the choice of lift $u_1(-\infty)$. There are two canonical sections w_x and w_y for any $(\bar{w}, \pi)$, the lift which ends at x and the lift which starts at y, and comparing these sections give a smooth $\mathbb{R}/\mathbb{Z}$ function f_{z_-,z_+}.

Claim. *For generic choice of extension P_η, the subset $w_x = w_y$ is transversally cut codimension 1 submanifold $\mathcal{R}(z_-, z_+) \subset \mathcal{M}_0(z_-, z_+) \times \mathcal{P}$.*

The argument is straightforward application of the Sard-Smale theorem; this uses that the flow lines $\pi \in \mathcal{P}$ pass through the region where we are allowed to vary P_η. Then $\mathcal{M}_0(y(z_-), x(z_+))$ is identified with $\mathcal{R}(z_-, z_+)$ via the projection map.

The rest of the argument is similar to §3.1. Let us recall that the linearized operator of §2.3.3 is defined on:

$$D_w : (\bigoplus_{\nu=1}^{k} W^{1,p,\delta}(u_\nu^* TSY, u_\nu^* TL)) \oplus \mathcal{I}_w \to \bigoplus_{\nu=1}^{k} L^{p,\delta}(u_\nu^* TSY),$$

where $\mathcal{I}_w$ is a finite dimensional vector space of coincident variations. To make a long story short, the domain and codomain of D_w split into a part tangent to $\mathbb{R}Z \oplus \mathbb{R}R \simeq \mathbb{C}$ and a part tangent to the distribution ξ. The operator is upper triangular with respect to this splitting. Following the same strategy as in §3.1, regularity boils down to:

Claim. *The kernel of the restriction of D_w to the vertical part is zero.*

This is sufficient because then $\dim \ker D_w$ is bounded from above by:

$$\dim \mathcal{R}(z_-, z_+) = \operatorname{index}(D_w),$$

using the same argument of differentiating the projection to $\mathcal{R}(z_-, z_+)$. To see that the kernel of the vertical restriction is zero, we appeal to the unique lifting §3.3 of disks. By analysis of the asymptotic convergence to $x(z_+)$ in future time, it follows that any variation which is purely vertical must be eventually zero (i.e., the vertical tangent vector at the last pearl $u_k(+\infty)$ must be zero). Then, by unique lifting, it follows that the tangent vector at $u_k(-\infty)$ is also zero, and by *coincident variations* (see §2.3.3) the variation at $u_{k-1}(+\infty)$ is also zero, etc. In this fashion, one concludes that the entire variation is zero, as desired. This completes the proof. □

3.5. *Computation of the differential and Theorem 3.7.* Abbreviate:

- $x = x(z_-)$, $y = y(z_-)$, $z = z_-$,
- $x' = x(z_+)$, $y' = y(z_+)$, $z' = z_+$.

It is sufficient to compute the outputs $d_{eq}(x)$ and $d_{eq}(y)$, which give four coefficients:

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} = \begin{bmatrix} \langle d_{eq}(x), x' \rangle & \langle d_{eq}(y), x' \rangle \\ \langle d_{eq}(x), y' \rangle & \langle d_{eq}(y), y' \rangle \end{bmatrix};$$

each coefficient is valued in $\mathbf{k}[q^{-1}, q]] \otimes \mathbf{k}[e]$. The rest of the proof is dedicated to computing these four coefficients.

First of all, write $A = A_0 + eA_1 + e^2 A_2 + \cdots$. The count A_0 equals the non-equivariant quantum differential, which by the arguments of §3.4 (see also [BK13, Section 7]), coincides with the count $\langle d(z), z' \rangle \in \mathbf{k}[q^{-1}, q]]$ where d is the ordinary quantum differential on $\mathrm{CM}(Q) \otimes \mathbf{k}[q^{-1}, q]]$.

Consider now the term A_j with $j \geq 1$, and let w be a solution of (3) joining x to x'. By the dimension formula (4) one gets (since $\mathrm{vdim}(w) = 0$):

$$\mu(w) + \mathrm{ind}_P(x') - \mathrm{ind}_P(x) + j - 1 = 0.$$

Recalling that $\mathrm{ind}_P(x') = \mathrm{ind}_Q(z')$ and $\mathrm{ind}_P(x) = \mathrm{ind}_Q(z)$ it follows that:

$$\mu(w) + \mathrm{ind}_Q(z') - \mathrm{ind}_Q(z) + j - 1 = 0.$$

Since the projection of w (appealing to Lemma 3.4) is a regular solution to (3), one sees that $\mathrm{vdim}(u) = -j$, and so the only solutions are the constant solutions with $\mathrm{vdim}(u) = -1$ when $j = 1$. In this case, $x' = x$.

We have therefore shown $A = d + eA_1$ and $A_1 = \left\langle d_1^+(x), x \right\rangle + \left\langle d_1^-(x), x \right\rangle$, where $d_1^\pm$ is shorthand for $\sum_{i,k} eq^i d_{i,1,k}^\pm$. To complete the proof we show the two terms in A_1 cancel each other. Indeed, these terms correspond to counting Morse trajectories over the gradient lines $\pi(s)$ joining $v_{0,+}$ to $v_{1,\pm}$. Each count must in fact be 1 because it is a coefficient in a continuation map $\mathrm{CM}(P_+|S^1) \to \mathrm{CM}(P_\pm|S^1)$ where $P_+ = P$ and $P_- = a^*P$. Indeed, the map defined by this count must be chain homotopic to the identity on the Morse complex of the circle with two critical points. However all the differentials on this complex are null, hence it must be a chain isomorphism, i.e., the count must be equal to 1. Thus $A_1 = 1 + 1 = 0$.

Next we consider the case $B = B_0 + eB_1 + e^2 B_2 + \ldots$. The count B_0 is zero by comparison with Morse theory of a circle; indeed, by index considerations the moduli space is empty unless the projection is constant and hence $z = z'$. In this case the count is still zero, as it corresponds to a coefficient in the Morse differential of our pseudogradient restricted to a circle, (the differential vanishes because there are only two generators x, y and it computes the homology of a circle).

Let us show $B_j = 0$ for $j > 0$. Suppose there is a solution w from x to y' solving (3) with $\mathrm{vdim}(w) = 0$. By using the index formula we have:

$$\mu(w) + \mathrm{ind}_P(y) - \mathrm{ind}_P(x) + j = 1$$

which comparing to indices on the base would imply:

$$\mu(w) + \mathrm{ind}_Q(z') - \mathrm{ind}_Q(z) = -j,$$

which does not happen by regularity of (Q, J). Thus $B_j = 0$ for $j > 0$.

Next we consider the case of $C = C_0 + eC_1 + e^2C_2 + \dots$. The count of C_0 defines an endomorphism F; moreover this endomorphism F is easily seen to coincide with the definition of the connecting homomorphism from [BK13, Theorem 4.1.1] (in their notation, it would correspond to picking a lift y of z under their "p", applying the differential and then pulling back x' to z' via their map "i" to the base).

The vanishing of the C_1 term is somewhat subtle. We write it as:

$$C_1 = C_{1,+} + C_{1,-} = \left\langle d_{1,+}(y), x' \right\rangle + \left\langle d_{1,-}(y), x' \right\rangle .$$

By index considerations, any solution w with $\mathrm{vdim}(w) = 0$ contributing to this term must live above (i.e., project to) a pearly trajectory $\bar{w}$ from z to z' which contributes to the quantum differential (so $\bar{w}$ lies in the one-dimensional moduli space of its translations). We will deform the equation w solves and show there is a compact cobordism between the solutions defining $C_{1,+}$ and $C_{1,-}$, and therefore $C_1 = 0$. The arguments are somewhat similar to the arguments used in consideration of the BV-operator in Floer cohomology; see [Abo15].

Let $[\pi_\pm]$ be the two flow lines (modulo translation) joining $v_{0,+}$ to $v_{1,\pm}$, and consider:

$$P_{\pm,s} = P_{\pi_\pm(s)}.$$

Note that $P_{\pm,s} = P_+$ for s sufficiently negative, while for s sufficiently positive it holds that $P_{+,s} = P_+$ and $P_{-,s} = P_-$. By the formulas $P_+ = P_{\epsilon,\theta}$, and $P_- = P_{\epsilon,\theta+0.5}$, we can therefore define an interpolation $P_\lambda = P_{\epsilon,\theta+0.5\lambda}$ and define a family over $\lambda \in [0,1]$:

$$P_{\lambda,s} = \begin{cases} P_+ \text{ for } s \text{ sufficiently negative,} \\ P_\lambda \text{ for } s \text{ sufficiently positive,} \\ P_{+,s} \text{ for } \lambda = 0 \ , \\ P_{-,s} \text{ for } \lambda = 1; \end{cases}$$

we suppose $P_{\lambda,s}$ is related to Q. Note that the changing positive asymptotic P_λ remains in the conjugacy class of P under the circle action; in particular, if one treats the asymptotic ends separately, the analysis of the equation is equivalent to one where the asymptotics are fixed.

Let us denote a representative underlying solution as $\bar{w} = (b; \bar{u}_1, \dots, \bar{u}_k)$ and write $b = (b_1, \dots, b_k) \in \mathcal{G}_k$. This leads to a parametric moduli space $\mathcal{M}(\bar{w})$ of solutions over the parameter $\lambda \in [0,1]$ and $\sigma \in \mathbb{R}$, given by data $(\lambda, \sigma; u_1, \dots, u_k)$ where:

- each u_i is in the symplectization end, is J_S-holomorphic, and projects to $\bar{u}_i$,
- the flow line of $P_{\lambda,s}$ on the interval $[\sigma + b_i, \sigma + b_{i+1}]$ takes $u_i(+\infty)$ to $u_{i+1}(-\infty)$,

- the flow lines on the end intervals similarly take $u_1(-\infty)$ and $u_k(+\infty)$ to y and x'.

Once we show that the solutions in the one-dimensional component of this moduli space form a compact cobordism between the counts for $\lambda = 0$ and $\lambda = 1$, we will be done, as the counts for $\lambda = 0, 1$ are $C_{1,\pm}$ by construction.

Let us observe that each u_i appearing in a solution $w \in \mathcal{M}(\bar{w})$ must have a bounded first derivative; indeed, u_i must remain the circle family of J_S-holomorphic lifts of $\bar{u}_i$ by Lemma 3.5. Therefore the only possible failure of compactness is if $\sigma \to \pm\infty$. However, if $\sigma \to \pm\infty$, one would conclude a solution to the quantum differential equation with fixed pseudogradient $P_\pm$ with negative virtual dimension, which cannot happen for a generic choice. Thus $\mathcal{M}_1$ forms a compact cobordism and $C_1 = 0$.

In the case C_j with $j \geq 3$, an index analysis rules out the existence of solutions w. Furthermore, this analysis also implies that the e^2 term consists only of equivariant Morse trajectories contained in the S^1 fibre over $z = z'$. Let us package the count of trajectories which remain in the S^1 fibre over z, with zero pearls, as defining a differential d_{loc} on the complex:

$$\mathrm{CM}_{eq}(S^1) \simeq \mathbf{k}[e]x \oplus \mathbf{k}[e]y.$$

Then, the count of trajectories C_2 represents the $\langle d_{loc}(y), x\rangle$.

The differential d_{loc} differential is just the Morse version of the equivariant cohomology differential, and standard arguments show that its homology $\mathrm{HM}_{eq}(S^1)$ is given by:

Lemma 3.9. *There is an isomorphism:*

$$\mathrm{HM}_{eq}(S^1) \cong \mathbf{k}[e]/(e^2)$$

of $\mathbf{k}[e]$*-modules.*

It follows by a dimension count that $\langle d_{loc}(y), x\rangle = e^2$ since all higher e terms vanish (and the e terms $\langle d_{loc}(x), x\rangle = \langle d_{loc}(y), y\rangle = 0$ vanish).

Finally we claim $D = d$. This is analogous to the case of A with no real modifications needed. □

Proof of Lemma 3.9. As in §2.4, one can show that $H^*_{eq}(S^1)$ is independent of the perturbation data. Consider the following vector field on S^1:

$$V_t = -\cos(2\pi t)\sin(2\pi t)\partial_t.$$

Note that it is $\mathbb{Z}/2\mathbb{Z}$-equivariant with respect to the involution. It has two index 0 critical points $x_0 = 0, x_1 = 0.5$ and two index 1 critical points $y_0 = 0.25$ and $y_1 = 0.75$. The usual Morse differential can be computed and is equal to:

$$\begin{cases} d_0(x_i) = y_0 + y_1, \\ d_0(y_i) = 0. \end{cases}.$$

Now we consider equivariant contributions. However, the vector field is equivariant and all relevant moduli spaces are transversely cut-out: hence the Borel data $P_\eta \equiv V$ for all $\eta \in S^\infty$ is a regular perturbation datum.

We compute the e term: this corresponds to constant solutions, but also their twist by the involution. Hence:

$$\begin{cases} d_1(x_j) = d_1^+(x_j) + d_1^-(x_j) = x_0 + x_1, \\ d_1(y_j) = y_0 + y_1. \end{cases}$$

Higher equivariant terms vanish for index reasons. Now construct the $k[e]$-linear map defined by:

$$\begin{aligned} \phi : k[e]/(e^2) &\mapsto H^*_{eq}(S^1) \\ 1 &\mapsto [x_0 + x_1] \end{aligned}$$

Note that $x_0 + x_1$ is indeed a cycle, so the assignment makes sense. To see that it is well defined, we compute

$$\phi(e^2) = [e^2(x_0 + x_1)] = [e \cdot e(x_0 + x_1)] = [e \cdot d_{eq}(x_0)] = [0].$$

Any class in $H^*_{eq}(S^1)$ is cohomologous to either $[x_0 + x_1]$ or $[e(x_0 + x_1)]$. Indeed, in degree 0 any cycle is of the form $\alpha x_0 + \beta x_1$ and by inspection this will be closed if and only if $\alpha = \beta = 1$. Moreover, any degree 1 cycle can be written $\alpha y_0 + \beta y_1 + e(\gamma x_0 + \chi x_1)$, and for this expression to be a cycle one needs either $\alpha = \beta = \gamma = \chi = 1$ or one of the pairs (α, β), (γ, χ) is identically zero. But then note that $y_0 + y_1 = e(x_0 + x_1) + \delta(x_0)$, which shows that the above class is either zero or cohomologous to $[e(x_0 + x_1)]$. This shows that the map is surjective.

To see that ϕ is injective note that by degree reasons $x_0 + x_1$ cannot be a coboundary. This concludes the proof. □

3.6. *Proof of Theorem 1.8.* We show the isomorphism:

$$\mathrm{QH}^*_{eq}(L) \cong \left(\mathrm{QH}^*(\bar{L}) \otimes \mathbf{k}[e]\right)/(e^2 + F),$$

of modules over $\mathbf{k}[q^{-1}, q]] \otimes \mathbf{k}[e]$, where F is as in §3.5. Keeping the splitting of the generators on the equivariant chain complex in mind (as in the proof of Theorem 3.7), we define a map:

$$i : z \in \mathrm{CM}(Q) \mapsto x(z) \in \mathrm{CM}(P),$$

which we extend to be a module homomorphism over $\mathbf{k}[q^{-1}, q]] \otimes \mathbf{k}[e]$ via extension of coefficients. By Theorem 3.7, i is a chain map provided one uses the non-equivariant quantum differential on the domain (with coefficients extended by tensoring $\mathbf{k}[e]$), and the equivariant quantum differential on the codomain. There is then an induced map on homology:

$$i : \mathrm{QH}^*(\bar{L}) \otimes \mathbf{k}[e] \to \mathrm{QH}^*_{eq}(L). \tag{16}$$

Let us compute the kernel. If a cycle ζ is sent to an exact cycle, then there are $\alpha, \beta \in \mathrm{CM}(Q) \otimes \mathbf{k}[q^{-1}, q]] \otimes \mathbf{k}[e]$, considered as a tuple (α, β) (via the identification (10)), such that:

$$d_{eq}(\alpha, \beta) = \begin{bmatrix} d\alpha + (e^2 + F)\beta \\ d\beta \end{bmatrix} = \begin{bmatrix} \zeta \\ 0 \end{bmatrix}.$$

The second coordinate shows that β is a cycle for the non-equivariant differential, and the first implies that ζ and $(e^2 + F)\beta$ are homologous. Hence the homology class $[\zeta]$ lies in the image of $(e^2 + F)$; the same argument in reverse shows that the kernel of i is equal to the image of $(e^2 + F)$. Thus i descends to a well-defined injective map:

$$i : \mathrm{QH}^*(\bar{L}) \otimes \mathbf{k}[e]/(e^2 + F) \to \mathrm{QH}^*_{eq}(L).$$

To show surjectivity of i, we inductively show that any cycle (α, β) in the equivariant chain complex is homologous to a cycle of the form $(\zeta, 0)$. The induction is on the degree $\deg(\alpha)$ (as a polynomial in the variable e) of α. First use Theorem 3.7:

$$d_{eq}(\alpha, \beta) = 0 \Leftrightarrow d\alpha + (e^2 + F)\beta = 0 \implies \deg(\beta) + 2 \leq \deg(\alpha).$$

This immediately gives the base cases $\det(\alpha) = 0, 1$ of the induction. In general, for $m = \deg(\alpha)$ with $m \geq 2$, write $\alpha = \alpha_0 + \cdots + \alpha_m e^m$, and then replace:

$$(\alpha', \beta') = (\alpha, \beta) + d_{eq}(0, -\alpha_m e^{m-2}),$$

which has $\deg(\alpha') < \deg(\alpha)$, and hence the induction proceeds. □

4. *The case of free actions and the proof of Theorem 1.12*

Let W and L be as in Theorem 1.12; here we recall that the $\mathbb{Z}/2\mathbb{Z}$ action on W is free, with smooth quotients $W' = W/\mathbb{Z}/2\mathbb{Z}$ and $L' = L/\mathbb{Z}/2\mathbb{Z}$.

Consider a regular pair of data (P', J') for defining $\mathrm{QH}^*(P', J')$ in the sense of Remark 2.9. This lifts to $\mathbb{Z}/2\mathbb{Z}$ equivariant data (P, J) on W. It is almost immediate to show that this lifted data is regular in the sense of §2.3.4: indeed, the linearized problem at a solution for (P, J) is identified with the linearized problem at a solution for (P', J') by simply projecting $W \to W'$.

Above each zero z of P' live two zeros of the same index, $x(z)$ and $y(z)$; unlike the proof of Theorem 3.7, there is no canonical way to distinguish these (and so we pick an arbitrary labeling). Define the operator a which swaps $x(z)$ with $y(z)$. Then:

$$d_{eq} = \bar{d} + e(1 + a),$$

where $\bar{d}$ is defined over $\mathrm{CM}_*(P) \otimes \mathbf{k}[q^{-1}, q]]$ and is the specialization $e = 0$ of the equivariant differential. The matrix entries of $\bar{d}$ cannot be determined explicitly because of the arbitrary choice of $x(z), y(z)$; any trajectory w

from a zero z to a zero z' has exactly two lifts, one starting at $x(z)$ and one starting at $y(z)$, and they land in $\{x(z'), y(z')\}$, however we cannot control which one. What we do know is that if one lift ends in $x(z')$, the other ends in $y(z')$ and vice-versa. In particular, one has the formula:

$$d_{eq}(x(z) + y(z)) = \bar{d}(x(z) + y(z)) = x(dz) + y(dz);$$

here d is the non-equivariant differential for P', L'. In particular, this shows that the map ϕ defined by $\phi(z) := x(z) + y(z)$ is a chain map. Theorem 1.12 is then implied by:

Lemma 4.1. *The map ϕ is a quasi-isomorphism.*

Proof. We prove it is injective and surjective on homology. To prove injectivity, suppose $\phi(\zeta) = d_{eq}(\beta)$ for some $\beta = \beta_0 + \cdots + e^n\beta_n$ and some cycle ζ (here ζ is a sum of critical points z of Q). We prove ζ is exact by induction on n. If $n = 0$, then:

$$d_{eq}(\beta) = \bar{d}(\beta_0) + e(1 + a)\beta_0 = \phi(\zeta).$$

Since $\phi(\zeta)$ has no e term, this implies $(1 + a)\beta = 0$, and hence β lies in the image of ϕ, and it follows that ζ is exact. Now assume the result holds for n and let $\beta = \beta_0 + \cdots + \beta_{n+1}e^{n+1}$ be such that $d_{eq}\beta = \phi(\zeta)$. It follows, by comparing e terms, that:

$$\begin{cases} (1 + a)\beta_n + \bar{d}\beta_{n+1} = 0, \\ (1 + a)\beta_{n+1} = 0. \end{cases}$$

In particular, β_{n+1} is symmetric, and hence can be written as:

$$\beta_{n+1} = (1 + a)\gamma_n.$$

Let $\tilde{\beta} = \beta_0 + \cdots + e^n(\beta_n + \bar{d}\gamma_n) = \beta + d_{eq}(e^n\gamma_n)$. Then $d_{eq}(\tilde{\beta}) = d_{eq}\beta = \phi(\zeta)$, and by our induction hypothesis it follows that z is exact, as desired.

Next we establish surjectivity. Let $\gamma = \gamma_0 + \cdots + e^n\gamma_n$ be a cycle in the equivariant chain complex. We prove that γ is cohomologous to a cycle of the form $\phi(\zeta)$ by induction on n. If $n = 0$, then $d_{eq}\gamma_0 = \bar{d}\gamma_0 + e(1+a)\gamma_0 = 0$ implies that γ_0 is symmetric and thus is in the image of ϕ. Assume the induction hypothesis for n and fix a cycle $\gamma = \gamma_0 + \cdots + \gamma_{n+1}e^{n+1}$. The fact that γ is closed implies:

$$\begin{cases} \bar{d}\gamma_{n+1} + (1 + a)\gamma_n = 0, \\ (1 + a)\gamma_{n+1} = 0. \end{cases}$$

Now choose δ_n so that $(1 + a)\delta_n = \gamma_{n+1}$. Then $\gamma - d_{eq}(e^n\delta_n)$ is a cycle, homologous to γ, of degree n. The result follows. □

Appendix A. *Transversality for pearl trajectories*

The goal is to prove Theorem 2.8, which concerns the generic regularity of Borel data. We will prove generic regularity for all solutions in (3) and solutions of the nodal equation §2.3.4, by induction on the virtual dimension (the induction terminates at $\mathrm{vdim}(w) = 1$). Recall the virtual dimension is given by:

$$\mathrm{vdim}(w) = \mu(w) + \mathrm{ind}(x_+) - \mathrm{ind}(x_-) + \dim \mathcal{P} - 1 - \mathfrak{n},$$

where $\mathfrak{n}$ is the number of nodes. If $\mathrm{vdim}(w)$ is sufficiently negative, there are no solutions at all, and so all solutions are regular.

Let us call a trajectory $w = (b, \pi, u_1, \ldots, u_k)$ *simple* provided the following holomorphic curves are simple:[15]

- the union of the holomorphic disks u_i over those indices i such that $\pi(b_i)$ is near the starting pole $v_{0,+}$,
- the union of the holomorphic disks u_i over those indices i such that $\pi(b_i)$ is near the ending pole $v_{j,\pm}$,
- the holomorphic curves u_i such that $\pi(b_i)$ is far from both the starting and ending pole.

Here we recall $\pi : \mathbb{R} \to S^\infty$ is the auxiliary flow line. The meaning of a point $\eta = \pi(b_i)$ being near a pole $v_{j,\pm}$ is that $\pm\eta_j \geq 4/5$, as per the definition of Borel data in §2.3.1. In the worst case, π is a constant flow line from v_0^+ to v_0^+; in this case simplicity implies the union of all the curves is required to be simple.

By adapting standard property that a suitable universal operator is a submersion at simple holomorphic curves (see [MS12, Proposition 3.4.2]), one shows that, for generic J, all simple pearl trajectories are regular[16] (in sense of Definition 2.5).

Morally, the strategy is now to show this automatically implies that all trajectories with $\mathrm{vdim}(w) \leq 1$ are simple. However, in the actual argument we will require further perturbation of J.

Suppose w is a non-simple trajectory: the idea, following [BC07, §3.2], is to extract from w an underlying simple trajectory. The key tool used in this part of the argument is the work of Lazzarini [Laz00, Laz11], which analyzes the structure of general J-holomorphic curves with boundary on a Lagrangian submanifold; see Figure 5.

To make a long story short:

Lemma A.1. *For generic J (the complex structure at the poles), a non-simple trajectory w from x to y with $\mathrm{vdim}(w) = d$ implies the existence of a simple*

[15] in the sense their injective points are dense in the domain, as in [MS12]

[16] Here we use the fact that J_η is allowed to vary in an η-dependent way when η is far enough from the poles.

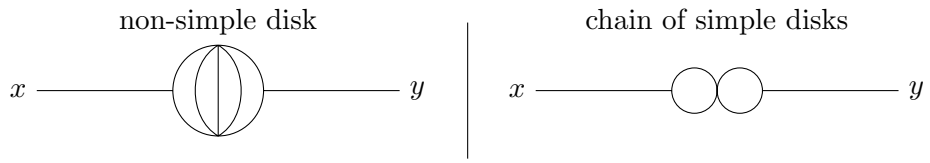


FIGURE 5. A non-simple pearl "shatters" into open holomorphic pieces separated by a certain graph G by the results of [Laz00, Laz11]; moreover, the interior of each holomorphic piece is a multiple cover of the interior of an underlying simple holomorphic disk. Therefore one can extract an underlying chain of simple disks as explained in [BC07, §3.2].

trajectory w' from x to y with:

$$\mathrm{vdim}(w') \leq d - 2.$$

In particular, if all simple trajectories w' are regular, than all trajectories with $\mathrm{vdim}(w) \leq 1$ *are regular.*

Proof. We refer the reader to the proof of [BC07, §3] for the details. Let us briefly indicate the argument in one case: if the union of disks u_i with $\pi(b_i)$ close to the starting pole $v_{0,+}$ is not a simple holomorphic curve, then the disks "overlap" each other (e.g., one disk can overlap itself and become non-simple, see Figure 5). This union of disks is $u_1, \ldots, u_i$ for some maximal i, and one considers the partial trajectory from $u_1(-\infty)$ to $u_i(+\infty)$. Using the simple pieces in Lazzarini's decomposition [Laz00, Laz11], one can extract a "short-cut" trajectory from $u_1(-\infty)$ to $u_i(+\infty)$. In the construction of this short-cut, the Maslov index must drop by at least 2. By attaching this to the rest of the old trajectory w, one obtains the desired trajectory w'. □

This completes the proof of Theorem 2.8. □

A.1. *Very regular continuation/deformation data exists.* The argument is analogous to the arguments used in the proof of Theorem 2.8.

Appendix B. *Elementary argument proving Theorem 1.1 when $n = 1$*

In this section we provide an elementary argument proving that, if $L \subset \mathbb{C}$ is the unit circle, then L cannot be displaced from itself, or from $\mathbb{R} \subset \mathbb{C}$, by an equivariant Hamiltonian isotopy. In fact, we can prove:

Proposition B.1. *Suppose that φ_η is a smooth family of area preserving diffeomorphisms parametrized by $\eta \in S^2$ so that:*

$$\varphi_{-\eta} = a\varphi_\eta a. \tag{17}$$

Then $\varphi_\eta(L) \cap L \neq \emptyset$ holds for at least two points $\eta \in S^2$.

Proof of Proposition B.1. Of course, it suffices to prove the statement for at least one point $\eta \in S^2$, since then η and $-\eta$ will be the desired two points.

Consider L as the image of the standard embedding $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$. The family φ_η yields a map $S^2 \times \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ given by:

$$G(\eta, t) = \varphi_\eta(\gamma(t)).$$

Notice that this map is equivariant in the following sense:

$$G(-\eta, t + 1/2) = a\varphi_\eta(\gamma(t)) = aG(\eta, t),$$

where we use that the standard embedding satisfies $\gamma(t + 1/2) = a\gamma(t)$.

In search of a contradiction, suppose that $G(\eta, t) \notin L$ for all η, t.

Consider the standard angular closed one-form:

$$\alpha = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2}.$$

It is well-known the the integrals of α over closed loops are valued in $2\pi\mathbb{Z}$. We claim that the integral of α over each loop $t \mapsto \gamma_\eta(t) = G(\eta, t)$ is zero. To show this, we claim the loop γ_η bounds a surface Ω_η disjoint from 0, and so we can apply Stokes' Theorem to the closed one form α. Clearly there exists a smooth surface Ω_η whose boundary is the image of γ_η — we can just take $\Omega_\eta = \varphi_\eta(D^2)$ — and this surface Ω_η has area π. If Ω_η contains 0, then Ω_η must be entirely contained in the interior of D^2 as it is not allowed to intersect $L = \partial D^2$. But this would imply the area of Ω_η is strictly smaller than π, which is a contradiction. This establishes the claim.

Thus $G^*\alpha$ is an exact one-form, since its integral vanishes over every loop in the domain $S^2 \times \mathbb{R}/\mathbb{Z}$. Thus $G^*\alpha = dF$ for some real-valued function F. We next claim that:

$$F(-\eta, t + 1/2) = F(\eta, t) + \pi + 2\pi k.$$

for some $k \in \mathbb{Z}$. Indeed, $F(-\eta, t + 1/2) - F(\eta, t)$ is the integral of α over an arc joining a point $z \in \mathbb{C}$ to its antipodal point $-z$ (this arc lies in the image of G and is therefore disjoint from 0). It is clear that the integral of α over an arc joining a pair of antipodal points is valued in $\pi + 2\pi k$ (one simply interprets α as the angular form). The integer k is independent of the point (η, t). However, this conclusion is impossible, because then:

$$F(\eta, t) = F(-\eta, t+1/2) + \pi + 2\pi k = F(\eta, t) + 2\pi + 4\pi k \implies 0 = 2\pi + 4\pi k,$$

which cannot hold for any integer k. This completes the argument. □

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