EQUIVARIANT FLOER COHOMOLOGY FOR CONTACTOMORPHISMS OF QUOTIENT SPACES

DYLAN CANT, ERIC KILGORE, AND JUN ZHANG

ABSTRACT. This paper establishes the orderability of contact manifolds which are quotients of fillable contact manifolds under finite group actions compatible with the filling, the prototypical example being $\mathbb{R}P^{2n-1}$ as the quotient of S^{2n-1}. Our approach employs an equivariant formulation of the contact Floer cohomology theory. This leads us to develop an analogue of Givental's nonlinear Maslov index using the $\mathbf{k}[[x]]$-module structure on an equivariant version of contact Floer cohomology. A key idea is that mapping cones of continuation maps detect crossings with the discriminant (recall that Givental's index is a continuous integer valued function on the complement of the discriminant). To properly handle the inherent non-canonicity in defining such mapping cones, we lift the structure of contact Floer cohomology to chain level by defining it as an ∞-functor on a suitable ∞-categorification of the Eliashberg–Polterovich orderability relation on the universal cover of the contactomorphism group.

1. Introduction

Many contact manifolds arising as quotients of finite group actions are shown to be orderable in the sense of [EP00], even when their covering spaces are non-orderable. The source of rigidity we will develop and exploit is equivariant Floer cohomology as in [SS10, Sei15, SZ21, She22]. Our paper resolves an old claim of Givental [Giv90, Giv91] that his generating function theory for RP^{2n-1} should admit a Floer theoretic analogue.

To be more precise, let G denote a finite group; our main results concern rigidity phenomena for contact manifolds Y which admit the following additional structure:

Definition 1.1. *A* G-filling *of a compact contact manifold Y is a convex-at-infinity symplectic manifold (W, ω) with a contact-at-infinity symplectic G-action whose ideal restriction acts freely on the ideal contact boundary ∂W, and a contactomorphism*[1] $Y \to \partial W/G$. *We say that Y admits an* aspherical G-filling *if the filling can be chosen so that ω vanishes on spherical homology classes.*

Date: August 26, 2026.

[1]Note that Y inherits a preferred coorientation from any G-filling.

In a sentence: we prove Y is orderable in the sense of [EP00] if it admits an aspherical G-filling W and the G-action on W is not free. This is noteworthy as the covering space ∂W of Y is non-orderable in many cases where our result applies (see [EKP06, HS24]). Moreover, we develop a framework of *spectral invariants* and *integer valued measurements* generalizing the rotation number for smooth isotopies S^1 and the Maslov index for paths in the linear symplectomorphism group.

1.1. **Background.** To start, let us recall some basic concepts in contact geometry. Let (Y^{2n-1}, ξ) be a contact manifold. The group of diffeomorphisms preserving its contact structure is denoted by $\mathrm{Cont}(Y)$; elements of this group are called *contactomorphisms*. A *contact isotopy*[2] is a path $\varphi_t \in \mathrm{Cont}(Y)$, defined for $t \in [0,1]$, based at $\varphi_0 = id$, and we denote by $\mathrm{CI}(Y)$ the group of all such contact isotopies. The quotient of $\mathrm{CI}(Y)$ by the subgroup of contractible loops is the universal cover of $\mathrm{Cont}(Y)$.

Attempts to study $\mathrm{Cont}(Y)$ and its universal cover have long been a major theme of contact geometry. In [EP00], Eliashberg and Polterovich propose to study these groups via the following relation: let us say that $\varphi_{0,t} \leq \varphi_{1,t}$ if there is a path in $\mathrm{CI}(Y)$ from $\varphi_{0,t}$ to $\varphi_{1,t}$, say $\varphi_{s,t}$, such that:

(1) $\quad s \mapsto \varphi_{s,1}(y)$ is never negatively transverse to ξ for each $y \in Y$.

Note that, in order for this to make sense, we require that ξ is cooriented.

If $\varphi_{0,t}$ and $\varphi_{1,t}$ differ by a contractible loop, then they are mutually related by $\leq$, and so the relation descends to the universal cover of $\mathrm{Cont}(Y)$. If the induced relation on the universal cover of $\mathrm{Cont}(Y)$ is a partial order, then Y is called *orderable*; otherwise, Y is called *non-orderable*.

It was shown in [EP00] that certain contact manifolds are orderable:

- real projective spaces $\mathbb{R}P^{2n-1}$ (with standard contact structure),
- prequantization bundles over certain symplectic manifolds,
- spherical cotangent bundles ST^*M, for certain M.[3]

Since [EP00], much research has been done concerning orderability of contact manifolds. A particularly noteworthy milestone is [EKP06] which establishes the non-orderability of the ideal boundary of stabilizations $W \times \mathbb{C}^d$ of Liouville manifolds W when $d \geq 2$; in particular, $S^3, S^5, \ldots$, with their standard contact structure, are all non-orderable.[4]

[2]To lighten the notation, we use the symbol φ_t rather than, say $\{\varphi_t : t \in [0,1]\}$, to denote contact isotopies. In the same way, a path of contact isotopies is denoted $\varphi_{s,t}$. This notational convention is the same as the one used in, e.g., [Can23, CHK23, Can24b].

[3]For the state of the art result that all spherical cotangent bundles are orderable we refer the reader to [CN16, Theorem 1.1.].

[4]See also [HS24] for the case of $W \times \mathbb{C}$ if W is T^*S^1 or is a Weinstein manifold with dimension at least 4.

The tool used to prove orderability of $\mathbb{R}P^{2n-1}$ is Givental's "non-linear generalization of the Maslov index" from [Giv90, Giv91], defined on the universal cover of the contactomorphism group of projective spaces using equivariant cohomology of generating functions. Let us recall here that Givental's index is a map $\mu : \mathrm{CI}(\mathbb{R}P^{2n-1}) \to \mathbb{Z}$ satisfying:

(G1) *(monotonicity)* $\mu(\varphi_{0,t}) \leq \mu(\varphi_{1,t})$ if $\varphi_{0,t} \leq \varphi_{1,t}$;
(G2) *(continuity from above)* $\mu(\varphi_t) = \lim_{s\to 0+} \mu(R_{st}\varphi_t)$ for any positive contact isotopy R_s;[5]
(G3) *(normalization)* $\mu(id) = 0$;
(G4) *(non-triviality)* μ attains arbitrarily large values;
(G5) *(discriminant)* μ is constant on the path-connected components of the complement of the discriminant.

Recall the *discriminant* in $\mathrm{CI}(Y)$ is the subset of those φ_t such that φ_1 has a fixed point x satisfying $\varphi_1^*\alpha_x = \alpha_x$, for some/any contact form α near x.

One of our main results (Theorem 1.11) involves the construction of invariants satisfying (G1) through (G5) for contact manifolds Y other than real projective spaces, and using Floer theory (settling Givental's claim in [Giv91, pp. 43] that a Floer theoretic approach to his index should be possible).

Remark 1.2. Givental's index famously generalizes the Maslov index: under the inclusion $\mathrm{Sp}(2n) \to \mathrm{Cont}(\mathbb{R}P^{2n-1})$ his index pulls-back to the Maslov index; in particular, this yields (G3) and (G4). The index we define in Theorem 1.11 also has this property, see Proposition 1.12.

Property (G5) suggests that "sufficiently positive paths" must intersect the discriminant. This is related to the dynamics of Reeb vector fields. Recall that a *Reeb vector field* is any vector field R whose flow preserves ξ and which is positively transverse[6] to the contact distribution ξ. Let us denote the time s flow of R by the symbol R_s. It then follows that:

$$\psi_{s,t} = \varphi_t^{-1} \circ R_{st} \text{ (defined for } s \in \mathbb{R})$$

satisfies $\lim_{s\to\pm\infty} \mu(\psi_{s,t}) = \pm\infty$. In particular, since Givental's index is locally constant on the complement of the discriminant, $\psi_{s,t}$ must intersect the discriminant for infinitely many values of s.

In Sandon's work [San11a, San11b, San12, San13], the values of s when $\psi_{s,t}$ intersects the discriminant were discovered to be the critical values of suitable generating functions and were called *lengths of translated points*.

Definition 1.3. *The* spectrum $\mathrm{Spec}_R(\varphi_t)$ *of* φ_t *relative to a Reeb flow* R*, is the set of numbers* $s \in \mathbb{R}$ *such that* $\varphi_t^{-1} \circ R_{st}$ *lies in the discriminant.*

[5] A positive contact isotopy R_s is one for which the curves $s \mapsto R_s(p)$ are positively transverse to the contact distribution

[6] It can be shown that each Reeb vector field satisfies $\alpha(R) = 1$ and $\mathrm{d}\alpha(R, -) = 0$ for a unique contact form α; indeed, many authors define Reeb vector fields in this way.

The work of [AM13, AM18] develops an elliptic Morse theory for a pair (R, φ_t) whose associated spectrum of critical values is $\mathrm{Spec}_R(\varphi_t)$; their theory is based on the Rabinowitz–Floer homology (RFH) theory of [CF09].

The approaches of Sandon and Albers–Merry led to the introduction of *spectral invariants*, namely, functions $c_R : \mathrm{CI}(Y) \to \mathbb{R}$ satisfying:

(R1) *(spectrality)* $c_R(\varphi_t) \in \mathrm{Spec}_R(\varphi_t)$.

The existence of such a spectral invariant is not guaranteed, as it implies the spectrum is non-empty, a fact which is not always true; see [Can24a].

More recently [Can23, DUZ23, Can24b, DUZ25] constructed such spectral invariants for certain contact manifolds Y using Hamiltonian Floer (co)homology in a symplectic filling; one requires that Y admits a filling W satisfying certain hypotheses; we refer the readers to [Can24b, DUZ25] for more details. This led to the construction of spectral invariants satisfying (R1) and four additional properties:

(R2) *(monotonicity)* $c_R(\varphi_{0,t}) \leq c_R(\varphi_{1,t})$ if $\varphi_{0,t} \leq \varphi_{1,t}$,
(R3) *(continuity from above)* $c_R(\varphi_t) = \lim_{s\to 0+} c_R(R_{st}\varphi_t)$, where R_s is any Reeb flow,
(R4) *(normalization)* $c_R(id) = 0$,
(R5) *(sub-additivity)* $c_R(\varphi_{0,t}\varphi_{1,t}) \leq c_R(\varphi_{0,t}) + c_R(\varphi_{1,t})$.

One of our main results (Theorem 1.10) guarantees the existence of such measurements under the assumption that Y admits a G-filling W with at least one point with non-trivial stabilizer (for the G-action).

Such a spectral invariant is sufficient to guarantee the orderability of Y:

Proposition 1.4. *The existence of c_R satisfying (R1) through (R5) implies Y is orderable.*

This statement is known in the literature (see, e.g., [AA26]). We quickly review the argument; the first step is to prove:

Proposition 1.5. *Any spectral invariant c_R satisfying axioms (R1) through (R5) also satisfies:*

$$|c_R(\varphi_{0,t}) - c_R(\varphi_{1,t})| \leq \mathrm{dist}_\alpha(\varphi_{0,t}, \varphi_{1,t}),$$

where dist_α *is* Shelukhin's Hofer distance [She17] *associated to the contact form α whose Reeb flow is R.*

Proof of Proposition 1.5. If $\varphi_{0,t} \leq R_{st}\varphi_{1,t}$ then:

$$c_R(\varphi_{0,t}) \leq c_R(R_{st}\varphi_{1,t}) \leq c_R(\varphi_{1,t}) + c_R(R_{st}) \leq c_R(\varphi_{1,t}) + s,$$

where the last step follows from:

$$c_R(R_{st}) \leq \lim_{k\to\infty} c_R(R_{st/k})k = (s/k)k = s;$$

here we appeal to the *spectrum gap for Reeb flows*[7] to conclude that s/k is the only number in the spectrum of $R_{st/k}$ which lies in some small neighborhood of 0. Thus:

$$\begin{aligned}|c_R(\varphi_{0,t}) - c_R(\varphi_{1,t})| &\leq \min\{s : \varphi_{0,t} \leq R_{st}\varphi_{1,t} \text{ and } \varphi_{1,t} \leq R_{st}\varphi_{0,t}\} \\ &\leq \mathrm{dist}_\alpha(\varphi_{0,t}, \varphi_{1,t}),\end{aligned}$$

where the latter inequality is well-known; see [AA26, Nak23]. □

Proof of Proposition 1.4. It follows easily from continuity and spectrality that $c_R(R_{st}) = s$, and hence $\mathrm{dist}_\alpha(1, R_{st}) \geq s$. This implies, by work of [Hed24] that Y is orderable. □

Remark 1.6. The emerging picture from the spectral invariants c_R is of a contact geometry analogue to the well-known theory of spectral invariants for Hamiltonian diffeomorphisms of symplectic manifolds initiated by [Vit92, Sch00, Oh05], except there is one such structure *for each Reeb flow* R (as with the Hofer-type norms in [She17]). However, certain closed contact manifolds, e.g., the non-orderable ones, do not admit such structures (this contrasts with the symplectic setting where all closed symplectic manifolds are supposed to admit spectral invariants).

Remark 1.7 (Hypertightness)**.** One can adapt the methods of this paper and [AFM15, AFM17, Oh21, Can24b, DUZ25] to show that any counterexample Y to the Weinstein conjecture admits invariants of type (R1) through (R5). The essential idea is to exploit the fact that any counterexample to the Weinstein conjecture is *hypertight*, that is, has no contractible Reeb orbits for some Reeb flow R. For such manifolds, one can "do Floer theory" directly in the symplectization SY, by appealing to the compactness results of [BEH$^+$03].

Remark 1.8 (Reeb orbits)**.** Let us comment on the relationship of the invariants μ satisfying (G1) through (G5) and the existence of Reeb orbits: any contact manifold Y admitting such a measurement μ must admit closed Reeb orbits. Indeed, since $\mu(R_{st})$ can only change when R_{st} crosses the discriminant, and μ attains arbitrarily large values, so R_{st} must intersect the discriminant for infinitely many values of s. This observation and the discussion in Remark 1.7 shows the invariants of type μ are, in a certain sense, rarer than the invariants of type c_R.[8]

1.2. **Statement of main results and examples.**

Definition 1.9. *Let $\varphi_t \in \mathrm{CI}(Y)$, and let W be a G-filling of Y. Define a* W-contractible discriminant point *of φ_t to be a discriminant point x of φ_1 such that $t \mapsto \varphi_t(x)$ lifts from Y to ∂W and any lift is a contractible loop*

[7] R_s has no discriminant points if $s \in (-\hbar, 0) \cup (0, \hbar)$ for some $\hbar > 0$.

[8] In fact, in certain cases, it will follow from our construction Theorem 1.11 that the values when $\mu(R_{st})$ changes are periods of *contractible* Reeb orbits (we need to assume the map $\pi_1(\partial W) \to \pi_1(W)$ is injective to make this deduction).

*in W. The W-*contractible discriminant *is the subset consisting of those $\varphi_t \in \mathrm{CI}(Y)$ for which φ_t has a W-contractible discriminant point.*

Theorem 1.10. *Let G be a finite group and suppose that Y is a contact manifold admitting an aspherical G-filling W. If the G-action on W possesses at least one point with non-trivial stabilizer, then there exists a contact spectral invariant c_R satisfying (R1) through (R5) for any Reeb flow on Y.*

Moreover, $c_R(\varphi_t)$ is the value of s for which $R_{st} \circ \varphi_t^{-1}$ has a W-contractible discriminant point.

Thus we prove that all such manifolds Y admit spectral invariants for each Reeb flow, in the sense of Remark 1.6, and are, in particular, orderable.

If we assume moreover that $\mathrm{SH}(W) = 0$ then our construction produces a measurement μ generalizing the Maslov class:

Theorem 1.11. *Assume the hypotheses of Theorem 1.10. If, in addition, the symplectic cohomology*[9] $\mathrm{SH}(W; \mathbb{Z}/p\mathbb{Z})$ *vanishes for some prime number p dividing the cardinality of a stabilizer of the G-action on W, then there is a measurement $\mu : \mathrm{CI}(Y) \to \mathbb{Z}$ satisfying properties (G1) through (G5).*

Moreover, μ can be chosen to satisfy the strengthened version of (G5): μ is constant on the path connected components of the complement of the W-contractible discriminant.

The obvious example of Y satisfying the hypotheses of Theorem 1.10 and 1.11 is $\mathbb{R}P^{2n-1}$, with its standard contact structure, since it admits the $\mathbb{Z}/2\mathbb{Z}$-filling $\mathbb{C}^n$ with a fixed point, and $\mathbb{C}^n$ is known to have vanishing symplectic cohomology (for any choice of coefficients $\mathbf{k}$). Similarly, our theorem applies to the standard lens spaces L_p, $p > 1$, obtained by quotienting S^{2n+1} by the $\mathbb{Z}/p\mathbb{Z}$ action generated by $z \mapsto e^{2\pi i/p} z$, and we recover some of the results of [GKPS21, AAS24].

The fact that our measurement generalizes the Maslov class can be made quite precise:

Proposition 1.12. *Suppose $W = \mathbb{C}^n$, with the antipodal $\mathbb{Z}/2\mathbb{Z}$-action. If φ_t is an isotopy valued in the subgroup $\mathrm{Sp}(2n)$ whose time-1 map does not have 1 as an eigenvalue, then $\mu(\varphi_t) = \mathrm{CZ}(\varphi_t) - n$, where the Conley-Zehnder index associated to a non-degenerate linear symplectic flow is as defined in, e.g.,* [PRSZ20, Chapter 8].[10]

[9] The symplectic cohomology $\mathrm{SH}(W; \mathbb{Z}/p\mathbb{Z})$ is a well-studied invariant for convex-at-infinity manifolds; see, e.g., [Vit99, Sei08a, Rit13]. To be concrete, we follow the conventions of [CHK23, Can24b], and work only in the free homotopy class of contractible loops. The orientation scheme needed to work with primes $p > 2$ will be explained in §2.4

[10] For μ to be normalized to zero, we require $\mathrm{CZ}(R_{\epsilon t}) = n$ for $\epsilon \in (0, 1)$, where R is the standard Reeb flow on the sphere S^{2n-1}, extended radially to $\mathbb{C}^n$. The quadratic Hamiltonian generating this linear isotopy is $\epsilon\pi\,|z|^2$.

The proof of this proposition is given in §5.4.8. In the following subsections we give concrete examples of manifolds to which both Theorem 1.10 and 1.11 can be applied.

1.2.1. **Quotients of stabilized cotangent bundles.** Let $W = T^*M \times \mathbb{C}^d$ and suppose that $\mathbb{Z}/p\mathbb{Z}$ acts smoothly on M with isolated fixed points. There is an induced action on T^*M by canonical transformations — importantly, this action preserves the Liouville form λ_{T^*M}. Then $\mathbb{Z}/p\mathbb{Z}$ acts diagonally on W, where the action on $\mathbb{C}^d$ is by multiplication by $e^{2\pi i/p}$. This action preserves $\lambda_{T^*M} + \lambda_{\mathbb{C}^d}$, and thus has an ideal restriction; this is a free action on ∂W because the action on M has isolated fixed points. Let $Y = \partial W/(\mathbb{Z}/p\mathbb{Z})$. In this setting, Theorem 1.10 applies if the action on M has at least one fixed point. If, in addition, $d > 0$, then Theorem 1.11 applies. The case when $M = \mathrm{pt}$ recovers the case of $\mathbb{R}P^{2d-1}$ and the lens spaces discussed above.

Interestingly, [EKP06, Theorem 1.16] and [HS24, Corollary 1.3] show the ideal boundary $\partial(T^*M \times \mathbb{C}^d)$ is non-orderable (for $d \geq 2$ by [EKP06], or $d \geq 1$ if $\dim M \geq 1$ by [HS24]), and hence does not admit measurements satisfying (R1) through (R5) or (G1) through (G5).

1.2.2. **Non-primitive prequantization spaces.** Let (B, ω) be a symplectic manifold, and suppose there is a unitary line bundle $\pi : W \to B$ satisfying[11] $c_1 = -[\omega]$. As explained in [McD91, pp. 656], there is a natural way to equip W with the structure of a symplectic manifold which is convex-at-infinity (the construction uses the unitary structure on W). The fiberwise action of $e^{2\pi i/p}$ is a symplectomorphism with fixed point set equal to the zero set B. If B is symplectically aspherical, it is known that the non-equivariant regular symplectic cohomology $\mathrm{SH}(W; \mathbb{Z}/p\mathbb{Z})$ vanishes (this is shown in, e.g., [CHK23] in the case $p = 2$). Therefore Theorem 1.10 and 1.11 apply to $Y = (\partial W)/(\mathbb{Z}/p\mathbb{Z})$, when B is aspherical.

The construction shows that ∂W is naturally identified with the unit circle bundle in W, and that Y is simply the ideal boundary of the negative line bundle $W^{\otimes p}$ with base space $(B, p\omega)$. The orderability of such Y was established already for a special class of aspherical B in [EP00, §1.3.C].

Note that in order for our methods to apply, we require that $Y = \partial W^{\otimes p}$ for $p > 1$, i.e., Y is "non-primitive," with respect to tensor powers. Some assumption is clearly necessary, since $Y = S^{2n-1}$ (which is a primitive prequantization space) is non-orderable [EKP06, HS24].

As a comparison, using the generating function approach of [Giv90, Giv91], the work of [BZ15, Zap20, GKPS21] establishes the existence of measurements satisfying (G1) through (G5) for certain special choices of Y — in these constructions the base B is a toric manifold; see also [San13, Ter21]

[11] Recall that c_1 is the deRham cohomology class of represented by any differential two-form $\mathbf{c}$ solving $\mathrm{d}\alpha = \pi^*\mathbf{c}$, where α is a *global angular form*, i.e., α is a one-form on W which restricts to $(2\pi)^{-1}(x\mathrm{d}y - y\mathrm{d}x)/(x^2 + y^2)$ in each fiber.

which uses generating functions to show $\mathrm{Spec}_R(\varphi_t) \neq \emptyset$ for all φ_t and a specific Reeb vector field R when the base B is a closed monotone symplectic toric manifold (provided that Y is not the standard contact sphere).

Floer theoretic approaches also have been developed to study the orderability (and construction of spectral invariants) for prequantization spaces; see [ASZ16], and the RFH approach[12] of [AF12, AM18, AK23, BKK24].[13]

1.2.3. **Milnor fibers and their stabilizations.** We can also apply our results to the links of isolated singularities of certain weighted homogeneous polynomials.[14]

Following classical singularity theory [Mil68, MO70], consider the polynomial $f_k : \mathbb{C}^{n+1} \to \mathbb{C}$ given by

$$f_k(z) = z_0^k + z_1^2 + \cdots + z_n^2,$$

and let $W_k = \left\{ z \in \mathbb{C}^{n+1} : f_k(z) = \epsilon \right\}$ be its *Milnor fiber* for some small $\epsilon > 0$. The space W_k is a Liouville manifold, and its ideal boundary $Y = W_k \cap \partial B(r)$ (for r sufficiently large relative to ϵ) is a contact manifold.

The manifold W_k carries a natural $\mathbb{Z}/2\mathbb{Z}$-action generated by the involution

$$\iota(z_0, z_1, \ldots, z_n) = (z_0, -z_1, \ldots, -z_n).$$

The fixed points of τ in W_k occur where $z_1 = \cdots = z_n = 0$. Imposing the defining equation $f_k(z) = \epsilon$, we find exactly k fixed points corresponding to the k-th roots of ϵ. Because we choose r such that $r^k \gg \epsilon$, these fixed points lie deep in the interior of W_k, so ι acts freely on the boundary Y. Consequently, W_k is a valid $\mathbb{Z}/2\mathbb{Z}$-filling for the quotient contact manifold $Y/(\mathbb{Z}/2\mathbb{Z})$. Because the action has non-trivial stabilizers, Theorem 1.10 applies to guarantee the existence of the spectral invariants c_R for $Y/(\mathbb{Z}/2\mathbb{Z})$.

It is worth noting that W_k typically does not satisfy the hypotheses of Theorem 1.11 for $k \geq 2$. As discussed in [KvK16, Theorem 6.3], the Milnor fibers of isolated singularities of weighted homogeneous polynomials (such as W_k) have non-vanishing symplectic cohomology (because they possess weakly exact Lagrangians).

However, to obtain a measurement μ satisfying properties (G1) through (G5), we can pass to the stabilization $W_k \times \mathbb{C}^d$ for $d \geq 1$, as in §1.2.1; this satisfies the hypotheses of Theorem 1.11.

---

[12] We emphasize that here the spectral invariants we construct are monotone (R2) and sub-additive (R5), and these properties are not established in the existing RFH literature.

[13] The paper [AF12] generalizes Givental's work [Giv90, Giv91] in a different direction from our paper. Their construction does not produce integer valued measurements for contact isotopies φ_t, but rather studies the "asymptotic growth rates" of Rabinowitz-Floer homology groups when one iterates positive paths.

[14] We thank Klaus Niederkrüger for bringing these examples of G-fillings to our attention.

1.3. **Infinity categorification of the Eliashberg–Polterovich relation.** The arguments used in the proofs of Theorems 1.10 and 1.11 are based on mapping cones of chain maps. We briefly preview the set-up:

- to each contact isotopy φ_t lying in the complement of the discriminant (see Definition 1.9), we will associate a chain complex $\mathrm{CF}_{\mathrm{eq}}(\varphi_t)$;
- to each non-negative path $\varphi_{s,t}$ (1) between $\varphi_{0,t}$ and $\varphi_{1,t}$ we will associate a chain map $\mathrm{CF}_{\mathrm{eq}}(\varphi_{0,t}) \to \mathrm{CF}_{\mathrm{eq}}(\varphi_{1,t})$;
- the mapping cone of this chain map is related to the intersections of the path $\varphi_{s,t}$ with the discriminant.

It is known that it is difficult to work with such cones on the level of homology groups (e.g., this leads to the theory of triangulated categories). Many arguments can be presented much more simply when one is working "on chain level." On the other hand, it is also known to be somewhat difficult to properly keep track of chain homotopy terms when working on chain level. A modern but well-established approach to working on chain level is the theory of ∞-categories following [Lur09, Lur17].

The first goal in this section (continued in §2.1) is to introduce an ∞-category $\mathcal{C}(Y)$ of contact isotopies. Roughly speaking, an infinity category is a collection of sets $\mathcal{C}_k(Y)$ of *k-simplices*, for $k = 0, 1, \ldots$, together with "face" and "degeneracy" relationships between these sets of simplices. The 0-simplices should be considered as the "objects" of the category and the 1-simplices should be considered as the "morphisms." In our construction, the 0-simplices $\mathcal{C}_0(Y)$ are contact isotopies φ_t lying outside the discriminant[15] and the 1-simplices $\mathcal{C}_1(Y)$ are essentially the non-negative paths $\varphi_{s,t}$ as in (1).[16] Each 1-simplex $\varphi_{s,t} \in \mathcal{C}_1$ has two *faces* in $\mathcal{C}_0(Y)$, and these are simply the endpoints of the non-negative path.

In a general ∞-category, the k-simplices represent homotopies between morphisms and their compositions. The definition of the set of k-simplices $\mathcal{C}_k(Y)$ in our ∞-category and the face and degeneracy maps will be explained in §2.1.

Our main structural theorem is that the *equivariant Floer cohomology complex*[17] induces an ∞-functor:

$$\mathrm{CF}_{\mathrm{eq}} : \mathcal{C}(Y) \to \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]]). \tag{2}$$

The right hand side is the *dg-nerve* of the category of graded[18] chain complexes over the ring $\mathbf{k}[[x]]$. The only piece of information we need about

[15]The construction admits a minor variation where $\mathcal{C}_0(Y)$ is the complement of the W-contractible discriminant.

[16]The precise definition of $\mathcal{C}_1(Y)$ uses the notion of a Moore path; see Definition 2.1.

[17]Here we are referring to the "Borel" version of equivariant cohomology, as advanced by [SS10, Sei15, SZ21, GMP23, Caz24, SC25].

[18]Our gradings will be valued in the group $\mathbb{Z}/2\mathbb{Z}$ (or sometimes the trivial group).

this dg-nerve is *how to define a functor valued in it.* Luckily the definition is rather compact:

Definition 1.13 (Taken from [Par16, §7.6])**.** *An ∞-functor* (2) *consists of the following assignments:*

- *each zero simplex $\sigma = \varphi_t$ is sent to a (graded) $\mathbf{k}[[x]]$-module:*
$$V_\sigma = \mathrm{CF}_{\mathrm{eq}}(\varphi_t)$$
with a differential d_σ of degree 1;
- *each n-dimensional simplex σ is sent to a map:*
$$\mathfrak{c}_\sigma : V_{\sigma|0} \to V_{\sigma|n}$$
of degree $1-n$; here $\sigma|j$ is the jth vertex of σ.

These maps are required to satisfy the structural equation:

$$\sum_{j=1}^{n-1}(-1)^j(\mathfrak{c}_{\sigma|[j\dots n]}\circ\mathfrak{c}_{\sigma|[0\dots j]}-\mathfrak{c}_{\sigma|[0\dots\hat{j}\dots n]}) = \mathfrak{c}_\sigma\circ d_{\sigma|0}+(-1)^n d_{\sigma|n}\circ\mathfrak{c}_\sigma; \tag{3}$$

additionally we require the following for degenerate simplices:

$$\mathfrak{c}_\sigma = \begin{cases} id & \text{if } \sigma \text{ is a degenerate 1-simplex,} \\ 0 & \text{if } \sigma \text{ is a degenerate } k\text{-simplex with } k>1. \end{cases} \tag{4}$$

Importantly, because 1-simplices are sent to chain maps, we can talk about their mapping cones in a simple and canonical way. We now state the theorem which will be used to prove Theorem 1.10 and 1.11:

Theorem 1.14. *If Y admits an aspherical $\mathbb{Z}/p\mathbb{Z}$-filling W with p prime, then there is an ∞-functor* (2) *for $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$ satisfying the following:*

(a) the output $\mathrm{CF}_{eq}(\varphi_t)$ is a finitely generated $\mathbf{k}[[x]]$-module, for each zero simplex $\varphi_t \in \mathcal{C}_0(Y)$;
(b) if $\varphi_{s,t}$ is a 1-simplex in $\mathcal{C}(Y)$ such that $\varphi_{s,t}$ does not lie on the W-contractible discriminant, for each s, then the associated chain map is a chain homotopy equivalence;
(c) the homology of the cone of the chain map associated to any 1-simplex in $\mathcal{C}(Y)$ is x-torsion (i.e., multiplication by x^d on the homology of the cone acts by 0, for some positive integer d);
(d) if the $\mathbb{Z}/p\mathbb{Z}$-action on W has at least one fixed point, then the homology of $\mathrm{CF}_{\mathrm{eq}}(\varphi_t)$ is not x-torsion, for any object $\varphi_t \in \mathcal{C}_0(Y)$;
(e) for ϵ small enough, the cone of $\mathrm{CF}_{\mathrm{eq}}(R_{-\epsilon t}) \to \mathrm{CF}_{\mathrm{eq}}(R_{\epsilon t})$ is not an acyclic complex.[19]

The construction of the ∞-functor is performed in §2. The proof of (c) uses a local Floer homology argument in §4. The proofs of (d) and (e) uses the PSS isomorphism (Theorem 1.16). Property (e) is added to ensure the

[19]Recall that an acyclic complex is one whose homology is zero.

theorem is not trivially satisfied by a constant functor; we will in fact prove something stronger and identify a particular homology class in $\mathrm{CF}_{\mathrm{eq}}(R_{\epsilon t})$ which projects to a non-zero element in the cone from (e).

1.3.1. **Homology level functor.** Taking homology of the ∞-functor in Theorem 1.14 gives an ordinary functor:

$$\mathrm{HF}_{\mathrm{eq}} : \mathrm{h}\mathcal{C}(Y) \to \mathrm{Mod}(\mathbf{k}[[x]]) \tag{5}$$

where $\mathrm{h}\mathcal{C}(Y)$ is the *homotopy category*[20] of $\mathcal{C}(Y)$ and $\mathrm{Mod}(\mathbf{k}[[x]])$ is the category of modules over $\mathbf{k}[[x]]$. Since $\mathrm{HF}_{\mathrm{eq}}$ is a regular functor defined on a regular category, we can take its colimit in the usual sense. This leads to the definition:

Definition 1.15. *The equivariant symplectic cohomology* $\mathrm{SH}_{\mathrm{eq}}(W)$ *is defined to be the colimit of* (5) *in the category of* $\mathbf{k}[[x]]$*-modules.*

The $\mathbf{k}[[x]]$-module $\mathrm{SH}_{\mathrm{eq}}(W)$ will be used to define the invariants c_R and μ. As we shall see, the invariant μ relies on the $\mathbf{k}[[x]]$-module structure in an important way.

1.4. **PSS map.** There is a larger ∞-category $\bar{\mathcal{C}}(Y)$ such that $\bar{\mathcal{C}}_0(Y) = \mathrm{CI}(Y)$ and $\mathcal{C}(Y) \subset \bar{\mathcal{C}}(Y)$ is the full subcategory spanned by the contact isotopies without discriminant points, (or W-contractible discriminant points, both variants work equally well).

Inside this larger ∞-category $\bar{\mathcal{C}}(Y)$, we will define another ∞-category $\mathcal{P}(Y)$, which we call the *PSS category.*[21] Roughly speaking, simplices in $\mathcal{P}(Y)$ have some of their vertices in $\mathcal{C}(Y)$, and the other vertices are fixed at the constant system *id* (see §3.3.1 for the precise definition). For the purposes of the introduction, it suffices to say:

- any 1-simplex $id \to \varphi_t$ in $\bar{\mathcal{C}}(Y)$, with $\varphi_t \in \mathcal{C}(Y)$, is in $\mathcal{P}(Y)$.

Such 1-simplices represent "continuation data" from the constant system *id* to some output system $\varphi_t \in \mathrm{CI}(Y)$, which must necessarily satisfy $id \leq \varphi_t$ in the sense of [EP00].

Theorem 1.16. *Assume the hypotheses of Theorem 1.14. Counting solutions of moduli spaces of type* [PSS96, FS07] *defines an* ∞*-functor:*

$$\mathcal{P}(Y) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathbf{k}[[x]])) \tag{6}$$

where:

- *non-identity* 0*-simplices* φ_t *are sent to the outputs* $\mathrm{CF}_{\mathrm{eq}}(\varphi_t)$ *of the functor from Theorem 1.14;*

[20] see [Lur09, §1.2.3] for the definition of $\mathrm{h}\mathcal{C}$.

[21] The name comes from [PSS96].

- *the identity* 0*-simplex id is sent to the equivariant Morse cohomology complex* $\mathrm{CM}_{\mathrm{eq}}(P)$ *associated to a* G*-equivariant Morse-Smale pseudogradient* P *which points outwards in the convex end of* W *(the definition is reviewed in 3.1.3).*

By virtue of how ∞*-functors work, the aforementioned* 1*-simplices in* $\mathcal{P}(Y)$ *joining id to a* $\varphi_t \in \mathcal{C}(Y)$ *are sent to chain maps:*

$$\mathrm{PSS} : \mathrm{CM}_{\mathrm{eq}}(X) \to \mathrm{CF}_{\mathrm{eq}}(\varphi_t).$$

If we apply this to a 1*-simplex of the form* $\varphi_{s,t} = R_{\epsilon st}$*, for* $\epsilon > 0$ *small enough, then:*

$$\mathrm{PSS} : \mathrm{CM}_{\mathrm{eq}}(X) \to \mathrm{CF}_{\mathrm{eq}}(R_{\epsilon t}).$$

is a chain homotopy equivalence which can be understood as an equivariant version of the PSS isomorphism of [PSS96, FS07].

1.4.1. **The negative slope PSS map.** There is a "reflected" version of the aforementioned set-up. Inside $\bar{\mathcal{C}}(Y)$, one can also define an ∞-category $\mathcal{N}(Y)$, which we call the *negative PSS category*; see §3.3.1 for the precise definition. For the purposes of the introduction, it suffices to say:

- any 1-simplex $\varphi_t \to id$ in $\bar{\mathcal{C}}(Y)$, with $\varphi_t \in \mathcal{C}(Y)$, is in $\mathcal{N}(Y)$.

Such 1-simplices represent "continuation data" from φ_t to the constant system id, and so the existence of a 1-simplex implies $\varphi_t \leq id$.

Theorem 1.17. *Assume the hypotheses of Theorem 1.14. Counting solutions of moduli spaces of type* [PSS96, FS07] *defines an* ∞*-functor:*

$$\mathcal{N}(Y) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathbf{k}[[x]])) \tag{7}$$

where:

- *non-identity* 0*-simplices* φ_t *are sent to the outputs* $\mathrm{CF}_{\mathrm{eq}}(\varphi_t)$ *of the functor from Theorem 1.14;*
- *the identity* 0*-simplex id is sent to the equivariant Morse cohomology complex* $\mathrm{CM}_{\mathrm{eq}}(-P)$ *associated to a* G*-equivariant Morse-Smale pseudogradient* $-P$ *which points* inwards *in the convex end of* W *(here* P *points outwards so* $-P$ *points inwards).*

By virtue of how ∞*-functors work, the aforementioned* 1*-simplices in* $\mathcal{N}(Y)$ *joining* $\varphi_t \in \mathcal{C}(Y)$ *to id are sent to chain maps:*

$$\mathrm{PSS} : \mathrm{CF}_{\mathrm{eq}}(\varphi_t) \to \mathrm{CM}_{\mathrm{eq}}(-X).$$

If we apply this to a 1*-simplex of the form* $\varphi_{s,t} = R_{\epsilon(s-1)t}$*, for* $\epsilon > 0$ *small enough, then:*

$$\mathrm{PSS} : \mathrm{CF}_{\mathrm{eq}}(\varphi_t) \to \mathrm{CM}_{\mathrm{eq}}(-X).$$

is a chain homotopy equivalence.

1.4.2. **Continuation across slope zero.** The two complexes $\mathrm{CM}_{\mathrm{eq}}(X)$ and $\mathrm{CM}_{\mathrm{eq}}(-X)$ are naturally related to one another, in that there is a canonical chain homotopy class of maps:

$$\mathfrak{c}_{\mathrm{CM}} : \mathrm{CM}_{\mathrm{eq}}(-X) \to \mathrm{CM}_{\mathrm{eq}}(X)$$

defined by counting Morse continuation lines; the definition is reviewed in §3.1.4. This Morse continuation map interacts with nicely PSS and the Floer cohomology continuation maps; this result is well-known to experts and similar statements appear already in [CFO10, Rit16, AK17, CHK23]. We include the precise formulation for completeness.

Theorem 1.18. *Let $\mathfrak{c}$ be the continuation map associated to the 1-simplex Φ in $\mathcal{C}(Y)$ such that Φ^{01} is the path $\varphi_{s,t} = R_{\epsilon(2s-1)t}$ joining $R_{-\epsilon t}$ to $R_{\epsilon t}$. Then the following diagram commutes up to chain homotopy:*

$$\begin{array}{ccc} \mathrm{CF}_{\mathrm{eq}}(R_{-\epsilon t}) & \xrightarrow{\mathfrak{c}} & \mathrm{CF}_{\mathrm{eq}}(R_{\epsilon t}) \\ \big\downarrow{\scriptstyle \mathrm{PSS}} & & {\scriptstyle \mathrm{PSS}}\big\uparrow \\ \mathrm{CM}_{\mathrm{eq}}(-X) & \xrightarrow{\mathfrak{c}_{\mathrm{CM}}} & \mathrm{CM}_{\mathrm{eq}}(X). \end{array}$$

The PSS maps are those from Theorem 1.16 and 1.17.

Remark 1.19. It is a bit awkard to have three PSS theorems; in an ideal world one imagines there is a single ∞-category containing both $\mathcal{P}(Y)$ and $\mathcal{N}(Y)$, with a single ∞-functor, such that Theorem 1.18 is a simple corollary rather than an independent theorem. However, the proof of Theorem 1.18 involves nodal gluing and it is not a priori clear how to package these nodal interfaces into a single infinity-category.

1.5. **On canonicity.** The reader may wonder whether our construction is "canonical," i.e., whether any two readers who follow our construction will obtain the "same" ∞-functor. Our invariant is not canonical, in the same way "the Morse complex" is not canonical: two readers can follow each step of the construction correctly but pick auxiliary data differently and end up with non-isomorphic complexes $\mathrm{CF}_{\mathrm{eq}}(\varphi_t)$ (they should, at least, be quasi-isomorphic).

Our invariant is canonical in a weaker sense: there exists a *canonical diagram* in the category of ∞-categories:

$$\begin{array}{ccc} \mathcal{D}(W) & \xrightarrow{\mathrm{CF}_{\mathrm{eq}}} & \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]]) \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathcal{C}(Y) & & \end{array}$$

where the vertical map is a *trivial Kan fibration*; see [Lur09, §2] or §1.9 for the definition. This diagram is canonical, in that two readers will agree on the result. This statement is of course not entirely mathematical (and

borders on the philosophical). A more mathematical formulation of "canonicity" is that the diagram is well-defined up to a canonical isomorphism. A similar statement holds with $\mathcal{C}(Y)$ replaced by the PSS category $\mathcal{P}(Y)$.

The ∞-functor from Theorem 1.14 is defined as a precomposition $\mathrm{CF}_{\mathrm{eq}} \circ s$, where s is a section of π (a similar statement holds for the ∞-functor in Theorem 1.16). Well-known results from [Lur09] imply these sections exist in abundance. Moreover, using basic properties of trivial Kan fibrations, one can relate the two functors $\mathrm{CF}_{\mathrm{eq}} \circ s$ and $\mathrm{CF}_{\mathrm{eq}} \circ s'$ by picking a "path" between s and s' in the "space" of sections and thereby prove the outputs are chain homotopy equivalent. Such a comparison between sections s, s' will prove the homology level functors agree up to natural isomorphism. In this sense, the lack of canonicity is isolated into the single "contractible choice" of section s.

Of course, such geometric language as "path" and "space" needs to be interpreted properly, and we return to this more precisely in §1.9.

1.6. **Definition of the measurements.** The first step is to appeal to the fact: *if a prime number p divides the order of the stabilizer of some point $w \in W$, then there is a $\mathbb{Z}/p\mathbb{Z}$-subgroup of the stabilizer.*[22] Then define Y' to be the quotient by the action of this $\mathbb{Z}/p\mathbb{Z}$ subgroup. Since Y' is a covering of Y, it is sufficient to prove Theorem 1.10 and 1.11 for Y' instead of Y. Thus, for the remainder of the discussion, we may suppose that $Y = Y'$ and $G = \mathbb{Z}/p\mathbb{Z}$. We can therefore apply Theorems 1.14 and 1.16.

The quantities c_R and μ satisfying Theorems 1.10 and 1.11 are defined in terms of a *unit element* $1 \in \mathrm{SH}_{\mathrm{eq}}$. It is constructed as follows: the PSS chain maps $\mathrm{CM}_{\mathrm{eq}}(X) \to \mathrm{CF}_{\mathrm{eq}}(\varphi_t)$ from Theorem 1.16, associated to a 1-simplex $\varphi_{s,t}$, yield cycles $1(\varphi_{s,t}) \in \mathrm{CF}_{\mathrm{eq}}(\varphi_t)$ corresponding to the sum of local minima in $\mathrm{CM}_{\mathrm{eq}}(X)$.

Lemma 1.20. *The image of the element $1(\varphi_{s,t})$ in the colimit $\mathrm{SH}_{\mathrm{eq}}$ is independent of the choice of 1-simplex $\varphi_{s,t}$.*

This lemma (proved in §5.1.1) is a consequence of the general structural result and furnishes a canonical element $1 \in \mathrm{SH}_{\mathrm{eq}}$. Following [Can23, DUZ23, Can24b, DUZ25] we define:

$$c_R(\varphi_t) := \inf\left\{s : 1 \text{ is in the image of } \mathrm{HF}_{\mathrm{eq}}(\varphi_t^{-1} \circ R_{st}) \to \mathrm{SH}_{\mathrm{eq}}(W)\right\}. \tag{8}$$

In §5.3 we prove that this definition of c_R satisfies (R1) through (R5).

Turning to μ, the new ingredient in Theorem 1.11 is the hypothesis that $\mathrm{SH}(W; \mathbf{k}) = 0$. We first explain how the classical symplectic cohomology fits into the framework of our paper. Define the *non-equivariant quotient* as a

[22] See, e.g., [Alu09, Theorem IV.2.1].

tensor product:[23]

$$\mathrm{CF}_{\mathrm{neq}}(\varphi_t) := \mathrm{CF}_{\mathrm{eq}}(\varphi_t) \otimes_{\mathbf{k}[[x]]} \mathbf{k}, \tag{9}$$

where x acts by 0 on $\mathbf{k}$. Its homology is denoted by $\mathrm{HF}_{\mathrm{neq}}(\varphi_t)$ and we define:

$$\mathrm{SH}_{\mathrm{neq}}(W) := \text{colimit of } \mathrm{HF}_{\mathrm{neq}}(\varphi_t) \text{ over } \varphi_t \in \mathrm{h}\mathcal{C}(Y).$$

We show in §5.4 that:

$$\mathrm{SH}(W;\mathbf{k}) = 0 \implies \mathrm{SH}_{\mathrm{neq}}(W) = 0, \tag{10}$$

where $\mathrm{SH}(W;\mathbf{k})$ agrees with standard definitions of symplectic cohomology, such as the one in [Sei08a].

From (9) and (10), there is a long exact sequence:

$$\cdots \longrightarrow \mathrm{SH}_{\mathrm{eq}} \xrightarrow{x} \mathrm{SH}_{\mathrm{eq}} \longrightarrow 0 \longrightarrow \cdots,$$

The measurement μ satisfying Theorem 1.11 is:

$$\mu(\varphi_t) := \sup\left\{ d \in \mathbb{Z} : x^{-d}1 \text{ is in the image of } \mathrm{HF}_{\mathrm{eq}}(\varphi_t) \to \mathrm{SH}_{\mathrm{eq}}(W) \right\};$$

we extend from $\mathcal{C}_0(Y)$ (the complement of the discriminant) to all of $\mathrm{CI}(Y)$ by continuity from above, i.e., in the unique way compatible with (G2).

In §5.4 we prove (10) and show that μ satisfies (G1) through (G5).

1.7. **Measurements associated to positive loop of contactomorphisms.** Following [FPR18], see also [Can24b, §1.4.7], one can use a positive loop of contactomorphisms in lieu of a Reeb flow when defining the spectral invariants; this naturally leads to an integer valued measurements.

Suppose the hypotheses of Theorem 1.10, and let ζ_t be a positive loop of contactomorphisms (which is necessarily non-contractible!). Then define:

$$\ell(\zeta_t;\varphi_t) := \sup\left\{ k : \mathrm{HF}_{eq}(\varphi_t\zeta_t^{-k}) \to \mathrm{SH}_{eq}(W) \text{ hits the unit} \right\}.$$

Then, as is well-known, $\ell(\zeta_t;-)$ is an unbounded, integer valued, monotone, conjugation invariant measurement, and is constant on the path connected components of the complement of the discriminant.

It is a natural question whether not the invariant μ we construct in Theorem 1.11 is related to such "positive loop" measurements; we plan to investigate this further in future work.

[23] Note that this is a chain level construction, which cannot be directly accessed from the homology level invariants.

1.8. **Local Floer cohomology and cones of continuation maps.** An important ingredient is part (c) of Theorem 1.14, which asserts that cones of continuation maps are always x-torsion. The key idea is that the cone of a continuation map can be understood via the *local Floer homologies* of isolated crossings with the discriminant. The rigorous development of this idea is the object of §4. Beyond the statement about cones of continuation maps being x-torsion, another important consequence of this theory is:

Theorem 1.21. *Consequently, if two isotopies $\varphi_{0,t}, \varphi_{1,t}$ can be joined in the complement of the W-contractible discriminant by* any *path (not assumed to be non-negative), then the morphisms $\mathrm{HF}_{\mathrm{eq}}(\varphi_{i,t}) \to \mathrm{SH}_{\mathrm{eq}}$ are isomorphic.*

The first part of the statement goes back to [UZ22]. The method of proof of the second part is to approximate any path by a zig-zag of positive and negative paths; see [Ulj23, §9] and [DUZ25, §2.1.3] for further discussion of such zig-zags. We prove Theorem 1.21 in §4.4.4.

Remark 1.22. One can localize the category $\mathrm{h}\mathcal{C}(Y)$ at the subcategory of morphisms which do not cross the discriminant (or the W-contractible discriminant) producing a new category $\mathcal{L}$ with a factorization:

$$\mathrm{h}\mathcal{C}(Y) \to \mathcal{L} \to \mathbf{k}[[x]]\text{-mod}.$$

By the zig-zag trick, the fundamental groupoid Π of the complement of the discriminant (without any positivity assumption) admits a functor into $\mathcal{L}$. Thus the restriction of $\mathrm{HF}_{\mathrm{eq}}$ to Π is a local system on the complement of the discriminant. Of course this discussion is rather speculative, and the details of the "localization of morphisms which do not cross the discriminant" step seems to be an interesting avenue of exploration.

1.9. **On ∞-categories.** Since the ∞-categorical language is not yet completely standard in symplectic geometry we recall here the reason for its appearance in the Floer theoretic setting, as well as some basic properties; the reader comfortable with ∞-categories and their role in Floer theory may skip this section.

One of the fundamental issues in Floer theory is that of *invariance*: how to show that the chain complexes we construct do not depend (in a suitable sense) on any choices auxiliary to the geometry at hand (e.g. almost complex structures, choices of Morse-Smale pseudogradient, etc). Morally, one expects these extra choices to be ultimately irrelevant whenever the space of such choices is contractible, and more generally, certain allowable paths in the data should give rise to chain maps.

In our setting we are naturally provided with the following setup: there is a topological category $\mathcal{C}_{\mathrm{top}}(Y)$ whose objects are contact isotopies, and whose morphism spaces consist of the non-negative paths connecting two given objects. Lying over this (via a natural forgetful functor) is a larger topological category $\mathcal{D}_{\mathrm{top}}(W)$ whose objects are all data necessary to define

the Floer cohomology complex, and whose morphism spaces consist of the spaces of regular continuation data with specified endpoints. However, this topological category $\mathcal{D}_{\mathrm{top}}(W)$ and the dg-category of chain complexes do not directly interact very nicely. The language of quasi-categories systematically developed in [Lur09] provides a setting into which both dg-categories and topological categories can be naturally integrated by means of taking the *topological and dg-nerves*, $\mathrm{N}_{\mathrm{top}}, \mathrm{N}_{\mathrm{dg}}$ respectively.

In the present paper the aforementioned ∞-category $\mathcal{C}(Y)$ models the topological nerve of the category $\mathcal{C}_{\mathrm{top}}(Y)$, and we define in §2.2.2 an ∞-category of Floer data $\mathcal{D}(W)$ modeling the topological nerve of $\mathcal{D}_{\mathrm{top}}(W)$; there is forgetful functor $\mathcal{D}(W) \to \mathcal{C}(Y)$. We prove in §2.2.6 that this forgetful map is a trivial Kan fibration:

Definition 1.23 (See [Lur09, §2])**.** *A* trivial Kan fibration *is a morphism of simplicial sets $\mathcal{D} \to \mathcal{C}$ such that following lifting diagram is satisfied:*

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{D} \\ \downarrow{\scriptstyle \iota} & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \Delta^n & \longrightarrow & \mathcal{C} \end{array}$$

for all $n \geq 0$, where ι is the obvious inclusion map. If $\mathcal{S} \to \Delta^0$ is a trivial Kan fibration, then $\mathcal{S}$ is called a contractible Kan complex*, and should be thought of as the simplicial set analog of a contractible space.*

Trivial Kan fibrations and contractible Kan complexes enjoy many convenient properties; the most important for our purposes is the following:

Proposition 1.24. *Let $\pi : \mathcal{D} \to \mathcal{C}$ be a trivial Kan fibration. Then π admits a section s. Moreover, the collection of such sections forms the zero simplices in a contractible Kan complex $\mathcal{S}$.*

Proof. Define an n-simplex in the space of sections $\mathcal{S}$ to be a map

$$s : \Delta^n \times B \to E$$

such that $\pi s = \mathrm{pr}_B$. There is an obvious structure of a simplicial set with these as the n-simplices. Since π is a trivial Kan fibration, monomorphisms (injections) of simplicial sets have the left lifting property against π; see [Lur09, Example 2.0.0.2]. By using this lifting property with the monomorphisms $\partial\Delta^n \times B \hookrightarrow \Delta^n \times B$, we prove that this simplicial set of sections $\mathcal{S}$ is a contractible Kan complex. □

The general strategy is then as follows: There is a canonical Floer complex functor:

$$\mathrm{CF} : \mathcal{D}(W) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathbf{k}[[x]]))$$

since the left hand side consists of exactly the data necessary to define Floer theory. Then every section s of π induces a functor:

$$\mathrm{CF}_{\mathrm{eq}} \circ s : \mathcal{C}(Y) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathbf{k}[[x]]))$$

by composition. Contractibility of the space of sections then says that any choice of s (i.e., any compatible collection of Floer data) induces homotopic Floer functors and, moreover, all such homotopies are themselves homotopic, etc. We return to this discussion in §2.2.6.

Carrying out such a "chain-level construction" has the advantage of making certain constructions (such as taking cones and tensor products[24]) considerably simpler. In our case, it is used in §4 to verify that a certain Floer complex may be expressed as a mapping cone for a very particular choice of data, and it then follows by the abstract machinery that any other choice of data extending the underlying system has the same property.

Such setups have been leveraged before in Floer theory to similar ends, cf., [Par16], [HLS20], [GPS20, §4], [Var21, §A].

2. Equivariant Floer complex as infinity functor

The goal of the first subsection §2.1 is to introduce the ∞-category of contact isotopies (continuing from the outline we gave in §1.3). In §2.2 we prove Theorem 1.14 in the case $p = 2$, which has various differences and simplifications from the general case $p \geq 3$, which we treat in §2.3 and §2.4.

2.1. **Infinity category of contact isotopies.** As an ∞-category, $\mathcal{C}(Y)$ is, first and foremost, a special type of *simplicial set*; see [Lur09, §A.2.7] for a review of simplicial sets. This means that $\mathcal{C}(Y)$ is a contravariant set-valued functor from the *combinatorial simplex category*, whose objects are non-negative integers $[0], [1], [2], \ldots$, and where the morphisms $[i] \to [j]$ are the non-decreasing maps $\{0, \ldots, i\} \to \{0, \ldots, j\}$. More prosaically, $\mathcal{C}(Y)$ is the data of sets $\mathcal{C}_0(Y), \mathcal{C}_1(Y), \mathcal{C}_2(Y), \ldots$, where $\mathcal{C}_j(Y)$ is called the set of *j-simplices*, together with morphisms:

$$f^* : \mathcal{C}_j(Y) \to \mathcal{C}_i(Y) \text{ for every morphism } f : [i] \to [j], \tag{11}$$

and such that the morphisms in (11) are functorial.

2.1.1. **Definition of the set of simplices.** We define[25] $\mathcal{C}_n(Y)$ to be certain coherent cubes of non-negative paths in the group of contact isotopies $\mathrm{CI}(Y)$.

The formal construction uses the space of non-negative "Moore paths" in $\mathrm{CI}(Y)$. Let us introduce $\mathrm{CI}(Y)^{\mathbb{R},+} \subset \mathrm{CI}(Y)^{\mathbb{R}}$ as the set of smooth non-negative paths as defined in (1).

[24]We use a chain level construction in the definition of $\mathrm{HF}_{\mathrm{neq}}(\varphi_t)$ as the homology of $\mathrm{CF}_{\mathrm{eq}}(\varphi_t) \otimes_{\mathbf{k}[[x]]} \mathbf{k}$.

[25]In this section, we primarily discuss the construction of $\mathcal{C}(Y)$. However, the discussion works equally well for constructing the larger category $\bar{\mathcal{C}}(Y)$ (see §1.4), the only difference being that 0-simplices in the full-subcategory $\mathcal{C}(Y) \subset \bar{\mathcal{C}}(Y)$ are required to be in the complement of the discriminant.

An element $(s \mapsto \varphi(s)) \in \mathrm{CI}^{\mathbb{R},+}(Y)$ is said to be *stationary* on an interval $I \subset \mathbb{R}$ provided $\varphi(s_1) = \varphi(s_2)$ for all $s_1, s_2 \in I$.

Definition 2.1. *A non-negative* Moore path *in* $\mathrm{CI}(Y)$ *is an element:*

$$(w, \varphi) \in [0, \infty) \times \mathrm{CI}(Y)^{\mathbb{R},+}$$

such that φ *is stationary on the intervals* $(-\infty, 0]$ *and* $[w, \infty)$*. The space of non-negative Moore paths shall be denoted* $\mathrm{NN}(Y)$*. The values of the path at* 0 *and* w *are called the* initial and terminal points *of the path. Two Moore paths* (w_1, φ_1) *and* (w_2, φ_2) *are said to be* composable *provided the terminal point of the former is the initial point of the latter; in this case, the* ℓ*-concatenation is denoted* $(w_1, \varphi_1)\#_\ell(w_2, \varphi_2)$ *which inserts a stationary cylinder of length* ℓ *in between.*

In other words, $(w, \varphi) \in \mathrm{NN}(Y)$ represents a non-negative paths of width w; we allow paths of width 0 which must be stationary on all of $\mathbb{R}$.

Definition 2.2. *A family of Moore paths parametrized by a cube*

$$(w, \varphi) : [0, 1]^{n-1} \to \mathrm{NN}(Y)$$

is said to be smooth *provided the underlying functions* $w : [0, 1]^{n-1} \to [0, \infty)$ *and* $\varphi : [0, 1]^{n-1} \to \mathrm{CI}(Y)^{\mathbb{R},+}$ *are smooth.*

Now we turn to the formal definition of $\mathcal{C}(Y)$. First of all, the set of zero simplices $\mathcal{C}_0(Y)$ is simply the subset of elements of $\mathrm{CI}(Y)$ which do not lie on the discriminant. For higher simplices, the definition will depend on the auxiliary choice of the smooth homeomorphism $\tau : [0, \infty) \to [0, \infty)$:

$$\tau(x) = \int_0^x e^{-1/y} dy \tag{12}$$

which:

- vanishes to infinite order at $x = 0$, and
- is bounded by $\tau(x) \leq x$.

We then have the following definition.[26]

Definition 2.3. *For* $n \geq 1$*, an* n*-simplex is a collection* Φ *of elements* $\Phi^{v_0 \cdots v_m}$*, one for each collection* $0 \leq v_0 < \cdots < v_m \leq n$ *of vertices, such that:*

- $\Phi^{v_0} \in \mathcal{C}_0(Y)$ *for each collection with* $m = 1$*,*
- $\Phi^{v_0 \cdots v_m} : [0, \infty)^{m-1} \to \mathrm{NN}(Y)$ *is a smooth cube of non-negative Moore paths joining* Φ^{v_0} *and* Φ^{v_m}*,*

These data are required to satisfy the following compatibilities:

(1) as $x_j \to \infty$*,*

$$\Phi^{v_0 \cdots v_m}(x) = \Phi^{v_0 \cdots v_j}(x_1, \ldots, x_{j-1}) \#_{\tau(x_j)} \Phi^{v_j \cdots v_m}(x_{j+1}, \ldots, x_{m-1}),$$

[26]We thank Joseph Helfer for suggesting such a model to the authors. See [GPS20, Definition 4.2] for similar definition.

(2) when $x_j = 0$,

$$\Phi^{v_0 \dots v_m}(x) = \Phi^{v_0 \dots \hat{v_j} \dots v_m}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{m-1}),$$

(3) on the face $x_j = 0$, $\partial_{x_j} \Phi^v(x)$ *vanishes to infinite order.*

The collection of all n*-simplices is denoted* $\mathcal{C}_n(Y)$.

2.1.2. **Definition of the face maps.** In this section, we define the face maps and verify the structural equations of a semi-simplicial set are satisfied. In the next section §2.1.3 we define the degeneracy maps and complete the proof that $\mathcal{C}(Y)$ is a genuine simplicial set.

Recall that any n-simplex Φ should have $n+1$ *faces* $d_0\Phi, d_1\Phi, \dots, d_n\Phi$ so that $d_i\Phi$ is the "pullback" of Φ by the simplicial map $\Delta^{n-1} \to \Delta^n$ which omits the ith vertex.

Definition 2.4. *Let* Φ *be an* n*-simplex, as in Definition 2.3.*

For $i = 0, \dots, n$*, the* i*th face is* $d_i\Phi$ *is defined by forgetting all those* $\Phi^{v_0 \dots v_m}$ *containing* $v_j = i$ *amongst the collection.*

One can now check that $\mathcal{C}(Y) = (\mathcal{C}_0(Y), \mathcal{C}_1(Y), \dots)$ with the face maps in 2.4 satisfies the axioms of a *semi-simplicial set*, namely that the structural relation holds:

$$d_i d_j = d_{j-1} d_i \text{ for } i < j;$$

see [Lur26, §1.1.1].

2.1.3. **Definition of the degeneracy maps.** A simplicial set is also equipped with *degeneracies*, $s_i : \mathcal{C}_n(Y) \to \mathcal{C}_{n+1}(Y)$, $i = 0, \dots, n$; here s_i corresponds to the simplicial map $\Delta^{n+1} \to \Delta^n$ for which the ith vertex has two preimages and all other vertices have one preimage; see [Lur26, §1.1.2].

We find it useful to introduce the following operation:

Definition 2.5. *Fix* $\tau : [0, \infty) \to [0, \infty)$*, as in* (12)*. The* smoothed max function *associated to* τ *is:*

$$\operatorname{smax}(x_1, \dots, x_k) := \tau^{-1}(\tau(x_1) + \cdots + \tau(x_k)).$$

Note that this function is not smooth at the origin, but it is smooth everywhere else.

Definition 2.6. *Let* Φ *be an* n*-simplex, as in Definition 2.3, and fix* τ *and its associated smoothed max function as in Definition 2.5. Suppose that* τ *vanishes to infinite order at the origin.*

For $i = 0, \dots, n$*, the* i*th degeneracy* $s_i\Phi$ *is an* $(n+1)$*-simplex defined as follows. Let* $\sigma_i : [n+1] \to [n]$ *be the surjective map where* $\sigma_i(j) = j$ *for* $j \leq i$ *and* $\sigma_i(j) = j - 1$ *for* $j > i$.

For any subset $V = \{0 \leq v_0 < \cdots < v_m \leq n+1\}$*, let* $U = \sigma_i(V) \subset [n]$*. The subcube* $(s_i\Phi)^V$ *is defined in two cases:*

Case 1: If V does not contain both i and $i+1$, then σ_i is injective on V. We define:

$$(s_i\Phi)^V(x_1,\dots,x_{m-1}) = \Phi^U(x_1,\dots,x_{m-1}).$$

Case 2: If V contains both i and $i+1$, then $v_k = i$ and $v_{k+1} = i+1$ for some $0 \le k < m$. The definition depends on the position of the doubled vertices:

- *If $0 < k < m-1$, the doubled vertices are both internal. We define:*

$$(s_i\Phi)^V(x) = \Phi^U(x_1,\dots,x_{k-1},\operatorname{smax}(x_k,x_{k+1}),x_{k+2},\dots,x_{m-1}).$$

- *If $k = 0$, the doubled vertices are at the beginning of V. We define:*

$$(s_i\Phi)^V(x) = id\#_{\tau(x_1)}\Phi^U(x_2,\dots,x_{m-1}).$$

- *If $k = m-1$, the doubled vertices are at the end of V. We define:*

$$(s_i\Phi)^V(x) = \Phi^U(x_1,\dots,x_{m-2})\#_{\tau(x_{m-1})}id.$$

If $m = 1$ and $V = \{i, i+1\}$, we interpret these formulas as yielding the stationary path of width zero based at the 0-simplex Φ^i, which acts as the identity morphism id.

Lemma 2.7. *For any n-simplex Φ and $0 \le i \le n$, the collection of subcubes $s_i\Phi$ from Definition 2.6 is a well-defined $(n+1)$-simplex (satisfying the axioms of Definition 2.3). Furthermore, the face and degeneracy operators satisfy the simplicial identities:*

$$s_i s_j = s_{j+1}s_i \text{ for all } 0 \le i \le j \le n,$$

and

$$d_j s_i = \begin{cases} s_{i-1}d_j & \text{if } j < i, \\ id & \text{if } j = i \text{ or } j = i+1, \\ s_i d_{j-1} & \text{if } j > i+1; \end{cases}$$

therefore $\mathcal{C}(Y)$ with these face and degeneracy operators defines a simplicial set.

Proof. First we show $s_i\Phi$ satisfies the $(n+1)$-simplex axioms. The proof of axiom (3), that $\bar{\Phi} = s_i\Phi$ restricts correctly to its subcubes as variables $x_k \to 0$, is identical to the proof of the identities $d_j s_i$ verified below.

To verify smoothness and the microcollaring axiom (4), observe that:

$$\operatorname{smax}(x_k, x_{k+1}) = \tau^{-1}(\tau(x_k) + \tau(x_{k+1}))$$

is smooth everywhere except when $x_k = x_{k+1} = 0$. However, because Φ satisfies microcollaring, its partial derivatives vanish to infinite order as variables approach 0. This infinite-order vanishing compensates for the non-smoothness of τ^{-1} at the origin, ensuring the pulled-back subcubes are smooth.

The breaking axiom (2) as $x_j \to \infty$ holds by direct calculation. For instance, if $x_k \to \infty$ in a subcube where $v_k = i$ and $v_{k+1} = i+1$, then

$\mathrm{smax}(x_k, x_{k+1}) \to \infty$. Using that $\tau(\mathrm{smax}(x_k, x_{k+1})) = \tau(x_k) + \tau(x_{k+1})$ we have:

$$\begin{aligned}\bar{\Phi}^V(x) &= \Phi^U(\dots, \mathrm{smax}(x_k, x_{k+1}), \dots) \\ &= \Phi^{U_-}(\dots)\#_{\tau(\mathrm{smax}(x_k, x_{k+1}))}\Phi^{U_+}(\dots) \\ &= \Phi^{U_-}(\dots)\#_{\tau(x_k)} id \#_{\tau(x_{k+1})}\Phi^{U_+}(\dots),\end{aligned}$$

which matches the formulas for the subcubes $\bar{\Phi}^{V_-}$ and $\bar{\Phi}^{V_+}$. The edge cases $k = 0$ and $k = m - 1$ follow by a similar argument.

We now verify the face relations. For $j \neq 0, n+1$, applying d_j to $s_i\Phi$ amounts to fixing the variable corresponding to the vertex j to 0. If $j = i$ or $j = i + 1$, we are evaluating a cube where the variable between i and $i + 1$ vanishes. Because $\mathrm{smax}(x, 0) = \mathrm{smax}(0, x) = x$, substituting 0 into $\mathrm{smax}(x_i, x_{i+1})$ leaves the remaining coordinate unchanged. For example, if we apply d_i to the top cube by setting $x_i = 0$, the i-th variable is omitted and x_{i+1} takes its place in the argument list, recovering the original top cube of Φ. For the boundary cases $i = 0$ and $i = n$, the internal faces $d_1 s_0 = id$ and $d_n s_n = id$ are handled similarly by setting the corresponding variable to zero, since $\tau(0) = 0$ means $\#_{\tau(0)} id$ acts as the identity. The outermost faces $d_0 s_0 = id$ and $d_{n+1} s_n = id$ follow from the breaking axiom (2) as the relevant variable tends to infinity. Thus $d_i s_i = id$ and $d_{i+1} s_i = id$ hold for all i.

For $j > i + 1$, setting the j-th variable to 0 omits it, while x_i and x_{i+1} are unaffected. Writing this out for the top cube yields:

$$\Phi(x_1, \dots, \mathrm{smax}(x_i, x_{i+1}), \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n).$$

The 0 occurs at the $(j-1)$-th position in the argument list passed to Φ. By Axiom (3) for Φ, this evaluates to the $j - 1$ face of Φ evaluated with the doubled i-th vertex, which is exactly $s_i(d_{j-1}\Phi)$. An identical argument handles $j < i$, proving $d_j s_i = s_{i-1} d_j$.

We now verify the degeneracy relations $s_i s_j = s_{j+1} s_i$ for $0 \leq i \leq j \leq n$. When $0 < i < j < n$, applying s_j then s_i (or vice-versa) results in an $(n+2)$-simplex containing the non-overlapping terms:

$$\Phi(x_1, \dots, \mathrm{smax}(x_i, x_{i+1}), \dots, \mathrm{smax}(x_{j+1}, x_{j+2}), \dots, x_{n+1}),$$

which is symmetric. When $i = j > 0$, both operators create a bracketed maximum (of the form $\mathrm{smax}(a, \mathrm{smax}(b, c))$). Since

$$\tau(\mathrm{smax}(a, b)) = \tau(a) + \tau(b),$$

we have:

$$\begin{aligned}\mathrm{smax}(\mathrm{smax}(x_i, x_{i+1}), x_{i+2}) &= \tau^{-1}(\tau(x_i) + \tau(x_{i+1}) + \tau(x_{i+2})) \\ &= \mathrm{smax}(x_i, \mathrm{smax}(x_{i+1}, x_{i+2})).\end{aligned}$$

In the boundary case $i = j = 0$, evaluating $s_0 s_0 \Phi$ introduces two prepended identity paths. Their concatenated length is $\tau(x_1) + \tau(x_2)$, giving:

$$id \#_{\tau(x_1)+\tau(x_2)} \Phi(x_3, \ldots, x_{n+1}).$$

Conversely, $s_1 s_0 \Phi$ yields:

$$id \#_{\tau(\mathrm{smax}(x_1,x_2))} \Phi(x_3, \ldots, x_{n+1}).$$

These are equal since $\tau(\mathrm{smax}(x_1, x_2)) = \tau(x_1) + \tau(x_2)$. The remaining boundary cases where $j = n$ are handled symmetrically. This completes the proof. □

2.1.4. **Verification of the ∞-category axiom.** An ∞-category is not merely a simplicial set, but is required to satisfy additional *horn-filling* properties; see [Lur09, Definition 1.1.2.4]. See also [Lur09, Notation A.2.7.2] and [Lur09, Example A.2.7.3].

Proposition 2.8. *The simplicial set $\mathcal{C}(Y)$ is an ∞-category.*

Proof. Fix $i \neq 0, n$, and fix $n-1$ simplices $\Phi_0, \ldots, \Phi_{i-1}, \Phi_{i+1}, \ldots, \Phi_n$ such that:

$$d_j \Phi_k = d_{k-1} \Phi_j$$

holds for all $j < k$, $j \neq i$. One needs to prove the equation:

$$d_j \Phi = \Phi_j \text{ for } j \neq i \tag{13}$$

admits a solution Φ. The key idea is that Φ is uniquely determined by its main cube $\Phi^{0\ldots n}$, and (13) specifies all faces of this main cube except for the $x_i = 0$ face. The existence of Φ then follows from the usual "Serre lifting property," which asserts that data specified on all faces of a cube, except for one, can be extended to the interior of the cube. The collaring conditions (2) and (4) for Φ_j ensure that one can pick an extension which is smooth, and which satisfies (2), (3), and (4). The details of this latter assertion go as follows: let P be the union of all boundary faces of a cube Q except for one face, and let p be the center point of this missing face. The "stereographic" projection map based at p maps $Q - p$ continuously onto P, and in fact defines a continuous retraction R of the inclusion $P \subset Q$. By precomposing with R, we can extend the data defined on P to data defined on $Q - p$. The crucial observation is that, even though R is not smooth at the points which are mapped to the faces of codimension two or higher, the fact the data on P satisfies (2) and (4) ensures that the pulled back data is in fact smooth. It is a routine affair to extend from $Q - p$ to all Q, by flowing by a vector field on Q which vanishes near P and which maps Q into $Q - p$. □

Remark 2.9. The same exact argument proves that the larger category $\bar{\mathcal{C}}(Y)$, where we do not require vertices lie off of the discriminant, is an infinity category.

2.1.5. **Equivalent morphisms.** In an ∞-category, two 1-simplices σ, σ' are *equivalent* if there is 2-simplex Φ satisfying $\Phi^{01} = \sigma$, $\Phi^{02} = \sigma'$, and $\Phi^{12} = id$, where id means a degenerate 1-simplex; see [Lur09, pp. 39].

Consider σ, σ' as elements in $\mathrm{NN}(Y)$. In our case, all the information of Φ is contained in the top dimensional cube:

$$\Phi^{012}(x_1) \in \mathrm{NN}(Y),$$

depending on $x_1 \in [0, \infty)$, satisfying:

- Φ^{012} has fixed endpoints,
- $\Phi^{012}(0) = \Phi^{02}$,
- $\Phi^{012}(x_1) = \Phi^{01} \#_{\tau(x_1)} id$,

for x_1 large enough. From this description, and standard facts about Moore paths, we see that: *two 1-simplices σ, σ' are equivalent if and only if they are homotopic in the space of non-negative paths $\mathrm{NN}(Y)$ with fixed endpoints.* It follows that the homotopy category $\mathrm{h}\mathcal{C}(Y)$ is the category considered in [CHK23, 2.2.9].

2.1.6. **On the complexity of $\mathcal{C}(Y)$.** In many appearances of ∞-categories in Floer theory, the ∞-category of data $\mathcal{C}$ is a contractible Kan complex (this is the case, in, e.g., the treatment of Hamiltonian Floer theory in [Par16]). In such cases, the ∞-category itself is not of significant independent interest, and its use is solely in achieving more canonical constructions (this is not to say the use of ∞-categories is unimportant in these contexts). In other appearances, such as [GPS20, Var21] the ∞-category of data $\mathcal{C}$ satisfies the property that the natural map to the nerve of the homotopy category:

$$\mathcal{C} \to \mathrm{N}(\mathrm{h}\mathcal{C}) \tag{14}$$

is a trivial Kan fibration (Definition 1.23); see [GPS20, §4.4] and [Var21, §3.2.2] for a related discussion. In these cases, the homotopy category $\mathrm{h}\mathcal{C}$ carries some amount of complexity, but no additional complexity is hidden in $\mathcal{C}$. It is notable that in our case, the category $\mathcal{C}(Y)$ does not satisfy this property.

Lemma 2.10. *The morphism* (14) *is not a trivial Kan fibration for $\mathcal{C} = \mathcal{C}(Y)$ if $\pi_k(\mathrm{Cont}(Y))$ is non-trivial for at least one number $k \geq 3$.*

Proof. The property we need about the nerve of the ordinary category $\mathrm{h}\mathcal{C}$ is that any map $\partial\Delta^k \to \mathrm{N}(\mathrm{h}\mathcal{C})$ extends to a map $\Delta^k \to \mathrm{N}(\mathrm{h}\mathcal{C})$, if $k \geq 3$.[27] In particular, if (14) is a trivial Kan fibration, then any map $\partial\Delta^k \to \mathcal{C}(Y)$

[27]Indeed, the information of a 3-simplex in $\mathrm{N}(\mathrm{h}\mathcal{C})$ is a sequence of three composable morphisms f, g, h, while the data of $\partial\Delta^3 \to \mathrm{N}(\mathrm{h}\mathcal{C})$ is the four sequences

- f, g on the $\{0, 1, 2\}$ face,
- f, gh on the $\{0, 1, 3\}$ face,
- fg, h on the $\{0, 2, 3\}$ face,
- g, h on the $\{1, 2, 3\}$ face.

extends to a map $\Delta^k \to \mathcal{C}(Y)$, for $k \geq 3$. We will use the existence of a non-trivial class $f \in \pi_k(\mathrm{Cont}(Y))$ to produce a boundary $\partial\Delta^k \to \mathcal{C}(Y)$ which cannot be filled, thereby proving the lemma.

It is more convenient to work in $\bar{\mathcal{C}}(Y)$ where the identity 0-simplex is allowed; this is without loss of generality, since the space of Moore paths joining id to id is naturally identified with the space of Moore paths joining φ_t to φ_t, via pointwise composition (using the group structure of $\mathrm{Cont}(Y)$).

First of all, by standard adjunction identifications, we can think of f as a $k-2$ cube in the space of Moore paths in the space of based loops of $\mathrm{Cont}(Y)$. In particular, this induces a $k-1$ simplex Φ in $\bar{\mathcal{C}}(Y)$ given by the formula:

$$\Phi(x_1, \dots, x_{k-2}) = f(x_1, \dots, x_{k-2}) \#_{\tau(x_1)+\dots+\tau(x_{k-2})} id.$$

Now consider $\Gamma(f) = \partial\Delta^k \to \bar{\mathcal{C}}(Y)$ which agrees with f on the $[1 \dots k]$ face, and agrees with id on each other face. Then:

Claim. *If $\Gamma(f)$ can be filled to a k-simplex in $\bar{\mathcal{C}}(Y)$, then f represents the trivial element in $\pi_k(\mathrm{Cont}(Y))$.*

If so, then there exists Ψ so that:

$$\Psi(x_1, \dots, x_{k-1}) = id \#_{\tau(x_1)} f(x_2, \dots, x_{k-1}) \#_{\tau(x_2)+\dots+\tau(x_{k-1})} id$$

for x_1 large enough, while $\Psi(0, x_1, \dots, x_{k-1}) = \Psi^{02\dots k}(x_2, \dots, x_{k-1}) = id$, and so $\eta \in [0, \infty) \mapsto \Psi(\eta, x_1, \dots, x_{k-1})$ acts as a homotopy between id and f. In general $\Psi(\eta, x_1, \dots, x_{k-1})$ is some non-negative loop based at id; since there do not exist C^0 small non-constant and non-negative loops of contactomorphisms,[28] the $t = 1$ endpoint of the isotopies must remain fixed at id. Thus the homotopy over $\eta \in [0, \infty)$ proves that f was the zero element in $\pi_k(\mathrm{Cont}(Y))$. □

Remark 2.11. In [CS16, Theorem 1] it is shown that $i : U(n) \to \mathrm{Cont}(S^{2n-1})$ is injective on homotopy groups. Thus Lemma 2.10 is not vacuous.

2.2. **Equivariant Floer cohomology over characteristic two.** We split the proof of Theorem 1.14 into the two cases $p = 2$ and $p \geq 3$. There are differences between the cases:

- in the case $G = \mathbb{Z}/2\mathbb{Z}$, we consider EG as $S^\infty \subset \mathbb{R}^\infty$ with the antipodal $\mathbb{Z}/2\mathbb{Z}$-action, and use the shift self-symmetry of $\mathbb{R}^\infty$:

$$\tau : (\eta_0, \eta_1, \dots) \mapsto (0, \eta_0, \eta_1, \dots);$$

One can simply take f, g from the $\{0, 1, 2\}$ face and h from the $\{1, 2, 3\}$ face, and prove that the three simplex (f, g, h) fills the map $\partial\Delta^3 \to \mathrm{N}(\mathrm{h}\mathcal{C})$; the verification uses the fact that the faces of the constituents of a map $\partial\Delta^3$ satisfy various equations amongst each other.

[28]The non-existence of C^0 small non-negative and non-constant loops of contactomorphisms follows from the existence of generating functions and action selectors for Legendrian isotopies in 1-jet spaces due to [Che96] see also [CS15].

- in the case $G = \mathbb{Z}/p\mathbb{Z}$ with $p \geq 3$, we consider EG as $S^\infty \subset \mathbb{C}^\infty$ with the $\mathbb{Z}/p\mathbb{Z}$-action inherited from the circle action on $\mathbb{C}^\infty$, and we use the shift self-symmetry of $\mathbb{C}^\infty$ instead of $\mathbb{R}^\infty$. This leads to various differences in the recipe for constructing equivariant Floer complexes.
- in the case $p = 2$, so $\mathbf{k} = \mathbb{Z}/2\mathbb{Z}$, we do not need to discuss orientations in Floer theory, whereas the case $p \geq 3$ requires such discussion.

In the remainder of this subsection §2.2 we focus solely on the case $p = 2$.

2.2.1. **Reminder of the Borel equivariant cohomology complex.** Our approach to defining equivariant Floer cohomology complexes follows [SS10, Sei15]. Their strategy is to emulate the Borel construction [Bor60] in the language of Morse theory in such a way that it carries over to Floer theory. We briefly recall the idea in the context of Morse theory: given a compact manifold M with a $\mathbb{Z}/2\mathbb{Z}$-action, one considers pseudogradients on $M \times_{\mathbb{Z}/2\mathbb{Z}} S^\infty$ of a specific type (some care is needed due to the non-compactness of S^∞). Appropriately counting the flow lines of such a pseudogradient defines the equivariant Morse differential.

One thinks of a pseudogradient P on $M \times_{\mathbb{Z}/2\mathbb{Z}} S^\infty$ as a $\mathbb{Z}/2\mathbb{Z}$-equivariant vector field on $M \times S^\infty$. It is convenient to fix a standard pseudogradient V on S^∞, and then only consider P which are related to V under the projection $M \times S^\infty \to S^\infty$. Such vector fields can be thought of as S^∞-families of vector fields on M satisfying a certain $\mathbb{Z}/2\mathbb{Z}$-equivariance. In the Floer theoretic setting, one similarly considers S^∞-families of auxiliary data as the necessary input to obtain a chain complex.

2.2.2. **Auxiliary category of Floer data.** As explained in §1.9, we do not directly define an infinity functor $\mathcal{C}(Y) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch})$, but rather construct a diagram of the form:

$$\begin{array}{ccc} \mathcal{D}^*(W) & \xrightarrow{\mathrm{CF}_{\mathrm{eq}}} & \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}) \\ {\scriptstyle \mathrm{forget}}\downarrow & & \\ \mathcal{C}(Y) & & \end{array} \tag{15}$$

where $\mathcal{D}^*(W)$ is a larger ∞-category whose objects consist of all the necessary data needed to define the Floer complex; for similar set-up, we refer the reader to [Par16, Eqn. 7.0.1]. We will first define $\mathcal{D}(W)$ and then define $\mathcal{D}^*(W) \subset \mathcal{D}(W)$ as a subcategory of "regular" simplices. The goal of the present section is to construct $\mathcal{D}(W)$ and the forgetful map. In §2.2.6, we show that the forgetful map in (15) is a *trivial Kan fibration* (see [Par16, Eqn. 7.5.3]). Precomposing $\mathrm{CF}_{\mathrm{eq}}$ with a section of this fibration induces an ∞-functor $\mathcal{C}(Y) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch})$; this is the ∞-functor satisfying Theorem 1.14.

The definition is in terms of *Borel data*, namely pairs $(\psi_{\eta,t}, J)$ where:[29]

- J is a G-invariant, ω-tame and Liouville-equivariant-at-∞ almost complex structure on W,
- $\psi_{\eta,t}$ is a family of Hamiltonian isotopies of W depending on a parameter $\eta \in S^\infty$;

Borel data is supposed to satisfy the following:

(B1) (*equivariance*) $g^{-1}\psi_{g\eta,t}g = \psi_{\eta,t}$,
(B2) (*shift invariance*) $\psi_{\tau(\eta),t} = \psi_{\eta,t}$, where $\tau : \mathbb{R}^\infty \to \mathbb{R}^\infty$ shifts by 1,
(B3) (*polar*) $\psi_{\eta,t}$ is independent of η in fixed neighborhoods of the poles,
(B4) (*ideal restriction*) the Hamiltonian generator $X_{\eta,t}$ of $t \mapsto \psi_{\eta,t}$ is η-independent and Liouville-equivariant outside of a compact set.

Consequently there is a contact isotopy arising as the ideal restriction, denoted $\varphi_t = \mathrm{IR}(\psi_{\eta,t})$.

As in Definition 2.1, one can define a *Moore paths* of Borel data to be a smooth map:[30]

$$s \in \mathbb{R} \mapsto (\psi_{s,\eta,t}, J_s)$$

together with a "width" $w \in [0, \infty)$ such that $\psi_{s,\eta,t}, J_s$ is s-independent on the intervals $[w, \infty)$ and $(-\infty, 0]$; such a path is said to be *non-negative* provided the ideal restriction lies in the space of non-negative paths $\mathrm{NN}(Y)$.

Definition 2.12. *The space of non-negative Moore paths of such Borel data is denoted* $\mathfrak{F}(W)$*, and admits an ideal restriction map to* $\mathrm{NN}(Y)$.

Then an n-simplex $\Phi \in \mathcal{D}_n(W)$ is defined exactly as in Definition 2.3 except:

- zero simplices $\mathcal{D}_0(W)$ are required to have ideal restrictions in $\mathcal{C}_0(Y)$.
- $\mathrm{NN}(Y)$ is replaced by $\mathfrak{F}(W)$.

The face and degeneracy maps are exactly as in Definition 2.4 and 2.6. The same proof of the "horn-filling axioms" given in Theorem 2.8 works here to show $\mathcal{D}(W)$ is an ∞-category. Moreover, there is an obvious "ideal restriction" ∞-functor $\mathcal{D}(W) \to \mathcal{C}(Y)$.

Inside of $\mathcal{D}(W)$ we will define a subcategory $\mathcal{D}^*(W)$ consisting of those simplices which achieve *regularity*, i.e., which render a certain (countable) collection of moduli spaces transverse. We defer the definition of $\mathcal{D}^*(W)$ and the proof that it is an ∞-category to the end of the next section §2.2.3.

[29] We refer the reader to [SS10, pp. 1468] for a similar set-up.

[30] The axioms (B3) and (B4) involve open neighborhoods, and these neighborhoods are assumed to be uniform along any smooth family.

2.2.3. **Moduli spaces.** First of all, we fix a pseudogradient V on $\mathbb{R}P^\infty$. In projective coordinates, the time s flow of V is given by:

$$[x_0 : x_1 : x_2 : \dots] \mapsto [x_0 : e^s x_1 : e^{2s} x_2 : \dots].$$

This pseudogradient on $\mathbb{R}P^\infty$ has one critical point v_k of each index k, and we refer to these as *poles*. The lift of V to the double cover S^∞ will also be denoted by V, and it has two critical points $v_{k,\pm}$ of each index, which we also call poles. Let $\mathcal{P}(v_{k,\pm})$ denote the space of parametrized flow lines of $-V$ on S^∞ joining $v_{k,\pm}$ to $v_{0,+}$. These form open manifolds of the expected dimension, namely k, and can be compactified to manifolds with corners in a reasonable way; see, e.g., [Sei15, §3.1].

Each triple $(\Phi, k, \pm)$, where Φ is an n-simplex in $\mathcal{D}(W)$, $k = 0, 1, 2, \dots$, and $\pm$ is a sign, determines a moduli space $\mathcal{M}(\Phi, k, \pm)$ of triples (x, π, u) of the following type:

- $x \in [0, \infty)^{n-1}$,
- $\pi \in \mathcal{P}(v_{k,\pm})$, and,
- u solves a suitable version of Floer's equation on $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$.

For technical use, we also fix a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that:

- $f'(t)$ is 1-periodic and non-negative,
- $f'(t)$ vanishes in a neighborhood of 0, and,
- $f(0) = 0$ and $f(1) = 1$.

and a $[0, 1]$-valued cut-off function ρ satisfying:

- $\rho'(t) \leq 0$,
- $\rho'(t)$ is supported in the interior of the region where $f = 1$,
- $\rho(0) = 1$ and $\rho(1) = 0$;

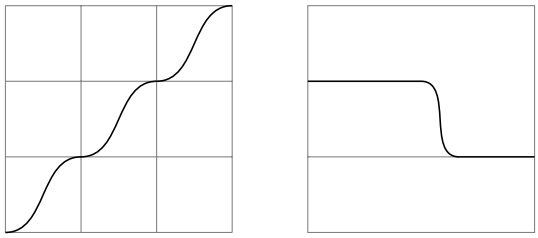


FIGURE 1. Graph of auxiliary functions $f : \mathbb{R} \to \mathbb{R}$ and $\rho : \mathbb{R} \to \mathbb{R}$. We require the derivative $\rho'(t)$ to be supported in the interior of the interval where $f = 1$.

For similar use of such auxiliary functions f, ρ in setting up the continuation equation, we refer the reader to [Can23, CHK23, Can24b].[31]

[31]In order for our constructions to be canonical, in the sense that any two readers of the paper will agree on all constructions, it is necessary that the readers pick the same

The role of f will be the following: if $\psi_{\eta,t}$ is a Hamiltonian isotopy, extended to all times $t \in \mathbb{R}$ by the rule $\psi_{\eta,t+1} = \psi_{\eta,t}\psi_{\eta,1}$, then we will consider the time-reparametrization $\psi_{\eta,f(t)}$.

Let us first consider $\mathcal{M}(\Phi, k, \pm)$ in the case when $n = 0$. Let $X_{\eta,t}$ be the generator of $\psi_{\eta,f(t)}$, and consider the following equation for a pair (π, u):

$$\begin{cases} u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to W \text{ and } \pi : \mathbb{R} \to S^\infty, \\ \pi'(s) = -V(\pi(s)) \text{ with } \pi(-\infty) = v_{k,\pm} \text{ and } \pi(+\infty) = v_{0,+}, \\ \partial_s u + J(u)(\partial_t u - X_{\pi(s),t}(u)) = 0, \\ \text{the integral of } \omega(\partial_s u, \partial_t u - X_{\pi(s),t}(u)) \text{ over the cylinder is finite.} \end{cases} \tag{16}$$

See Figure 2 for an illustration:

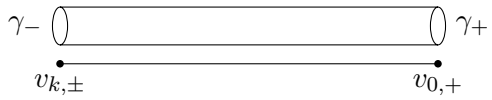


FIGURE 2. Illustration of a solution to (16). With our cohomological conventions, γ_+ will be considered as the "input."

Let us call Borel data *non-degenerate* provided the time-1 map $\psi_{\eta,1}$ has only non-degenerate fixed points when η is a pole. In this case, it follows from (B3) that the solutions u are asymptotic to stationary solutions over non-degenerate orbits of $X_{v_{k,\pm},t}$ and $X_{v_{0,+},t}$ at the two ends.

At this stage, we impose our first regularity assumption for an n-simplex Φ:

(∗1) The vertices of Φ are non-degenerate Borel data.

This ensures the asymptotics of solutions to (16) behave well for all 0-simplices.

We next define $\mathcal{M}(\phi, k, \pm)$ for a single non-negative Moore path $\phi \in \mathfrak{F}(W)$, expressed in the form $(\psi_{\eta,\sigma,t}, J_\sigma)$ with $\sigma \in \mathbb{R}$. Write $X_{\eta,s,t}$ and $Y_{\eta,s,t}$ for the generators of $\psi_{\eta,-s,f(t)}$ with respect to t and s, and (abusing notation) let us relabel $J_s = J_{-s}$.

Then $\mathcal{M}(\phi, k, \pm)$ is defined to be the solutions (π, u) to:

$$\begin{cases} u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to W \text{ and } \pi : \mathbb{R} \to S^\infty, \\ \pi'(s) = -V(\pi(s)) \text{ with } \pi(-\infty) = v_{k,\pm} \text{ and } \pi(+\infty) = v_{0,+}, \\ (\partial_s u - \rho(t) Y_{\pi(s),s,t}(u)) + J_s(u)(\partial_t u - X_{\pi(s),s,t}(u)) = 0, \\ \text{the integral of } \omega(\partial_s u - \rho(t) Y_{\pi(s),s,t}(u), \partial_t u - X_{\pi(s),s,t}(u)) \text{ is finite.} \end{cases} \tag{17}$$

cut-off functions f, ρ. To make this choice canonical, we should give a precise formula for the cut-off functions f and ρ. Such formulas can be found in any advanced calculus text.

Finally, we define $\mathcal{M}(\Phi^{v_0\cdots v_m}, k, \pm)$, $n \geq 1$, as the moduli space of triples (x, π, u) such that:

$$\begin{cases} x \in [0,\infty)^{m-1},\ \pi : \mathbb{R} \to S^\infty,\ \text{and } u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to W, \\ (\pi, u) \in \mathcal{M}(\Phi^{v_0\cdots v_m}(x), k, \pm), \end{cases} \tag{18}$$

where $\mathcal{M}(\Phi^{v_0\cdots v_m}(x), k, \pm)$ is defined in (17). Let us note that $(*1)$ implies that each solution u is asymptotic to non-degenerate orbits at its two ends.

Remark 2.13. As is typical in Floer theory, it is necessary to show these moduli spaces have certain compactness properties; this boils down to establishing a priori estimates on certain energy integrals; see, e.g., [Can24b, §2.2.6] and [CHK23, §2.2.5] for some related discussion. We will not discuss this aspect of the theory further, as it is by now well-established.

This leads to the second regularity requirement:

- $(*2)$ the moduli space of continuation cylinders $\mathcal{M}(\Phi^{v_0\cdots v_m}, k, \pm)$ is cut transversally (in the parametric sense).

See, e.g., [Sei15, §3.2] for related discussion. We define $\mathcal{D}^*(W) \subset \mathcal{D}(W)$:

Definition 2.14. *An n-simplex Φ is regular provided $(*1)$ holds and $(*2)$ holds for all subcubes $\{v_0, \ldots, v_m\} \subset \{0, \ldots, n\}$. The collection of regular simplices is denoted $\mathcal{D}^*(W)$*

Proposition 2.15. *The subset of regular simplices is closed under taking degeneracy maps.*

Proof. Let us consider the ith degeneracy $s_i\Phi$. Each subcube of $s_i\Phi$ is equal to one of the subcubes of Φ, perhaps with two adjacent variables compressed together via $\mathrm{smax}(x_i, x_{i+1})$, or perhaps with an additional $id\#_{\tau(x_1)}(-)$ or $(-)\#_{\tau(x_n)}id$; see Definition 2.6.

The equations associated to φ and $id\#_\tau\varphi$ are naturally identified (due to the obvious translation symmetry of the set-up). It follows that the equations associated to $s_i\Phi$ and its subcubes are pulled back from analogous equations for Φ along a fibration $\pi : [0,\infty)^k \to [0,\infty)^{k'}$, where either:

- $k' = k$ (in which case the fibration is an isomorphism) or
- $k' = k + 1$ (in which case π is defined using smax, or is a projection map in the edge cases s_0 and s_n).

This implies that $s_i\Phi$ and its faces satisfy $(*2)$ are surjective, since the we assume Φ satisfies $(*2)$; the other item $(*1)$ is more obvious. □

Proposition 2.16. *The subset of regular simplices is a sub-∞-category.*

Proof. This follows from standard results asserting that "transversality is generic;" indeed, the horn filling problem can be solved in $\mathcal{D}(W)$, and simplices can be perturbed so as to lie in $\mathcal{D}^*(W)$ (the perturbation can be made relative the horn if all of the faces in the horn are already in $\mathcal{D}^*(W)$).

We comment on how exactly we achieve transversality for the moduli space $\mathcal{M}(\Phi, k, \pm)$. It is a parametric moduli space over $[0,\infty)^{n-1}$, and solutions "break" into solutions of the lower parametric families as one coordinate x_i approaches ∞ or 0. One "recursively" achieves transversality (in a manner similar to [Sei08b]). We only perturb in a compactly supported way in W to avoid influencing the ideal restriction. Since the perturbation is allowed to depend on the location in the parameter space $[0,\infty)^{n-1}$, the problem boils down to one in [SS10, pp. 1468] and [Sei15, §4] where transversality for families over the flow line spaces $\mathcal{P}(v_{k,\pm})$ is considered, (one also appeals to standard techniques such as those in [MS12, §8]). □

2.2.4. **Definition of the equivariant chain complex.** Let $\Phi = (\psi_{\eta,t}, J)$ be a single Borel datum in $\mathcal{D}_0^*(W)$. Following [SS10, Sei15], we define:

$$\mathrm{CF}_{\mathrm{eq}}(\psi_{\eta,t}, J) := \mathrm{CF}(\psi_t) \otimes \mathbf{k}[[x]],$$

where $\psi_t = \psi_{v_{0,+},t}$ and where $\mathrm{CF}(\psi_t)$ is the $\mathbf{k}$-vector space generated by contractible 1-periodic orbits of ψ_t. The differential is defined as a power series:

$$d_{eq} = \sum x^k d_k = \sum x^k (d_{k,+} + d_{k,-})$$

where $d_{k,\pm}$ counts solutions in the family $\mathcal{M}(\Phi, k, \pm)$, as follows:

$$d_{k,+}(\gamma_+) := \sum \{u(-\infty) : (\pi, u) \in \mathcal{M}_1(\Phi, k, +)/\mathbb{R} \text{ and } u(+\infty) = \gamma_+\},$$

$$d_{k,-}(\gamma_+) := \sum \{g(u(-\infty)) : (\pi, u) \in \mathcal{M}_1(\Phi, k, -)/\mathbb{R} \text{ and } u(+\infty) = \gamma_+\}.$$

Note that the $d_{k,-}$ term twists the orbit by g, which is necessary for the count to make sense. Note as well that $d_{0,+}$ is the usual Floer differential and $d_{0,-} = 0$.

The fact that $d_{eq}^2 = 0$ is essentially an argument in family Floer cohomology, similar to some discussion in [BC25, Lemma 42], and is explained in [SS10] and [Sei15]. We briefly review the argument:

Proof that the equivariant differential squares to zero. As is usual in Floer theory, we prove $d_{eq}^2 = 0$ by considering the non-compact ends of the 1-dimensional moduli spaces $\mathcal{M}_2(\Phi, k, \pm)/\mathbb{R}$. Here an *end* is a proper embedding of $[0,\infty)$ into the moduli space, understood via the unbounded sequences (ℓ_n, π_n, u_n) which remain in the end. Of course, in this case Φ is a single choice of Borel data so $\ell_n \in \mathcal{M}(\Delta^n)$ is entirely irrelevant. We consider two kinds of ends (π_n, u_n):

(i) $[\pi_n] \in \mathcal{P}(v_{k,\pm})/\mathbb{R}$ converges (but u_n does not converge),
(ii) $[\pi_n]$ breaks into a limit in $\mathcal{P}(v_{l_1,\pm})/\mathbb{R} \times \mathcal{P}(v_{l_2,\pm})/\mathbb{R}$ where $l_1 + l_2 = k$ and both $l_i > 0$.

As in [SS10, Sei15] the fact that the pseudogradient is invariant under the self-similarity map and the involution implies that:

$$\{\text{flows lines from } v_{l_1,\epsilon_1} \text{ to } v_{l_2,\epsilon_2}\} \simeq \mathcal{P}(v_{l_2-l_1,\epsilon_1\epsilon_2}),$$

and it is with respect to this identification that only spaces of flow lines starting at $v_{0,+}$ appear in the formula (ii).

The linearization of the parametric equation is a linear map:

$$T\mathcal{P}(v_{k,\pm}) \times W^{1,p}(u_n^* TW) \to L^p(u_n^* TW),$$

and the restriction to $W^{1,p}$ is a Cauchy-Riemann operator D_{u_n} whose index depends on the asymptotic orbits γ_-, γ_+ and the homology class of $[u]$ relative γ_-, γ_+. We record the formula for its index:

$$\text{index}(D_{u_n}) = 2c_1^{\mathfrak{s}}(u) + \text{CZ}_{\mathfrak{s}}(\gamma_+) - \text{CZ}_{\mathfrak{s}}(\gamma_-), \tag{19}$$

where:

- $\mathfrak{s}$ is any generic section of $\det_{\mathbb{C}}(TW)$ non-vanishing along $\gamma_\pm$,
- $c_1^{\mathfrak{s}}(u)$ is the signed intersection number $\text{PD}(\mathfrak{s}^{-1}(0)) \cap [u]$,
- the Conley-Zehnder indices $\text{CZ}_{\mathfrak{s}}(\gamma_\pm)$ are computed using trivializations of $TW|_{\gamma_\pm}$ which render $\mathfrak{s}$ constant;

see, e.g., [Sal97]; we also refer the reader to [Can22] and the references therein for further details on this formula. In particular, along any sequence in $\mathcal{M}_2(\Phi, k, \pm)$ it holds that $\text{index}(D_{u_n}) = 2 - k$. In case (i), one picks representatives:

$$(\pi_n, u_n) \in \mathcal{M}_2(\Phi, k, \pm)/\mathbb{R}$$

so that π_n converges (after passing to a subsequence). Then u_n converges on compact subsets by standard Floer theory to a solution u_∞ in $\mathcal{M}_d(\Phi, k, \pm)$ with $d \leq 2$. By $(*2)$ it holds that:

$$\cdots = \mathcal{M}_{-1}(\Phi, k, \pm) = \mathcal{M}_0(\Phi, k, \pm) = 0, \tag{20}$$

if $k \neq 0$, while if $k = 0$ then $\mathcal{M}_0(\Phi, 0, +)$ consists of only the stationary cylinders. Note that if $k = 0$, then π_n is irrelevant, and the argument boils down to Floer's original argument and shows $d_0^2 = 0$. Assume $k > 0$ for the rest of the proof.

If u_n does not converge uniformly, then it must break off non-stationary Floer differential cylinders (solutions to the s-independent equation) at one of the ends. Each Floer differential cylinder must have a positive index (19), by $(*1)$ and $(*2)$. It then follows from (20) that only one non-stationary Floer cylinder of index 1 breaks off, and the other limit u_∞ is in $\mathcal{M}_1(\Phi, k, \pm)$. For each $k \geq 1$, the count of these breakings (interpreted in the usual Floer theoretic sense) gives:

$$d_0 d_k + d_k d_0.$$

Standard Floer theory gluing proves each hypothetical breaking in the fiber product inside of $\mathcal{M}_1/\mathbb{R} \times \mathcal{M}_1/\mathbb{R}$ actually does appear as the end of the moduli space $\mathcal{M}_2/\mathbb{R}$. In general we will suppress discussion of gluing.

Turning now to the other non-compact ends (ii). These breakings only occur when $k \geq 2$. In this case, if $[\pi_n]$ breaks into $[\pi_-], [\pi_+]$, we claim that $[\pi_n, u_n]$ must converge (after a subsequence) into a broken configuration $[\pi_-, u_-]$

and $[\pi_+, u_+]$ in $\mathcal{M}_1/\mathbb{R} \times \mathcal{M}_1/\mathbb{R}$. This is the only possibility compatible with (ii) and the index formula (19). There is some small technicality regarding how the breakings are interpreted, when the self-similarity and involution maps are taken into account; see [Sei15, pp. 972] for related discussion. It suffices to say that this technicality explains that the breakings of type (ii) correspond to the sum of compositions $d_{l_1} d_{l_2}$ where $l_1 + l_2 = k$ and $l_1, l_2 > 0$, extending the case $d_k d_0 + d_0 d_k$ to the intermediate values. In total, one shows:

$$\sum_{l_1+l_2=k} d_{l_1} d_{l_2} = 0, \text{ for each } k$$

which is equivalent to $d_{eq}^2 = 0$. □

2.2.5. **Chain homotopies associated to higher simplices.** We continue from §1.3. Recall that a map $\mathcal{D}^*(W) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathbf{k}[[x]]))$ is the data of:

(a) a finitely generated graded[32] $\mathbf{k}[[x]]$-module V_Φ with differential d_Φ of degree 1 for each zero simplex Φ,
(b) a map $\mathfrak{c}_\Phi : V_{\Phi|0} \to V_{\Phi|n}$ of degree $1 - n$ for each n-simplex Φ.

In the present case where $\mathrm{char}(\mathbf{k}) = 2$, these are supposed to satisfy:

$$\sum_{j=1}^{n-1} \mathfrak{c}_{\Phi|[j\dots n]} \circ \mathfrak{c}_{\Phi|[0\dots j]} + \mathfrak{c}_{\Phi|[0\dots \hat{j}\dots n]} = \mathfrak{c}_\Phi \circ d_{\Phi|0} + d_{\Phi|n} \circ \mathfrak{c}_\Phi, \tag{21}$$

and, concerning degeneracies,

$$\mathfrak{c}_{s_i\Phi} = \begin{cases} 0 & \text{for each } n\text{-simplex } \Phi \text{ with } n \geq 1, \\ id & \text{for each 0-simplex } \Phi. \end{cases} \tag{22}$$

The first stage (a) is done: each zero simplex $\Phi = (\psi_{\eta,t}, J)$ is sent to its equivariant Floer cohomology complex $V_\Phi = \mathrm{CF}_{\mathrm{eq}}(\psi_{\eta,t}, J)$ with differential $d_\Phi = d_{eq}$ as defined in §2.2.4.

The second stage (b) is satisfied by the following definition:

$$\mathfrak{c}_\Phi = \sum x^k \mathfrak{c}_k = \sum x^k (\mathfrak{c}_{k,+} + \mathfrak{c}_{k,-})$$

where $\mathfrak{c}_{k,\pm}$ are defined by:

$$\mathfrak{c}_{k,+}(\gamma_+) = \sum \left\{ u(-\infty) : (x, \pi, u) \in \mathcal{M}_0(\Phi^{0\dots n}, k, +) \text{ with } u(+\infty) = \gamma_+ \right\},$$
$$\mathfrak{c}_{k,-}(\gamma_+) = \sum \left\{ g(u(-\infty)) : (x, \pi, u) \in \mathcal{M}_0^{0\dots n}(\Phi, k, -) \text{ with } u(+\infty) = \gamma_+ \right\};$$

here the "0" subscript in $\mathcal{M}_0$ signifies the dimension of the moduli space. It remains only to verify the equations (21) and (22). To simplify the notation, let us denote by: $\mathcal{M}_d(\Phi) = \mathcal{M}_d(\Phi^{0\dots n}, k, \pm)$.

[32] We need to use $\mathbb{Z}/2\mathbb{Z}$-gradings in certain signs when the characteristic of the field is $p \geq 3$; in the case $p = 2$ we can use ungraded modules.

The equation (21) is proved by analyzing the 1-dimensional component $\mathcal{M}_1(\Phi)$. The moduli space admits a map:

$$x : \mathcal{M}_1(\Phi) \to [0,\infty)^{n-1},$$

which records the underlying cubical coordinates; this is used to assign algebraic interpretations to the non-compact ends:

(i) $\mathfrak{c}_{\Phi|[j\dots n]} \circ \mathfrak{c}_{\Phi|[0\dots j]}$: ends where one of the cubical coordinates x_j diverges to ∞,
(ii) $\mathfrak{c}_{\Phi|[0\dots\hat{j}\dots n]}$: ends where one of the cubical coordinates x_j converges to 0,
(iii) $\mathfrak{c}_\Phi \circ d_{\Phi|0}$ and $d|_{\Phi|n} \circ c_\Phi$: ends where the cubical coordinate x converges in the interior $(0,\infty)^{n-1}$.

In the remainder of this section we make this sketch more precise.

Proof of equation (21). The moduli space $\mathcal{M}(\Phi, k, \pm)$ defined in (18) admits a "forgetful" map to $\mathcal{P}(v_{k,\pm}) \times [0,\infty)^{n-1}$.

Referring to the Fredholm index in (19), it holds that:

$$\text{index}(u) + k + n - 1 = d \text{ for } (x,\pi,u) \in \mathcal{M}_d(\Phi, k, \pm), \tag{23}$$

As in (i) through (iii), let us study the parameter map $x : \mathcal{M}_1 \to [0,\infty)^{n-1}$. We begin with:

Claim 2.17. *Along any end $(x_\nu, \pi_\nu, u_\nu) \in \mathcal{M}_1(\Phi, k, \pm)$ such that the $x_{\nu,i}$ converges to 0 but the other coordinates converge in the interior $(0,\infty)$, and π_n converges, some subsequence will converge to a solution $(x_\infty, \pi_\infty, u_\infty)$ for the restriction $\mathcal{M}_0(\Phi|_{[0\dots\hat{i}\dots n]}, k, \pm)$.*

Proof of claim. If follows from (3) that the PDE which u_n solves converges to the one for the restricted simplex; and then, by standard compactness results, u_n also converges (after passing to a subsequence) in the stated sense, yielding the claim. The dimension drops by 1 by inspecting (23). □

These ends lead to the terms of the form $\mathfrak{c}_{\Phi|[0\dots\hat{i}\dots n]}$ appearing in (21). All such contributions do appear as ends of one-dimensional moduli spaces by standard Floer theoretic "gluing" (here we do not actually glue together two solutions, but one still appeals to the inverse function theorem; see [BC24, §4] for related discussion).

The same argument used for the claim works if $x_i \to 0$ holds for multiple values of i, (i.e., all coordinates converge in $[0,\infty)$ and π_n converges). Then the index will drop by the number of coordinates which converged to 0. For generic data, we can therefore assume that only one coordinate ever converges to 0, provided the other coordinates converge in $[0,\infty)$ and π_n converges.

Let us now analyze the case when one of the coordinates, say x_i, diverges to ∞, but the other coordinates converge in $[0,\infty)$.

Then the property (2), stating that:

$$\Phi^{0\dots n}(x) = \Phi^{0\dots i}(x_1,\dots,x_{i-1})\#_{\tau(x_i)}\Phi^{i\dots n}(x_{i+1},\dots,x_{n-1})$$

enables us to extract two limits. The first u_+ is just the limit of the original sequence on compact subsets. The second u_- is the limit of retranslations by $\tau(x_i)$. Moreover, using the same parametrizations, π also converges to some pair of flow lines π_-,π_+ of indices k_-,k_+ on S^∞ (after passing to a subsequence). By compactness, the other cubical coordinates converge and we write $x_- = (x_1,\dots,x_{i-1})$ and $x_+ = (x_{i+1},\dots,x_n)$ for these limits. We claim:

- $(u_-,x_-,\pi_-) \in \mathcal{M}_{d_-}(\Phi|_{[i\dots n]},k_-,\pm)$,
- $(u_+,x_+,\pi_+) \in \mathcal{M}_{d_+}(\Phi|_{[0\dots i]},k_+,\pm)$,

for appropriate $\pm$ signs modulo applying the involution and using the shift self-symmetry of S^∞, as in [SS10, Sei15]. Here $d_+ + d_- \leq 0$. The only possibility is $d_+ = d_- = 0$, and $k_- + k_+ = k$ (rather than $k_- + k_+ \leq k$). This implies that π_n actually converges in the generalized Morse sense to the configuration π_-,π_+ we already found.

Therefore, the pair $(u_-,x_-,\pi_-),(u_+,x_+,\pi_+)$ contributes to the term:

$$\mathfrak{c}_{\Phi|[i\dots n]}\mathfrak{c}_{\Phi|[0\dots i]};$$

all such contributions actually do appear as ends of one-dimensional moduli spaces by standard Floer theoretic gluing for parametric families of continuation cylinders, similarly to [Sei15, pp. 972].

Essentially the same argument implies that one cannot have two coordinates which diverge to infinity, without contradicting regularity.

All of the other ends are of the form where x converges in $(0,\infty)^{n-1}$. If π_n converges to (π_-,π_+) with positive indices k_-,k_+ in the Morse theoretic sense, then one concludes the terms of the form:

$$\mathfrak{c}_{k_-}d_{k_+} + d_{k_-}\mathfrak{c}_{k_+}.$$

In all other cases permitted by regularity, π_n converges in $\mathcal{P}(v_{k,\pm})/\mathbb{R}$, and one concludes the terms of the form $\mathfrak{c}_0 d_k + d_k\mathfrak{c}_0$. Summing together all of these possible ends gives the desired relation (21). □

Proof of the degeneracy equation (22). The case of a degenerate 1-simplex is just the well-known fact that the continuation map associated to stationary continuation data is equal to the identity map on chain level.

For degenerate n-simplices with $n > 1$, the argument is essentially the same as the one given in the proof of Proposition 2.15. The top cube of $s_i\Phi$ is pulled back from the top cube of Φ by a submersion:

$$\pi : [0,\infty)^n \to [0,\infty)^{n-1},$$

defined by an explicit formula. The moduli spaces associated to different points in the same fiber of π are canonically identified, and the linearizations at solutions above $x \in [0,\infty)^n$ are surjective if the linearizations of above $\pi(x)$ are surjective. In particular, there is an empty zero dimensional component of the moduli space above $[0,\infty)^n$ associated to $s_i\Phi$, and so it holds that $\mathfrak{c}_{s_i\Phi} = 0$, as desired. □

2.2.6. **Definition of the ∞-functor.** Most of the arguments needed in this subsection were already explained in §1.9. The main technical result of this subsection is:

Lemma 2.18. *The forgetful map $\mathcal{D}^*(W) \to \mathcal{C}(Y)$ is a trivial Kan fibration.*

Proof. One shows that $\mathcal{D}(W) \to \mathcal{C}(Y)$ is a trivial Kan fibration by direct construction. Using a G-invariant radial function r, one can cut-off the generating Hamiltonians functions from the symplectization end to the filling and explicitly construct extensions; e.g., one can use a formula of the form $f(r)H$ where f vanishes outside the symplectization end. Extensions constructed in this way will not be regular, but they can be perturbed on the compact part to be made regular. □

Therefore, by Proposition 1.24, there is a contractible Kan complex of sections $\Sigma : \mathcal{C}(Y) \to \mathcal{D}^*(W)$ of this forgetful map. We define:

$$\mathrm{CF}_{\mathrm{eq}} \circ \Sigma : \mathcal{C}(Y) \to \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]]),$$

as the ∞-functor satisfying Theorem 1.14. While this depends on a choice of section, this choice is "contractible" in the ∞-category sense, and any two choices give equivalent functors (in an appropriate sense).

It is perhaps reassuring to observe the following: for any $(\psi_{\eta,t}, J) \in \mathcal{D}_0^*(W)$, there is a section Σ such that $\Sigma(\varphi_t) = (\psi_{\eta,t}, J)$, where φ_t is the ideal restriction of $\psi_{\eta,t}$. Indeed, this follows from the lifting property for the diagram:

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & \mathcal{D}^*(W) \\ \downarrow & \overset{\Sigma}{\nearrow} & \downarrow \\ \mathcal{C}(Y) & \xrightarrow{id} & \mathcal{C}(Y). \end{array}$$

This means that, given any desired regular extension of φ_t to Borel data $(\psi_{\eta,t}, J)$, one can assume that the output of $\mathrm{CF}_{\mathrm{eq}} \circ \Sigma(\varphi_t)$ agrees with the equivariant Floer complex $\mathrm{CF}_{\mathrm{eq}}(\psi_{\eta,t}, J)$ from §2.2.4 "on the nose." In any case, the chain level constructions we will appeal to are insensitive to the precise choice of Σ.

2.3. **The case when $p \geq 3$.** For the rest of this section we fix a prime $p \geq 3$ and let $G = \mathbf{k} = \mathbb{Z}/p\mathbb{Z}$.

The main contents of the construction are similar to the $\mathbb{Z}/2\mathbb{Z}$-case. The principal difference lies in the underlying Morse model for BG. As noted at

the start of §2.2, in the case $p \geq 3$, we work on $S^\infty \subset \mathbb{C}^\infty$ with the diagonal $G \subset U(1)$ action, and use the shift operator τ associated to the standard $\mathbb{C}$-valued coordinates. The usual perfect Morse function on $\mathbb{C}P^\infty$ pulls back to a $U(1)$ invariant Morse–Bott function on S^∞ given by:

$$f(z_0, z_1, \ldots) = \sum_{n=0}^{\infty} n\,|z_n|^2\,. \tag{24}$$

Well chosen G-equivariant perturbations turn this into a Morse function on BG (see [SZ21, §4] for discussion of this construction, indeed we are happy to work with the same model). Moreover, this may be done compatibly with the action of τ.

Ultimately we are concerned with the pseudogradient as in §2.2.3. We take the pseudogradient B on $\mathbb{C}P^\infty$ whose time-s flow is given by:

$$[z_0 : z_1 : z_2 : \ldots] \mapsto [z_0 : e^s z_1 : e^{2s} z_2 : \ldots].$$

Fixing a connection on the line bundle $S^\infty \to \mathbb{C}P^\infty$, we obtain a unique lift to a vector field B on S^∞ taking values in the horizontal distribution of the connection; this vector field is Morse-Bott with Lyapunov function given by (24). One "Morsifies" B by adding a vector field F tangent to the fibers of the map $S^\infty \to \mathbb{C}P^\infty$. One considers $V = B + F$ as the $p > 2$ analogue of the vector field V considered in §2.2.3. For convenience, one can assume that the connection is flat in a neighborhood of the poles of $\mathbb{C}P^\infty$.

One picks F and the connection so that:

- the restriction of F to the fiber above a critical point of B is as in Figure 3;
- F and the connection are $\mathbb{Z}/p\mathbb{Z}$ equivariant;
- F and the connection is invariant under the shift map:

$$\tau : (z_0, z_1, z_2, \ldots) \mapsto (0, z_0, z_1, \ldots).$$

- the *total vector field* $V = B + F$ is Morse-Smale;

We leave the construction of such F and such connection to the reader, and refer to [SZ21] for further discussion.

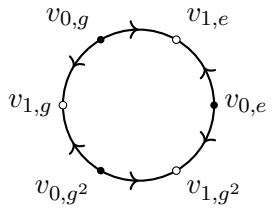


FIGURE 3. Restriction of F to the critical circle has $2p$ critical points (shown for $p = 3$). Here $v_{1,e}$ is required to be a positive pole.

This V then has exactly p zeros of index n for each $n \geq 0$, which are related to each other by the G action, and τ acts by shifting the index by 2. Choosing distinguished critical points $v_{0,e}, v_{1,e}$, of index 0 and 1 respectively, we have a natural labeling of the rest $v_{n,g}$ by the data of their index n, and the element $g \in G$ which sends $v_{n,g}$ to the appropriate shift of $v_{0,e}$ or $v_{1,e}$. Coherent orientations of the finite dimensional spaces of flow lines completes the Morse setup; we describe this in greater detail in the sequel. To keep things shorter we fix such a model on BG once and for all as input to the rest of the construction.[33]

Having fixed a Morse model on BG, the category of auxiliary data $\mathcal{D}(W)$, and moduli spaces are essentially as in §2.2, with the following modifications:

- In addition to conditions (B1) through (B4) above, auxiliary data must be identical near $v_{0,e}, v_{1,e}$.
- Associated to a k-simplex of Borel data Φ we have moduli spaces:
$$\mathcal{M}(\Phi, k, g, i)$$
where $i \in \{0, 1\}$ identifies the input point (otherwise as in §2.2.3).

Fixing a Borel datum $\Phi = (\psi_{\eta,t}, J)$, the equivariant chain complex now has underlying $\mathbf{k}[[x]]$ module:

$$\mathrm{CF}_{eq}(\Phi) := \mathrm{CF}(\psi_t) \otimes_{\mathbf{k}} (\mathbf{k}[[x]] \oplus \mathbf{k}[[x]][1]) .$$

Here $\mathrm{CF}(\psi_t)$ is the direct sum of the $\mathbf{k}$-orientation lines $\mathfrak{o}(\gamma)$ of contractible 1-periodic orbits γ of $\psi_t = \psi_{(1,0,0,\dots),t}$. This module carries a differential, given by:

$$d_{eq}(\gamma \otimes 1) = \sum_{k \geq 0} x^k d_0^{2k}(\gamma) \otimes 1 + x^k d_0^{2k+1}(\gamma) \otimes \theta \tag{25}$$

$$d_{eq}(\gamma \otimes \theta) = \sum_{k \geq 0} x^k d_1^{2k+1}(\gamma) \otimes \theta + x^{2k+2} d_1^{2k+2}(\gamma) \otimes 1, \tag{26}$$

where $1, \theta$ denote the generators of $\mathbf{k}[[x]] \oplus \mathbf{k}[[x]][1]$, and:

$$d_i^k(\gamma) = \sum_{g \in G} \sum_{u \in \mathcal{M}(\Phi,k,g,i)} \mathfrak{o}(u).$$

The inner sum is over all $u \in \mathcal{M}(\Phi, k, g, i)$ which are rigid up to translation and which are asymptotic to γ at their input end. The symbol $\mathfrak{o}(u)$ can be considered in two ways:

(1) $\mathfrak{o}(u) = \pm\gamma'$, where γ' is the output asymptotic of u, and the sign $\pm$ is determined by comparing the orientation lines of $\mathcal{M}(\psi, k, g, i)$ with a system of coherent orientations, as in [FH93];

[33]Unlike the case of $B\mathbb{Z}/2\mathbb{Z} = \mathbb{R}P^\infty$, the Morse-Smale gradient on BG is not canonically defined, as we have written it. This is a minor loss of canonicity in our construction, and it seems the optimal resolution would be to encode the remaining ambiguity as auxiliary data to be included in the category $\mathcal{D}(W)$. For reasons of space, we leave the details of this optimization to the reader.

(2) $\mathfrak{o}(u) \in \mathrm{Hom}(\mathfrak{o}(\gamma), \mathfrak{o}(\gamma'))$ is a canonical isomorphism constructed by the linear gluing theory of [FH93] (see, e.g., [Abo15, Par16]).

That d_{eq} squares to 0 follows from an analysis of the moduli spaces identical to that of 2.2.4 above, replacing the $\mathbb{Z}/2$ counting of that section with the usual orientation line formalism (see 2.4).

Chain homotopies from higher simplices, and the definition of the ∞-functor are also built as above, mutatis mutandis.

In the next section, we digress and explain the set-up of orientations; the reader who is comfortable with the framework of [Abo15, pp. 290], and/or [Par16, §C.13.2] can safely skip to §3.

2.4. **Gradings, asymptotic operators, and orientation lines.** The essential ingredient in the construction is [FH93, Theorem 10] and their kernel gluing lemma. We will also appeal to [Par16, §C.13.1] for orientations on "spaces of flow lines on Δ^n," and [SZ21, §A] for discussion of the orientations on the spaces of Morse flow lines on BG, for $G = \mathbb{Z}/p\mathbb{Z}$.

2.4.1. **Asymptotic operators.** An *asymptotic operator* is a first order differential operator $\mathfrak{A}$, acting on $\mathbb{R}^{2n}$-valued functions on $\mathbb{R}/\mathbb{Z}$, of the form:

$$\mathfrak{A} = -J\partial_t - S(t)$$

where $t \in \mathbb{R}/\mathbb{Z} \mapsto S(t)$ is a smooth loop of symmetric matrices and J is the standard complex structure on $\mathbb{R}^{2n}$. Such operators are elliptic and self-adjoint. If $\mathfrak{A}$ induces an isomorphism $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \to C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ then we say $\mathfrak{A}$ is *non-degenerate*.

2.4.2. **Cauchy-Riemann operators with asymptotics.** Given non-degenerate asymptotic operators $\mathfrak{A}_-, \mathfrak{A}_+$, we denote by $C(\mathfrak{A}_-, \mathfrak{A}_+)$ the space of operators on the cylinder $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ of the form:

$$\partial_s + J\partial_t + S(s,t).$$

such that $-J\partial_t - S(s,t)$ converges to $\mathfrak{A}_\pm$ as $s \to \pm\infty$. The convergence should be in a suitable topology whose details are not so relevant to the present discussion.

Then $C(\mathfrak{A}_-, \mathfrak{A}_+)$ is a convex space of Fredholm operators D, see [Sal97], all of which have the same Fredholm index $i(\mathfrak{A}_-, \mathfrak{A}_+)$. Moreover there is a continuous real-line bundle:

$$\det(\mathfrak{A}_-, \mathfrak{A}_+) \to C(\mathfrak{A}_-, \mathfrak{A}_+)$$

whose restriction to any stratum with fixed kernel dimension agrees with:

$$\det(\ker D \oplus (\mathrm{coker}\, D)^\vee),$$

The fact that these bundles patch together across the different strata is a fundamental property of Fredholm operators, and is discussed in [MS12].

2.4.3. **Orientation lines.** The real line bundle $\det(\mathfrak{A}_-, \mathfrak{A}_+)$ admits two orientations $O, \bar{O}$ (since the base space is contractible). Let us denote by $\mathfrak{o}(\mathfrak{A}_-, \mathfrak{A}_+)$ the *orientation line* generated by $O, \bar{O}$ modulo the relation that $O + \bar{O} = 0$. This object $\mathfrak{o}(\mathfrak{A}_-, \mathfrak{A}_+)$ is a free $\mathbb{Z}$-module of rank 1.

2.4.4. **Automatic isomorphism.** In [Sal97] it is shown that $\partial_s - \mathfrak{A}$ is an isomorphism for any non-degenerate asymptotic operator $\mathfrak{A}$. Thus $\mathfrak{o}(\mathfrak{A}, \mathfrak{A})$ is *canonically* identified with $\mathbf{k}$, because the determinant line at $D = \partial_s - \mathfrak{A}$ is the determinant line of zero, which is canonically oriented.

2.4.5. **Kernel gluing.** The kernel gluing theorem of [FH93] gives isomorphisms:

- $\mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \otimes \mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_2) \to \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_2)$

which are associative, and such that "multiplication" by the canonical positive generator of $\mathfrak{o}(\mathfrak{A}, \mathfrak{A})$ acts by the identity.

Moreover, the pairing:

$$\mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \otimes \mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_0) \to \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_0) \simeq \mathbb{Z}$$

canonically identifies $\mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \to \mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_0)$, as follows: one sends a generator of $\mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1)$ to the unique generator of $\mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_0)$ which pairs with it to produce 1. Associativity proves the composition of the canonical identifications $\mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \to \mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_0) \to \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1)$ is the identity map.

2.4.6. **Twisting identifications.** Given $\mathfrak{A}$ and a loop $\varphi : \mathbb{R}/\mathbb{Z} \to \mathrm{U}(n)$ consider the conjugation:

$$\varphi_* \mathfrak{A} := \varphi \mathfrak{A} \varphi^{-1}.$$

Multiplication of kernel and cokernel elements by φ yields an identification:

$$\mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \to \mathfrak{o}(\varphi_* \mathfrak{A}_0, \varphi_* \mathfrak{A}_1). \tag{27}$$

Unpacking the construction from [FH93] shows (27) is compatible with the kernel gluing maps from §2.4.5, in the sense that the diagram commutes:

$$\begin{array}{ccc} \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_1) \otimes \mathfrak{o}(\mathfrak{A}_1, \mathfrak{A}_2) & \longrightarrow & \mathfrak{o}(\varphi_* \mathfrak{A}_0, \varphi_* \mathfrak{A}_1) \otimes \mathfrak{o}(\varphi_* \mathfrak{A}_1, \varphi_* \mathfrak{A}_2) \\ \downarrow & & \downarrow \\ \mathfrak{o}(\mathfrak{A}_0, \mathfrak{A}_2) & \longrightarrow & \mathfrak{o}(\varphi_* \mathfrak{A}_0, \varphi_* \mathfrak{A}_2) \end{array}$$

where the vertical maps are the kernel gluing maps, and the horizontal maps are the twisting identifications.

The twisting identification between $\mathfrak{o}(\mathfrak{A}, \mathfrak{A})$ and $\mathfrak{o}(\varphi_* \mathfrak{A}, \varphi_* \mathfrak{A})$ preserves the canonical generators from §2.4.4, since the empty basis is conjugated to the empty basis.

2.4.7. **Parallel transport.** Suppose that $\mathfrak{A}_{0,s}, \mathfrak{A}_{1,s}$, $s \in [0,1]$, are families of asymptotic operators which remain non-degenerate. Then there is a *parallel transport map*:

$$\mathfrak{o}(\mathfrak{A}_{0,0}, \mathfrak{A}_{1,0}) \to \mathfrak{o}(\mathfrak{A}_{0,1}, \mathfrak{A}_{1,1}), \tag{28}$$

defined by considering any continuous family $D_s \in C(\mathfrak{A}_{0,s}, \mathfrak{A}_{1,s})$ and the associated determinant line bundle over this family.

2.4.8. **Reference asymptotic operator.** Let $\mathfrak{R} = -J\partial_t - \delta$ for $\delta \in (0, 2\pi)$. This acts as a reference operator.

It will be important for us to consider the *complex linear* orientation:

$$\mathfrak{o}(\varphi_*\mathfrak{R}, \mathfrak{R}) \to \mathbb{Z},$$

The space of complex linear D is an affine subspace, so the resulting orientation of $\det(\varphi_*\mathfrak{R}, \mathfrak{R})$ is independent of the choice of complex linear D.

Consequently, we obtain canonical isomorphisms:

$$\mathfrak{o}(\mathfrak{A}, \mathfrak{R}) \to \mathfrak{o}(\varphi_*\mathfrak{A}, \varphi_*\mathfrak{R}) \to \mathfrak{o}(\varphi_*\mathfrak{A}, \mathfrak{R}), \tag{29}$$

where in the latter step we multiply by the generator of $\mathfrak{o}(\varphi_*\mathfrak{R}, \mathfrak{R})$ corresponding to the complex linear orientation.

2.4.9. **Linearization procedure.** Given a non-degenerate orbit γ of a Hamiltonian system in a symplectic manifold W:

$$\gamma(t) - X_t(\gamma(t)) = 0.$$

Let us consider travelling symplectic coordinates $e_t : B(\epsilon) \to W$, where $B(\epsilon)$ is the standard symplectic ball of capacity ϵ, such that $e_t(0) = \gamma(t)$. Then nearby loops can be expressed as $e_t(\eta(t))$, and the non-linear functional is represented by:

$$\eta'(t) + \mathrm{d}e_t(\eta(t))^{-1} \left[\partial_t e_t(\eta(t)) - X_t(e_t(\eta(t))\right].$$

The linearization of this is equal to $J\mathfrak{A}_e$ where $\mathfrak{A}_e$ is an asymptotic operator.

Changing e amounts to conjugating $J\mathfrak{A}_e$ by a linear symplectic map $\Psi(t)$,

$$J^{-1}\Psi(t)J\mathfrak{A}_{e'}\Psi(t)^{-1} = \mathfrak{A}_e.$$

Because $\mathrm{Sp}(2n)$ deformation retracts to $\mathrm{U}(n)$, one obtains a canonical parallel transport isomorphism:

$$\mathfrak{o}(\mathfrak{R}, \varphi_*\mathfrak{A}_{e'}) \to \mathfrak{o}(\mathfrak{R}, \mathfrak{A}_e). \tag{30}$$

This can be precomposed with the morphism constructed in §2.4.8:

$$\mathfrak{o}(\mathfrak{R}, \mathfrak{A}_{e'}) \to \mathfrak{o}(\mathfrak{R}, \varphi_*\mathfrak{A}_{e'}). \tag{31}$$

Claim 2.19. *The assignment $e \mapsto \mathfrak{o}(\mathfrak{R}, \mathfrak{A}_e)$, with the morphisms obtained by composing* (31) *and* (30) *defines a functor from the indiscrete groupoid spanned by choices of symplectic coordinates.* □

Definition 2.20. *The* orientation line $\mathfrak{o}(\gamma)$ *is the limit of $e \mapsto \mathfrak{o}(\mathfrak{R}, \mathfrak{A}_e)$.*

A similar linearization procedure can be done at any solution of the Floer-type PDEs considered in this text (without varying parameters), and one obtains canonical identifications:

$$\mathfrak{o}(D_{u,e}) \to \mathfrak{o}(\mathfrak{A}_{-,e}, \mathfrak{A}_{+,e}) \to \mathfrak{o}(\gamma_-) \otimes \mathfrak{o}(\gamma_+),$$

where e is a travelling family of symplectic coordinates defined along the domain of u which restrict to families of coordinates at the asymptotic ends γ_-, γ_+ (in general, one tensors the orientation lines of all asymptotics).

Claim 2.21. *The identifications $\mathfrak{o}(D_{u,e}) \to \mathfrak{o}(\gamma_-) \otimes \mathfrak{o}(\gamma_+)$ are compatible with the conjugation isomorphisms arising from changing e.* □

2.4.10. **Note on parametric moduli spaces.** Recall that in the construction of the equivariant cohomology infinity-functor, we count solutions (x, π, u) of parametric moduli spaces where:

- $x \in [0, \infty)^{n-1}$,
- π is a flow line on S^∞ for a G- and τ-invariant pseudogradient,
- u solves Floer's continuation cylinder equation for data which depends on ℓ and π;

see (18) from §2.2.3 for the precise details.

Let us denote by $D_{x,\pi,u}$ the linearization associated to the parametric problem. Parallel transport from $D_{x,\pi,u}$ to D_u in the space of Fredholm operators produces an identification :

$$\mathfrak{o}(D_{x,\pi,u}) \to \mathfrak{o}(T_x) \otimes \mathfrak{o}(T_\pi) \otimes \mathfrak{o}(D_u) = \mathfrak{o}(T_x) \otimes \mathfrak{o}(T_\pi) \otimes \mathfrak{o}(\gamma_-) \otimes \mathfrak{o}(\gamma_+),$$

where T_x, T_π are the (finite-dimensional) tangent spaces to:

- the cube $[0, \infty)^{n-1}$ at x, and,
- the space of flow lines on BG with the same asymptotics as π.

We comment here that [Par16, §C.13.1] considers orientations on T_x, in a slightly different context, while [SZ21, §A] considers orientations on T_π.

2.4.11. **The definition of the infinity-functor.** Fix the Morse model on BG as described in §2.3. Given a zero-simplex $(\varphi_{\eta,t}, J)$ in the ∞-category of regular Borel data $\mathcal{D}(W)$, we define:

$$\mathrm{CF}_{\mathrm{eq}}(\varphi_{\eta,t}, J) = \bigoplus \mathfrak{o}(\gamma) \otimes \mathfrak{o}(y) \otimes \mathbf{k}[[x]],$$

where the direct sum is over pairs γ, y where:

- y is a critical point in the fiber of BG over the minimum on $\mathbb{C}P^\infty$;
- γ is an orbit of the system $H_{y,t} = H_t$ over the distinguished lift of y; see Figure 3 for details on distinguished lifts $v_{0,e}$ and $v_{1,e}$.

Here $\mathfrak{o}(y)$ is the orientation line for the unstable manifold of y. Standard Morse theory and the self-similarity transformations give isomorphisms:

$$\mathfrak{o}(T_\pi) \to \mathfrak{o}(y_-) \otimes \mathfrak{o}(x^k y_+) \simeq \mathfrak{o}(y_-) \otimes \mathfrak{o}(y_+) \otimes \mathfrak{o}(x^k),$$

where we abuse notation and let x^k denote the critical point of index $2k$ on the base space $\mathbb{C}P^\infty$, and let $x^k y$ denote the "shift" of y by the self-similarity map k times.

The count of rigid-up-to-translation trajectories can be considered as:

$$d_{eq} : \mathrm{CF}_{\mathrm{eq}}(\varphi_{\eta,t}, J, \mathbf{k}) \to \mathrm{CF}_{\mathrm{eq}}(\varphi_{\eta,t}, J, \mathbf{k})$$

provided we pick orientations for $\mathfrak{o}(x^k)$ (and provided we follow the overall "twisting asymptotics by the group action" scheme of [SS10, Sei15, SZ21], similarly to in §2.2).

Claim 2.22. *If one picks the complex linear orientations of $\mathfrak{o}(x^k)$, then it holds that $d_{eq}^2 = 0$.*

Proof. This is a simplified version of what is shown in [SZ21, §A]. □

This complex produces the first half of an infinity-functor. The second half concerns the morphisms associated to n-simplices Φ for $n \geq 1$. Counting the rigid elements (x, π, u) produces a map:

$$\mathfrak{c}_\Phi : \mathrm{CF}_{\mathrm{eq}}(\Phi|_0) \to \mathrm{CF}_{\mathrm{eq}}(\Phi|_n),$$

provided one also picks orientations of the $n-1$ cube $[0,\infty)^{n-1}$. Then:

Claim 2.23. *If one orients the j-dimensional cubes in the standard way, so that $\partial_1, \ldots, \partial_j$ is an oriented basis, for $j < n$, it holds that:*

$$\sum_{j=1}^{n-1} (-1)^j (\mathfrak{c}_{\Phi|[j\ldots n]} \circ \mathfrak{c}_{\Phi|[0\ldots j]} - \mathfrak{c}_{\Phi|[0\ldots \hat{j}\ldots n]}) = \mathfrak{c}_\Phi \circ d_{\Phi|0} + (-1)^n d_{\Phi|n} \circ \mathfrak{c}_\Phi.$$

Proof. This follows from the boundary classification and orientation properties established by Pardon in [Par16, §C]. □

This completes our description of the infinity-functor in the case $p \geq 3$.

3. **PSS isomorphism and localization in Morse theory**

In this section we prove Theorem 1.16, and part (d) of Theorem 1.14.

Let W be an aspherical G filling of a closed contact manifold Y, for some prime cyclic group G of order p. Throughout we use the field $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$, which we generally suppress from the notation.

The usual PSS map in symplectic cohomology, as constructed in [FS07], yields an isomorphism:

$$\mathrm{HM}^*(W) \cong \mathrm{HF}(R_{\epsilon t})$$

where the left hand side is the Morse cohomology of W, and the right hand side is the Floer cohomology of any linear-at-infinity Hamiltonian whose slope ϵ with respect to a fixed Reeb flow R on ∂W is sufficiently small

and positive. We'll implement this in our equivariant setup, and lift the construction to the chain level statement stated in Theorem 1.16.

The strategy to prove Theorem 1.16 is four-fold:

(1) define the Morse equivariant cohomology complex $\mathrm{CM}_{\mathrm{eq}}(P_\eta)$ for suitable "Borel families" of pseudogradients P_η on W; see §3.1;
(2) Prove a localization result in §3.2, where fixed points of the G-action in W are shown to imply the equivariant cohomology $\mathrm{HM}_{\mathrm{eq}}(P_\eta)$ is not x-torsion;
(3) construct the ∞-categories $\mathcal{P}(Y), \mathcal{N}(Y)$ and the ∞-functors appearing in the statements of Theorems 1.16 and 1.17; see §3.3;
(4) prove the resulting PSS chain maps, are quasi-isomorphisms, provided the Floer data is a small Reeb flow; this is achieved by constructing chain-homotopy inverses to the PSS maps, as in [FS07]; see §3.4.

3.1. **The equivariant Morse complex.** In this section, we are concerned with defining the G-equivariant Morse complex $\mathrm{CM}_{\mathrm{eq}}(\pm P_\eta)$ and developing its properties. As above, $G = \mathbb{Z}/p\mathbb{Z}$, and we work over a base field $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$.

3.1.1. **Morse–Borel data.** The auxiliary data is analogous §2.2.2.

Definition 3.1. *A* Morse–Borel datum *is a family of vector fields P_η on W parameterized by $\eta \in EG = S_\infty$ such that:*

(M1) $g_* P_\eta = P_{g\eta}$*, for all* $g \in G$*,*
(M2) P_η *agrees with* aZ*, where* Z *is the Liouville vector field and* $a \in \mathbb{R}$*, outside a compact set,*[34]
(M3) $P_{\tau(\eta)} = P_\eta$*, where* τ *is the self-similarity map,*
(M4) P_η *is independent of* η *on neighborhoods of the critical points of the model* G*-invariant Morse pseudogradient* V *of* $S^\infty = EG$*.*
(M5) P_η *is a Morse pseudogradient, whenever* $\eta \in EG$ *is a zero of* V*.*

Here the model G-invariant Morse pseudogradient on EG is as described in §2.2 in the case $p = 2$, or in §2.3 in the case $p \geq 3$.

Remark 3.2. Note there are two types of Morse–Borel data, those which point outwards ($P_\eta = aZ$ for $a > 0$ in the end) versus those which point inwards ($P_\eta = aZ$ with $a < 0$ in the end). We say the former are of *type* $+$, while the latter are of *type* $-$.

[34] We use the Liouville flow here only because it is a canonical outwards pointing vector field; any other outwards pointing vector field would suffice.

3.1.2. **Spaces of flow lines.** Associated to such Morse–Borel data P_η one can form the moduli spaces $\mathcal{M}(P_\eta)$ of flowlines $(q,\pi):\mathbb{R}\to W\times S^\infty$ satisfying:

$$\begin{cases} q'(s) = -P_{\pi(s)}(q(s)), & \lim_{s\to\pm\infty} q(s) \text{ exists,} \\ \pi'(s) = -V(\pi(s)). \end{cases}$$

Any such flow line has well-defined asymptotics $(q_\pm,\eta_\pm)$.

Definition 3.3. *If the moduli spaces $\mathcal{M}(P_\eta)$ are smooth manifolds of the expected dimension given by:*

$$\operatorname{index}(q_-) - \operatorname{index}(q_+) + \operatorname{index}(\eta_-) - \operatorname{index}(\eta_+),$$

then we say that P_η is regular *Morse–Borel data.*

The existence and genericity of regular data follow from the usual Morse-theoretic transversality arguments.

3.1.3. **Definition of the Borel-equivariant Morse complex.** The equivariant Morse complex associated to Morse–Borel data P_η is defined to be the direct sum:

- $\mathrm{CM}_{\mathrm{eq}}(P_\eta) := \bigoplus \mathfrak{o}(q)\otimes\mathfrak{o}(y)\otimes\mathbf{k}[[x]]$,

similarly to the Floer case (see §2.2.4 and §2.4.11), where the direct sum runs over pairs (q,y) such that:

- y is a critical point of the pseudogradient on BG lying above the pole $[1:0:\dots]$ in projective space,[35]
- q is a critical point of P_η, where η is the distinguished lift of y from BG to EG. The word "distinguished" is to be understood vis-a-vis Figure 3 from §2.3: the distinguished lifts were denoted $v_{0,e}, v_{1,e}$.

The differential d_{eq} on this complex is defined by counting the 1-dimensional components in $\mathcal{M}(P_\eta)$ and interpreting the count as follows: if $\pi(s)$ is asymptotic to $g\tau^k(\eta_-)$ and η_+, and $q(s)$ is asymptotic to gq_-, q_+, then (q,π) is registered as contributing towards the coefficient:

$$\mathfrak{o}(q_+)\otimes\mathfrak{o}(y_+)\mapsto x^k\mathfrak{o}(q_-)\otimes\mathfrak{o}(y_-),$$

where $y_\pm$ are the projections of $\eta_\pm$ to BG. This is all in a manner similar to §2.4.11. The differential squares to zero by the same arguments used in [SS10, Sei15, SZ21] and in §2.2, §2.3, and §2.4.

[35] With the correct interpretation of "projective space" (as either $\mathbb{R}P^\infty$ or $\mathbb{C}P^\infty$), this handles both cases $p=2$ and $p\geq 3$; in the former case, there is a unique choice of y, while in the latter case, one can arrange the construction so there are two choices of y.

3.1.4. **Definition of the Morse continuation map.** Consider the space of Moore paths $\mathfrak{M}$ consisting of pairs of:

- a real number $w \in [0, \infty)$ and
- a smooth path $s \in \mathbb{R} \mapsto P_{\eta,s}$ of vector fields on W (parametrized by η) such that $\partial_s P_{\eta,s} = 0$ for s outside $[0, w]$,

such that (M1), and (M3) are satisfied for all s.

Definition 3.4. *It will be important in the sequel to consider families $P_{\eta,s}$ such that $P_{\eta,s} = 0$ holds identically. In this case we say that $P_{\eta,s}$* remains at 0.

If we fix two endpoints, say, P_η and P'_η, then we let $\mathfrak{M}(P_\eta, P'_\eta)$ be the subspace of Moore paths whose endpoints are P_η and P'_η. Each of these morphism spaces in $\mathfrak{M}$ is a convex space.

Definition 3.5. *A path $\phi \in \mathfrak{M}$ is said to be* admissible *for the Morse continuation map provided that the endpoints $P_{\eta,0}$ and $P_{\eta,w}$ of ϕ are valid Morse–Borel data, and either:*

- *$P_{\eta,0}, P_{\eta,w}$ are both of type $+$ or both of type $-$,*
- *$P_{\eta,0}$ is of type $-$ and $P_{\eta,w}$ is of type $+$.*

In other words, paths from type $+$ to type $-$ are not admissible.

Associated to an admissible path $\phi \in \mathfrak{M}$, expressed as $P_{\eta,s}$, one can form the moduli space $\mathcal{M}(\phi)$ of solutions (q, π) to the following equation:

$$\begin{cases} q'(s) = -X_{-s,\pi(s)}(q(s)), & \lim_{s\to\pm\infty} q(s) \text{ exists}, \\ \pi'(s) = -V(\pi(s)). \end{cases} \tag{32}$$

Definition 3.5 is used to prove an a priori bound for solutions of (32): any sequence q_k of solutions to (32) must remain in a uniform compact set.

By counting the rigid elements (as in §3.1.3) associated to the moduli spaces $\mathcal{M}(\phi)$ for generic admissible paths, one defines chain maps between the equivariant Morse complexes:

$$\mathfrak{c}_\phi : \mathrm{CM}_{\mathrm{eq}}(P_\eta) \to \mathrm{CM}_{\mathrm{eq}}(P'_\eta). \tag{33}$$

The map is independent of the choice of morphism ϕ from P_η to P'_η in $\mathfrak{M}$, up to chain homotopy. Moreover, standard gluing arguments then prove that continuation maps commute with each other up to chain homotopy, and consequently one shows all chain complexes $\mathrm{CF}_{\mathrm{eq}}(P_\eta)$ of the same type (either type $-$ or type $+$) are chain homotopy equivalent (via continuation maps). Note that the map in the statement of Theorem 1.18 is a special case of (33), when P_η, P'_η are of different types.

The direct limit of the homology groups $\mathrm{HM}_{\mathrm{eq}}(P_\eta)$ over the diagram of choices P_η of the same type (connected by continuation maps) yields two invariants:

- type +: *the equivariant cohomology* $\mathrm{HM}_{\mathrm{eq}}(W)$,
- type −: *the relative equivariant cohomology* $\mathrm{HM}_{\mathrm{eq}}(W, \partial W)$,

and the morphisms (33) in $\mathfrak{M}$ connecting type − to type + yield a map:

$$\mathrm{HM}_{\mathrm{eq}}(W, \partial W) \to \mathrm{HM}_{\mathrm{eq}}(W).$$

3.1.5. **The unit element in equivariant Morse cohomology.** Let us comment on the construction of a class $1 \in \mathrm{HM}_{\mathrm{eq}}(W)$. Let P_η be Morse–Borel data of type +. Each generating pair $(q, y) \in \mathrm{CM}_{\mathrm{eq}}(P_\eta)$ where $\mathrm{index}(q) = \mathrm{index}(y) = 0$ admits canonical isomorphisms:

$$\mathbb{Z} \simeq \mathfrak{o}(q) \text{ and } \mathbb{Z} \simeq \mathfrak{o}(y),$$

because the orientation line of a zero-dimensional space has a canonical generator (corresponding to the empty basis). Taking their tensor product with $1 \in \mathbf{k}[[x]]$, and direct summing over such pairs (q, y) produces an element in the complex $\mathrm{CM}_{\mathrm{eq}}(P_\eta)$, see §3.1.3, denoted by the symbol 1.

Lemma 3.6. *The element* 1 *satisfies* $d_{eq}(1) = 0$*, and is preserved under the continuation maps of §3.1.4.*

Proof. The proof is based on two standard ideas in Morse theory:

- the sum of all local minima is a cycle,
- the continuation map from one Morse complex to another sends the sum of all local minima to the sum of all local minima.

The details are left to the reader. □

3.1.6. **Equivariant pseudogradients.** Let us now focus on a special class of Borel–Morse data of type +, namely, data $P = P_\eta$ which is globally independent of η, and which satisfy the following additional axiom:

(EP) around each critical point q in the fixed submanifold $F \subset W$ of the $G = \mathbb{Z}/p\mathbb{Z}$ action, there is a coordinate chart $B^{2r} \times B^{2n-2r}$ so that F is identified with $B^{2r} \times \{0\}$, and $g \in G$ acts by $(z_1, z_2) \mapsto (z_1, gz_2)$ for some unitary representation of G on $\mathbb{C}^{n-r}$, and so that P appears as a direct sum $P_1 + P_2$ of two linear vector fields, and P_2 points radially outwards.

Lemma 3.7. *There exist vector fields* P *satisfying (EP) which define regular Borel–Morse data of type +.*

Proof. See [SS10, §2.2]. The idea is to pick P first on the fixed submanifold $F \subset W$, and then extend it to a tubular neighborhood of F by requiring it points outwards (from the perspective of Lyapunov functions, we add a positive definite quadratic function on each fiber of the normal bundle NF). Then we extend P equivariantly to the rest of W (generically), bearing in mind it should agree with Z in the convex end.

If the choice of extension from NF to W is done sufficiently generically, then all flow lines asymptotic to a critical point in $W \setminus F$ will be cut transversally.

Moreover, if the initial choice of P on F is done sufficiently generically, then all flow lines contained entirely in F will be cut transversally *inside of* F. However, the assumption that P points outwards along fibers of NF ensures that such flow lines are also cut transversally when considered in W (there is no change in the index difference).

The cases covered in the two preceding paragraphs cover all possible flow lines, and the proof is complete. □

3.2. **Localization in equivariant Morse theory.** *Localization* refers to the phenomenon that the ring-theoretic localization of the equivariant cohomology ring of a G-space at certain elements recovers the regular cohomology of the fixed point set. This statement can be made quite sophisticated (we refer the reader to [AB84] for a refined result). In our paper, we will require only a small input from this theory (for Theorem 1.14 part (d)):

Proposition 3.8. *If the action of $G = \mathbb{Z}/p\mathbb{Z}$ on a G-filling W has at least one fixed point, then the unit element $1 \in \mathrm{HM}_{\mathrm{eq}}(W)$ is not x-torsion.*

The group $\mathrm{HM}_{\mathrm{eq}}(W)$ is a finitely generated $\mathbf{k}[[x]]$-module, so it admits a torsion submodule and a free quotient; the proposition guarantees the free quotient is non-zero (in the presence of at least one fixed point). More refined localization results identify exactly the rank of this free quotient.

The somewhat involved Morse-theoretic arguments of [SS10, §2], which are carried out in the case of a closed manifold with a $\mathbb{Z}/2$-action, prove something stronger. We will take a simpler approach[36] than that of [SS10].

3.2.1. **Proof of Proposition 3.8.** We assume $p \geq 3$, as the case $p = 2$ is simpler (and is essentially established in [SS10]).

Without loss of generality, we pick $P_\eta = P$ satisfying (EP). Since we assume there is at least one fixed point, the fixed point submanifold F is non-empty, and, by construction of P, there is at least one index 0 critical point q_0 in the fixed point submanifold.

Let us introduce symbols v_0, v_1 as the index 0 and index 1 critical points on BG; all other critical points are of the form $\tau^k(v_0)$ or $\tau^k(v_1)$, and each such critical point has p lifts to critical points on EG (shown in Figure 3 using notation $v_{0,g}$ and $v_{1,g}$ for the lifts of v_0, v_1).

[36] Another reasonable approach is to essentially translate each step of [AB84] into Morse theoretic terms. This requires four ingredients: *embedding functoriality* (i.e., pull-back maps), *the relative cohomology of a pair*, *the Thom isomorphism*, and the *ring structure on (equivariant) cohomology*. The interpretation of these steps within Morse theory is more or less standard. The recipe in [AB84] then gives a refined localization result.

Recall from §3.1.3 that:

$$\mathrm{CM}_{\mathrm{eq}}(P) := \bigoplus_q (\mathfrak{o}(v_0) \otimes \mathfrak{o}(q) \otimes \mathbf{k}[[x]]) \oplus (\mathfrak{o}(v_1) \otimes \mathfrak{o}(q) \otimes \mathbf{k}[[x]]), \tag{34}$$

where the sum is over all critical points q (zeros of P). Let us consider the projection map onto the summand:

$$Q(P, q_0) := (\mathfrak{o}(v_0) \otimes \mathfrak{o}(q_0) \otimes \mathbf{k}[[x]]) \oplus (\mathfrak{o}(v_1) \otimes \mathfrak{o}(q_0) \otimes \mathbf{k}[[x]])$$

associated to the chosen index 0 critical point $q_0 \in F$. This map is a quotient by a subcomplex because any flow line $(q(s), \pi(s))$ which ends with $q(-\infty) = q_0$ must also start with $q(+\infty) = q_0$.

The induced differential on $Q(P, q_0)$ yields a complex which computes the equivariant Morse cohomology of a point $\mathrm{HM}_{\mathrm{eq}}(\mathrm{pt})$ (this is obvious, since the quotient differential on $Q(P, q_0)$ only considers flow lines $(q(s), \pi(s))$ for which $q(s) = q_0$ while $\pi(s)$ is allowed to vary arbitrarily on EG).[37] It is also clear from the definitions that this quotient map sends $1 \in \mathrm{HM}_{\mathrm{eq}}(W)$ to $1 \in \mathrm{HM}_{\mathrm{eq}}(\mathrm{pt})$. We now invoke the following lemma:

Lemma 3.9. *The element $1 \in \mathrm{HM}_{\mathrm{eq}}(\mathrm{pt})$ is not x-torsion; here it is important to assume that $G = \mathbf{k} = \mathbb{Z}/p\mathbb{Z}$.*

Proof. This is obvious provided one knows that $\mathrm{HM}_{\mathrm{eq}}(q_0)$ is the ordinary Borel G-equivariant cohomology $H^*_G(q_0, \mathbf{k})$. However, it also follows from a direct computation of the differentials for the special pseudogradient BG; see, e.g., [SZ21, §4]. □

This completes the proof of Proposition 3.8. □

3.3. **The equivariant PSS construction.** The goal is to give a precise description of the ∞-categories $\mathcal{P}(Y), \mathcal{N}(Y)$ introduced in §1.4, and to construct the ∞-functors:

$$\mathcal{P}(Y) \to \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]]) \quad \mathcal{N}(Y) \to \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]])$$

with the properties stated in Theorems 1.16 and 1.17.

The aspects of the construction involving the negative PSS category $\mathcal{N}(Y)$ are entirely analogous to the aspects involving the positive PSS category $\mathcal{P}(Y)$, and for this reason we focus our discussion on the case of $\mathcal{P}(Y)$.

3.3.1. **The PSS category.** In this section we define the PSS category $\mathcal{P}(Y)$. Recall from §1.4 and Remark 2.9 the ∞-category $\bar{\mathcal{C}}(Y)$.

Definition 3.10. *An n-simplex in $\mathcal{P}_n(Y)$ is a pair (j, Φ) where $-1 \leq j \leq n$, and Φ is an n-simplex in $\bar{\mathcal{C}}(Y)$ such that:*

- *the vertices $j+1, \ldots, n$ lie in $\mathcal{C}(Y)$ (the non-discriminant part),*

[37] The quotient map $\mathrm{CM}_{\mathrm{eq}}(P) \to Q(P, q_0)$ is the Morse-theoretic realization of the cohomological pullback $H^*_G(W) \to H^*_G(\{q_0\})$ induced by the inclusion $\{q_0\} \to W$.

- *the subsimplex spanned by vertices* $0, \dots, j$ *remains at id in the sense of Definition 3.11.*

In particular, the simplices with $j = -1$ *form a copy of* $\mathcal{C}(Y)$*.*

Definition 3.11. *A* j*-simplex* Φ *in* $\bar{\mathcal{C}}(Y)$ remains at *id provided all of its subcubes are mapped into the subset of* $\mathrm{NN}(Y)$ *consisting of the constant paths at id (of any length).*

Definition 3.12 (Face and degeneracy maps)**.** *If* $(j, \Phi) \in \mathcal{P}_n(Y)$*, then the* i*th face map is given by:*

$$d_i(j,\Phi) = \begin{cases} (j, d_i\Phi) & \text{if } j < i, \\ (j-1, d_i\Phi) & \text{if } j \geq i. \end{cases}$$

The i*th degeneracy map is given by:*

$$s_i(j,\Phi) = \begin{cases} (j, s_i\Phi) & \text{if } j < i, \\ (j+1, s_i\Phi) & \text{if } j \geq i. \end{cases}$$

Lemma 3.13. *The collection of* n*-simplices* $\mathcal{P}_n(Y)$*, with the above face and degeneracy maps, defines an* ∞*-category.*

Proof. The face and degeneracy maps are closely related to the *join operation*; see [Lur09, §1.2.8]. It is an easy combinatorial verification to track the j variable and bootstrap the known ∞-category structure on $\bar{\mathcal{C}}(Y)$ to conclude $\mathcal{P}(Y)$ is also an ∞-category. □

Remark 3.14. The negative PSS category $\mathcal{N}(Y)$ is defined analogously, except now the subsimplex spanned by the last vertices $n-j, \dots, n$ is required to be remain at *id*, and the first $0, \dots, n-j-1$ vertices are required to lie in $\mathcal{C}_0(Y)$.

3.3.2. **The category of Morse–Floer data.** As with the definition of the ∞-functor in §2, we will actually define a canonical diagram:

$$\begin{array}{ccc} \mathcal{Q}^*(W) & \longrightarrow & \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]]), \\ \big\downarrow & & \\ \mathcal{P}(Y) & & \end{array} \tag{35}$$

and prove that the vertical arrow is a trivial Kan fibration. The category $\mathcal{Q}^*(W)$ contains auxiliary information; most importantly, the fibers of $\mathcal{Q}^*(W)$ over $id \in \mathcal{P}(Y)$ represent choices of regular Morse–Borel data.

Similarly to the discussion in §2.2.2, we will first define a larger category $\mathcal{Q}(W)$ and then define $\mathcal{Q}^*(W)$ as the subcategory of "regular" simplices.

Definition 3.15. *Let us define* $\mathfrak{MF}$ *to be the space of triples* (w, ϕ, φ) *where:*

- $(w, \phi) \in \mathfrak{M}$*, as in §3.1.4, (a Moore path of Morse–Borel data),*
- $(w, \varphi) \in \mathfrak{F}$*, as in §2.2.2, (a Moore path of Floer–Borel data),*

satisfying either:

(i) ϕ remains at 0 (Definition 3.4), and φ has non-degenerate endpoints,
(ii) ϕ has non-degenerate endpoints of type $+$, and φ remains at id (Definition 3.11),
(iii) ϕ joins type $+$ data to the 0 vector field, and φ joins id to non-degenerate data.

The space $\mathfrak{MF}$ should be thought of as the space of Moore paths in the space of hybrid Morse-Floer data. The concatenation operations $\#_\tau$ (insertion of a stationary path) is well-defined between any two paths in $\mathfrak{MF}$ with composable endpoints ((iii) composed with (i) is again of type (iii), etc).

Definition 3.16. *A zero simplex in $\mathcal{Q}_n(W)$ is either a Morse–Borel datum P_η of type $+$, as in §3.1.1, or a Floer–Borel data $(\varphi_{\eta,t}, J)$ as in §2.2.2.*

An n-simplex Ψ in $\mathcal{Q}_n(W)$, for $n \geq 1$, is a collection of zero simplices $\Psi^i \in \mathcal{Q}_0(W)$ for $i = 0, \dots, n$, and a collection smooth cubes:

$$\Psi^{v_0 \dots v_m} : [0, \infty)^{m-1} \to \mathfrak{MF},$$

for all $0 \leq v_0 < \cdots < v_m \leq n$ and $m > 0$, such that, as in Definition 2.3,

(1) the endpoints of $\Psi^{v_0 \dots v_m}(x)$ are Ψ^{v_0} and Ψ^{v_m}, for all x; this is understood in the appropriate sense that we ignore the component which is set to 0 or id in Definition 3.15,
(2) as $x_j \to \infty$,

$$\Psi^{v_0 \dots v_m}(x) = \Psi^{v_0 \dots v_j}(x_1, \dots, x_{j-1}) \#_{\tau(x_j)} \Psi^{v_j \dots v_m}(x_{j+1}, \dots, x_{m-1}),$$

(3) when $x_j = 0$,

$$\Psi^{v_0 \dots v_m}(x) = \Psi^{v_0 \dots \hat{v_j} \dots v_m}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{m-1}),$$

(4) on the face $x_j = 0$, $\partial_{x_j} \Psi^v(x)$ vanishes to infinite order.

The definition of the simplicial structure on $\mathcal{Q}(W)$ and the verification that it is an infinity category proceeds exactly the same as in §2.1.

3.3.3. **The subcategory of regular Morse–Floer data.** The next step is to define the category $\mathcal{Q}^*(W)$ as the subcategory of simplices which render a certain collection of countably many moduli spaces tranverse.

To each simplex we will associate the moduli spaces for each subcollection of vertices $\{v_0 < \cdots < v_m\}$. The primary case of interest is when v_0 is Morse type and v_m is Floer type, as these are associated to the hybrid moduli spaces. Let us therefore focus on this case.

From each element $(w, \phi, \varphi) \in \mathfrak{MF}$ joining a Morse type vertex to a Floer type vertex, one can extract:

- Hamiltonian vector fields $Y_{\eta,s,t}$ and $X_{\eta,s,t}$, as in §2.2.3,
- Morse pseudogradients $P_{\eta,s}$, as in §3.1.4,

defined for $s \in \mathbb{R}$ and $t \in [0,1]$. Using these, we associate to w, ϕ, φ the moduli space $\mathcal{M}(w, \phi, \varphi)$ of solutions (π, q, u) to:

$$(36)\quad \begin{cases} \pi : \mathbb{R} \to BG, \quad q : \mathbb{R} \to W, \quad u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to W, \\ \pi'(s) = -V(\pi(s)), \\ q'(s) = -P_{-s,\pi(s)}(q(s)), \quad \lim_{s\to\infty} q(s) \text{ exists}, \\ (\partial_s u - \rho(t) Y_{\pi(s),s,t}(u)) + J_s(u)(\partial_t u - X_{\pi(s),s,t}(u)) = 0, \\ \text{the integral of } \omega(\partial_s u - \rho(t) Y_{\pi(s),s,t}(u), \partial_t u - X_{\pi(s),s,t}(u)) \text{ is finite}, \\ u(+\infty) = q(-\infty). \end{cases}$$

This is considered as a hybrid between (32) and (17); see Figure 4.[38]

Given $\Psi \in \mathcal{Q}_n(W)$, and any $\{v_0 < \cdots < v_m\} \subset \{0, \ldots, n\}$ we consider the cubes:

$$\Psi^{v_0 \ldots v_m} : [0,\infty)^{m-1} \to \mathfrak{M}\mathfrak{F}.$$

This is associated with the parametric moduli space of tuples (x, π, q, u):

$$(37)\quad \begin{cases} x \in [0,\infty)^{m-1}, \\ \Psi^{v_0 \ldots v_m}(x) = (w, \phi, \varphi), \\ (\pi, q, u) \in \mathcal{M}(w, \phi, \varphi) \text{ as in (36)}. \end{cases}$$

There are two edge cases; if all vertices $v_0, \ldots, v_m$ are of Morse type, then φ is just a constant path at id, so we ignore u and only ask that:

- (π, q) solves equation (32).

On the other hand, if $v_0, \ldots, v_m$ are all Floer type vertices, then we ignore the Morse part and only ask that (π, u) solves:

$$(38)\quad \begin{cases} \pi : \mathbb{R} \to BG \quad u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to W, \\ \pi'(s) = -V(\pi(s)), \\ (\partial_s u - \rho(t) Y_{\pi(s),s,t}(u)) + J_s(u)(\partial_t u - X_{\pi(s),s,t}(u)) = 0, \\ \text{the integral of } \omega(\partial_s u - \rho(t) Y_{\pi(s),s,t}(u), \partial_t u - X_{\pi(s),s,t}(u)) \text{ is finite}. \end{cases}$$

Definition 3.17. *If the parametric moduli spaces* (37)*, or* (32) *or* (38)*, are transverse for each subcube determined by $\Psi \in \mathcal{Q}_n(W)$, then we say Ψ is a regular n-simplex. The collection of all regular simplices is denoted $\mathcal{Q}^*(W)$.*

Lemma 3.18. *The regular simplices $\mathcal{Q}^*(W)$ form a sub ∞-category.*

Proof. It follows, essentially by construction, that $\mathcal{Q}^*(W)$ is closed under face maps (since we require transversality for all subcubes in Definition 3.17). In fact, $\mathcal{Q}^*(W)$ is also closed under degeneracy maps, as follows from the same argument used in Proposition 2.15. Briefly: the actual equations which are associated to a degenerate simplex are the same equations which appear for

[38]The negative PSS version of this equation, used for Theorem 1.17, is a reflected version, with $u(-\infty) = q(\infty)$.

the underlying non-degenerate simplex, but the parameter space has been enlarged with additional degrees of freedom. This implies that regularity is preserved under taking degeneracies.

The fact that $\mathcal{Q}^*(W)$ satisfies the horn filling property follows from Lemma 3.19 below. □

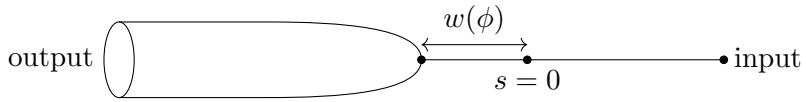


FIGURE 4. Illustration of the hybrid Morse–Floer equation (36).

3.3.4. **The trivial Kan fibration property.**

Lemma 3.19. *There is a commutative square of simplicial maps:*

$$\begin{array}{ccc} \mathcal{D}^*(W) & \xrightarrow{\subset} & \mathcal{Q}^*(W) \\ \downarrow & & \downarrow \\ \mathcal{C}(Y) & \xrightarrow{\subset} & \mathcal{P}(Y) \end{array}$$

and the vertical morphisms are trivial Kan fibrations.

Proof. This boils down to the following problem: given a $[0,\infty)^\infty \to \mathrm{NN}(Y)$ arising from a simplex in $\mathcal{P}(Y)$, and a lift of the boundary of the cube to $\mathfrak{MF}$, can one extend the lift to the interior? The Floer part can be handled in the same way as Lemma 2.18. The Morse part is handled straightforwardly by convexity in the space of Moore paths of vector fields. □

3.3.5. **Definition of the PSS infinity functor.** In this section we complete the first part of Theorem 1.16 (and, by analogy, also Theorem 1.17), and construct the ∞-functor $\mathcal{Q}^*(W) \to \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]])$; the passage to a functor defined on $\mathcal{P}(Y)$ (by choosing sections of a trivial Kan fibration) follows the same lines as §2.2.6.

Each n-simplex $\Psi \in \mathcal{Q}^*(W)$ yields a collection of subcubes; let us label the important ones for the purposes of verifying Definition 1.13:

$$\Psi^{0\dots n},\ \Psi^{0\dots \hat{i}\dots n},\ \Psi^{0\dots i},\ \text{and},\ \Psi^{i\dots n},\ \text{for } i = 1,\dots,n-1 \tag{39}$$

Each of these subcubes determines a parametric moduli space, which is assumed to be regular (as we start with $\Psi \in \mathcal{Q}^*(W)$).

The overall strategy is as follows:

(a) vertices Ψ of Morse type are sent to $\mathrm{CM}_{\mathrm{eq}}(\Psi)$, while vertices of Floer type are sent to $\mathrm{CF}_{\mathrm{eq}}(\Psi)$; to unify notation, let us denote this chain complex by $\mathrm{C}_{\mathrm{eq}}(\Psi)$;

(b) n-simplices Ψ with $n \geq 1$ between zero-simplices $\Psi|_0$ and $\Psi|_n$ are sent to the map $\mathfrak{c}_\sigma : \mathrm{C}_{\mathrm{eq}}(\Psi|_0) \to \mathrm{C}_{\mathrm{eq}}(\Psi|_n)$ obtained by counting the rigid solutions of (37) for the top subcube (corresponding to the full set of vertices $0, \ldots, n$) in an appropriate fashion (recalled below);

Then, to prove the ∞-functor relations, one argues as follows:

(c) the relation between the one-dimensional components of (37), for the top subcube, and the rigid elements in the moduli spaces associated to the lower dimensional subcubes listed in (39) is used to show that the first part of Definition 1.13 is satisfied;
(d) the rigid components of the moduli space associated to the top subcube of a degenerate n-simplex are shown to either yield the identity chain map (if $n = 1$) or to the 0 map (if $n > 1$); the latter case is proved by exhibiting a degree of freedom in the moduli space, similarly to Lemma 3.18.

There is not much left to explain for (a); the complexes are as defined in §2.2.4, §2.3, §2.4.11 (Floer), and §3.1.3 (Morse).

The counting scheme used for (b) follows mostly the same lines as §2.4.11 and §3.1.3. We only explain the case when there are both types of vertices; the cases when the equation is entirely Morse or entirely Floer follow simpler lines and are left to the reader.

Each solution (x, π, q, u) determines an linearized operator:

$$D_{\pi,q,u} : T_x \oplus T_\pi \oplus T_q \oplus W^{1,p}(u^*TW) \to L^p(u^*TW) \oplus TW_{u(\infty)},$$

where T_x, T_π, T_q are shorthands for the tangent spaces at x, π and q. The map to the second factor is the linearization of the constraint $u(\infty) = q(w)$; if $(\delta x, \delta\pi, \delta q, \delta u)$ is a variation, the map to $TW_{u(\infty)}$ is given by:

$$q'(w(x))dw_x(\delta x) + \delta q(w(x)) - \delta u(\infty),$$

while the map to the first factor is defined by a standard linearization procedure, as discussed in §2.2.4 and §2.4.10.

This linearized operator is surjective, by definition of regularity, and the kernel of this operator is the tangent space of the moduli space at (x, π, q, u).

Thus, in the case of the zero-dimensional moduli spaces, one can use the orientability of the moduli space and the parallel transport maps for Fredholm determinants to produce generators of:

$$\mathfrak{o}(T_x) \otimes \mathfrak{o}(T_\pi) \otimes \mathfrak{o}(T_q) \otimes \mathfrak{o}(D_u) \otimes \mathfrak{o}(TW_{u(\infty)}),$$

and then use the canonical gluing isomorphisms from §2.4.9 to identify:

$$\mathfrak{o}(D_u) \simeq \mathfrak{o}(\gamma_-) \text{ and } \mathfrak{o}(T_\pi) \simeq \mathfrak{o}(y_-) \otimes \mathfrak{o}(y_+),$$

and consequently, as in §2.4.11, we have a canonical identification:

$$\mathfrak{o}(T_\pi) \otimes \mathfrak{o}(T_q) \otimes \mathfrak{o}(D_u) \otimes \mathfrak{o}(TW_{u(\infty)}) \simeq \mathfrak{o}(\gamma_-) \otimes \mathfrak{o}(y_-) \otimes \mathfrak{o}(q_+) \otimes \mathfrak{o}(y_+),$$

where $\mathfrak{o}(q_+)$ is the orientation line of the *unstable manifold* at $q_+ = q(+\infty)$ (note that T_q is naturally identified with the stable manifold at q_+).

In this fashion, the rigid solutions in the moduli space associated to the top subcube of σ are packaged into a linear map:

$$\mathfrak{c}_\sigma : \bigoplus \mathfrak{o}(q_+) \otimes \mathfrak{o}(y_+) \otimes \mathbf{k}[[x]] \to \bigoplus \mathfrak{o}(\gamma_-) \otimes \mathfrak{o}(y_-) \otimes \mathbf{k}[[x]].$$

This completes our discussion of (b). The discussion of (c) is analogous to §2.2.5 and §2.4.11; we only briefly comment on the novel aspects introduced by the hybrid equation (37).

Let us consider the case of an end in the 1-dimensional moduli space where $x_i \to \infty$; there are two cases to consider, depending on whether the vertex v_i is of Morse type or Floer type. In either case, $\Psi(x)$ breaks into:

- $\Psi^{0\dots i}(x_1, \dots, x_{i-1}) \#_{\tau(x_i)} \Psi^{i\dots n}(x_{i+1}, \dots, x_{n-1})$,

and the only difference is which piece of $\mathfrak{MF}$; if v_i is Morse type, then:

- $\Psi^{0\dots i}$ takes values in the Morse-Morse part of $\mathfrak{MF}$,
- $\Psi^{i\dots n}$ takes values in the Morse-Floer part of $\mathfrak{MF}$,

otherwise, if v_i is of Floer type, then:

- $\Psi^{0\dots i}$ takes values in the Morse-Floer part of $\mathfrak{MF}$,
- $\Psi^{i\dots n}$ takes values in the Floer-Floer part of $\mathfrak{MF}$.

These is the expected breaking, and terms with this type of breaking contribute the term $\mathfrak{c}_{\sigma|[i\dots n]} \circ \mathfrak{c}_{\sigma|[0\dots i]}$ which appears in (3).

Let us explain in a bit more detail the degenerating sequence in the case when v_i is of Morse type. A sequence of solutions (x_k, π_k, q_k, u_k) will converge, after passing to a subsequence, and on compact subsets of the domain (either $\mathbb{R}$ or $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$), to an equation for the subcube $\Psi^{0\dots i}$. Note that the Floer part converges to a constant, as it solves the holomorphic curve equation and has finite energy. To catch the other limit, one needs to translate the solutions by the amount:

$$w(\Psi^{0\dots i}(x_1, \dots, x_{i-1})) + \tau(x_i),$$

where w stands for the width of a Moore path. After this translation is done, the sequence of solutions will converge on compact subsets to a solution in the equation for the subcube $\Psi^{i\dots n}$ (after passing to a subsequence).

The rest of the details of (c) follows similar lines to §2.2.5, and the necessary deferal to the orientation line gluing arguments as in §2.4.11.

Finally, (d), concerning the counts associated to degenerate simplices, follows easily from the discussion in Lemma 3.18 and the same argument used in §2.2.5. Namely, when one degenerates, the parameter space is simply enlarged by one dimension, without changing the actual underlying solutions; this precludes the formation of rigid solutions. The case of a degenerate 1-simplex is an edge case, but it is standard and handled, as in, say, [HS95].

This completes our construction of the functor $\mathcal{Q}^*(Y) \to \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]])$. The passage to $\mathcal{P}(Y)$ follows the strategy of §2.2.6 and §1.9, namely by picking sections of the trivial Kan fibration from §3.3.4.

3.4. **The low-slope PSS isomorphism.** We explain why the maps:

$$\mathrm{PSS} : \mathrm{CM}_{\mathrm{eq}}(P_\eta) \to \mathrm{CF}_{\mathrm{eq}}(R_{\epsilon t})$$

associated to specific 1-simplices $\varphi_{s,t} = R_{s\epsilon t}$ joining id to $R_{\epsilon t}$, for small positive slopes ϵ, are chain-homotopy equivalences. The analogous statement for the negative PSS map of Theorem 1.17 is proved in the exact same way, and so we only focus on the "positive" PSS map. The argument we will use follows [FS07, pp. 24-25]; the idea is to construct a chain-homotopy inverse. Due to the similarity with [PSS96, FS07], we only sketch the results.

3.4.1. **Geometric set up for the inverse of PSS.** Let r be a radial coordinate on the convex end of W, so that $\Omega = \{r \leq 1\}$ is an aspherical symplectic domain with contact-type boundary, and so that $r \geq 1$ is identified with the positive half of the symplectization of ∂W. We suppose that r is G-invariant. Then X_r has an ideal restriction to ∂W as a Reeb flow R which lifts a Reeb flow on $Y = \partial W/G$.

Let $h : \mathbb{R} \to \mathbb{R}$ be a convex cut-off function so that:

- $h(r) = r$ for $r \geq 2$,
- $h'(r) > 0$ for $r > 1$,
- $r = 3/2$ on $\Omega = \{r \leq 1\}$,

and let $H = \epsilon h(r)$. We assume ϵ is small enough that the only orbits of X_H are the constant orbits in Ω. For a choice of almost complex structure J as in §2.2.2, the pair (H, J) is equivariant Borel data (certainly not regular) whose ideal restriction is $R_{\epsilon t}$.

We also pick a vector field P which agrees with the Liouville vector field Z in the convex end and is a regular equivariant Morse–Borel data as in §3.1.6.

3.4.2. **Reverse PSS cylinders.** Let $(\psi_{\eta,t}, J)$ be regular Borel data whose ideal restriction is $R_{\epsilon t}$. As in §2.2.3, let us denote by $X_{\eta,t}$ the generator of $\psi_{\eta,f(t)}$, where f is the cut-off function from Figure 1. We may assume, without loss of generality, that $X_{\eta,t} = f'(t)X_H$ outside Ω (i.e., we assume $\psi_{\eta,t} = R_{\epsilon t}$ outside Ω). We use this $f(t)$ only to be consistent with the set-up of §2.2.3, rather than for any purpose internal to this section.

Introduce the continuation data:

$$X_{\eta,s,t} := (1 - f(s))f'(t)X_H + f(s)X_{\eta,t} + E_{s,t},$$

for $s \in [0, 1]$, extended to the rest of the line by s-independence, where:

- $E_{s,t}$ is a C^∞-small Hamiltonian pertubation, supported in Ω.

Introduce the moduli space $\mathcal{R}$ of *reverse PSS continuation cylinders*:

$$\begin{cases} u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to W,\ q : (-\infty, 0] \to W, \text{ and } \pi : \mathbb{R} \to EG, \\ \pi'(s) = -V(\pi(s)),\ q'(s) = -P(q(s)), \\ \partial_s u + J(u)(\partial_t u - X_{\pi(s),s,t}(u)) = 0, \\ u(-\infty) = q(0). \end{cases} \tag{40}$$

This solution is asymptotic at the input to a Hamiltonian orbit, and has a flow line of $-P$ connected at the output end.

Importantly, because $X_{\pi(s),s,t}(u)$ is independent of s, outside of Ω, the finite energy solutions obey the necessary a priori energy estimates needed for the usual Floer theory compactness results.

We only consider those solutions $(u, q, \pi) \in \mathcal{R}$ so that:

- u has finite energy, which implies the left asymptotic is a removable singularity $u(-\infty) \in \Omega$ (see [FS07, Figure 3]);[39]
- $q(-\infty)$ converges to a zero of P (i.e., we do not consider solutions where $q(-\infty)$ drifts off in the non-compact end);

By counting the rigid such solutions in $\mathcal{R}$ in the same way as in §3.3.5 (for generic choice of the perturbation term $E_{s,t}$), we obtain a map:

$$\mathrm{CF}_{\mathrm{eq}}(\psi_{\eta,t}, J) \to \mathrm{CM}_{\mathrm{eq}}(P); \tag{41}$$

consideration of the 1-dimensional part of $\mathcal{R}$ proves this is a chain map.

3.4.3. **The chain homotopy inverse.** We now explain why (41) is the chain homotopy inverse of the PSS map of §3.3.5.

To define the PSS map, one ultimately uses the equation (36). In the case of the 1-simplex $s \mapsto R_{\epsilon st}$, one needs to pick a path of Borel data $s \mapsto (\psi_{\eta,s,t}, J)$ whose ideal restriction is $R_{\epsilon st}$. As in §3.4.2, we pick $\psi_{\eta,s,t}$ agreeing with $R_{\epsilon st}$ outside of Ω.

Unpacking (36) leads to a family of vector fields $Y_{\eta,s,t}$ and $X_{\eta,s,t}$ so that:

(i) $Y_{\eta,s,t} = 0$ outside of $s \in [-w(\varphi), 0]$,
(ii) $X_{\eta,s,t} = X_{\eta,t}$ for $s \leq -w(\varphi)$, where $X_{\eta,t}$ is as in §3.4.2,
(iii) $X_{\eta,s,t} = 0$ for $s \geq 0$,
(iv) both $X_{\eta,s,t}, Y_{\eta,s,t}$ are η independent and lie in the line spanned by the Reeb vector field R, outside of Ω,

[39] This point is slightly subtle since $f'(t)X_H$ has every point in the domain Ω as a constant 1-periodic orbit, and has no other orbits. For any sequence $s_n \to -\infty$, there is a subsequence of $u(s_n, t)$ which converges to a point q. Then, by some maximum principle, e.g., the one of [AS10, Lemma 7.2], we conclude that u takes values entirely in Ω. It follows that u is a genuine holomorphic curve on the half-cylinder $s \leq 0$; having established this, the analysis follows the usual PSS arguments of [PSS96].

(v) the curvature of the Hamiltonian connection[40] on the complement of Ω determined by $X_{s,t}, Y_{s,t}$ is non-positive.

One counts the rigid solutions of the parametric moduli space where η is constrained to equal $\pi(s)$ for flow lines $\pi : \mathbb{R} \to EG$, as usual, to define the chain map. This PSS chain map can be post-composed with (41). By standard Floer theory, the composition is chain homotopic to the count of rigid cylinders solving Floer's equation for glued Hamiltonian connections (with flow lines attached at both ends, in the PSS sense).

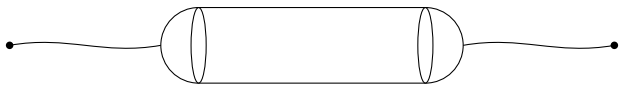


FIGURE 5. Cartoon of the composition of PSS followed by (41)

The glued Hamiltonian connections are similarly determined by vector fields $X_{R,\eta,s,t}, Y_{R,\eta,s,t}$, where R is a gluing parameter. These glued connections still satisfy (iv) and (v). In addition, they satisfy the property that:

(vi) $X_{R,\eta,s,t} = f'(t)X_H$ for $s \leq -s_R$, and $X_{R,\eta,s,t} = 0$ for $s \geq s_R$, for some s_R.

The key idea is that the space of connections on the cylinder (depending on the auxiliary parameter η) satisfying (iv), (v) and (vi) is a convex space.[41] Thus we can deform the glued data $X_{R,\eta,s,t}, Y_{R,\eta,s,t}$ in a one parameter family until it agrees with the following:

- $X_{s,t} = (1 - f(s))X_H$ on $s \in [0, 1]$, and extended by s-independence.

Indeed, we can simply travel in a straight line through the space of affine Hamiltonian connections.

The solutions for this final Hamiltonian connection are as follows:

(1) a cylinder u solving $\partial_s u + J(u)(\partial_t u - X_{s,t}(u)) = 0$,
(2) a flow line of $-P$ on the interval $[1, \infty)$ starting at the removable singularity $u(+\infty)$,
(3) a flow line of P on the interval $(-\infty, 0]$ ending at the removable singularity $u(-\infty)$,
(4) an underyling flow line of $\pi : \mathbb{R} \to EG$, which does not influence the equation because we use equivariant data.

By the maximum principle, e.g., [AS10, Lemma 7.2], u must lie entirely in the domain Ω, and so must be a holomorphic sphere, and therefore must be constant. Thus all we are doing is counting the flow lines of P. The index

[40] see [BC25, §3.2] for an introduction to Hamiltonian connections

[41] this uses (iv) in a crucial way, otherwise having non-positive curvature is a non-linear condition; for related discussion, see [BC25, §3.4.2]

zero requirement implies the only solutions we will count are the constant solutions, and so the resulting map is the identity map $\mathrm{CM}_{\mathrm{eq}}(P) \to \mathrm{CM}_{\mathrm{eq}}(P)$.

Thus, by the usual chain homotopy argument, we conclude the composition of the PSS map with (41) is chain homotopic to the identity map, as desired.

Thus we have proved (41) is the left inverse of the PSS map. A similar argument to the one given in [PSS96, FS07] proves that (41) is also the right inverse. We omit the details of this half of the argument and refer the reader to the aforementioned references. This completes the proof of Theorem 1.16. □

3.5. **Proof of Theorem 1.18.** The argument involves similar (and standard) analysis as the proof that (41) is the right inverse of PSS; namely it involves nodal gluing between PSS cylinders whose removable singularities are coincident. The details are similarly left to the reader. For further discussion, we refer the reader to [CHK23, §2.3.3]. □

4. **Local Floer cohomology as the cone of a simple crossing**

Local Floer cohomology is associated to a suitable set C of orbits. What is important is that Floer cylinders which start and end in C occupy a certain minimal amount of energy (in a way which persists under perturbations). This is the perspective developed in [Flo89]. Some special cases are:

- C is a single isolated fixed point; see [GG10, Zha19, SZ21, She22];
- C is an S^1 family of orbits; see, e.g., [CFHW96, Mcl12].

In our setting, the families of orbits for which we will develop a local Floer cohomology are associated to crossings with the discriminant.

4.1. **Extenders.** To any compact contact manifold Y we will associate a class of "simple" morphisms $\mathcal{E} \to \mathcal{C}_1(Y)$ and a map:

$$\mathrm{CF}_{\mathrm{loc}} : \mathcal{E} \to \mathrm{Ch}(\mathbf{k})$$

by applying local Floer cohomology theory in the symplectization SY.

Define an *extender*, denoted $\psi_{s,t} \in \mathcal{E}$, to be a Hamiltonian isotopy on SY with s,t-generating Hamiltonians $K_{s,t}, H_{s,t}$ satisfying:

(E1) $H_{s,t} = H_{0,t}$ and $K_{s,t} = 0$ on the negative end,
(E2) $X_{H_{0,t}}$ is Liouville equivariant everywhere,
(E3) $X_{H_{s,t}}$ is Liouville equivariant on the positive end,
(E4) $K_{s,1}$ is non-negative,
(E5) $X_{H_{1,t}}$ generates a Hamiltonian diffeomorphism with a compact set of fixed points, all of which have negative action,
(E6) there is at most a single action value attained in (E5).



FIGURE 6. The domain of an extender is the symplectization SY decomposed into a negative end, a positive end, and a compact part in between.

There is an ideal restriction $\mathcal{E} \to \mathcal{C}_1(Y)$ given by the flow generated by $H_{s,t}$ on the positive end.

Given an extender $\psi_{s,t} \in \mathcal{E}$, and a subset $\kappa \subset \pi_0(\Lambda Y)$ of free homotopy classes of loops, we will define in §4.4 the *local Floer chain homotopy type*[42] $\mathrm{CF}_{\mathrm{loc}}(\psi_{s,t}; \kappa)$ for $H_{1,t}$ following [Flo89]. Then:

Theorem 4.1. *Consider the infinity functor constructed in §2:*

$$\mathrm{CF}_{\mathrm{eq}} : \mathcal{C}(Y) \to \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]]),$$

assuming the hypotheses of Theorem 1.14. For $\psi_{s,t} \in \mathcal{E}$ with ideal restriction $\varphi_{s,t} \in \mathcal{C}_1(Y)$, the cone of the continuation map associated to $\varphi_{s,t}$ lies in the chain homotopy class $\mathrm{CF}_{\mathrm{loc}}(\psi_{s,t}; \kappa)$ where κ is the collection of W-contractible orbits in Y (namely, those orbits in Y which lift to orbits in ∂W which are contractible in W).

With this result in mind, we say an extender $\psi_{s,t}$ is *trivial relative W* if $X_{H_{1,t}}$ has no W-contractible 1-periodic orbits.

The proof of Theorem 4.1 is given in §4.4.3. In order to apply the theorem, it is necessary to show that there is an abundance of such extenders:

Theorem 4.2. *For any $\varphi \in \mathcal{C}_1(Y)$ which is a strictly positive Moore path, there exists an n-simplex Φ in $\mathcal{C}_n(Y)$ whose $[0, n]$ edge is σ, and whose $[i, i+1]$ edge is one of two types:*

(i) the ideal restriction of an extender, or,
(ii) an isomorphism in $\mathcal{C}_1(Y)$, i.e., projects to an isomorphism in the homotopy category,

If $\varphi \in \mathrm{NN}(Y)$ never intersects the W-contractible discriminant, then one can pick the extenders in (i) to be trivial relative W, and the continuation map associated to φ is a quasi-isomorphism.

This result together with Theorem 4.1 implies part (c) of Theorem 1.14. It also implies, together with §4.2.6, Theorem 1.21.

4.2. **Decompositions of continuation data.** The goal of this section is to prove Theorem 4.2 on the decomposability of 1-simplices into simpler pieces.

[42]Here a chain homotopy type is a chain complex up to chain homotopy equivalence.

4.2.1. **Decompositions.** We say that a 1-simplex σ is a *composition* of simplices $\sigma_1 \dots \sigma_n$ provided there is an n-simplex whose $[i-1, i]$ edge equals σ_i and whose $[0, n]$ edge equals σ. Then $[\sigma]$ is equal to the composition of $[\sigma_1], \dots, [\sigma_n]$ in the homotopy category $\mathrm{h}\mathcal{C}$. We refer to this process as *decomposition of* σ.

Our first observation is rather basic, but useful nonetheless:

Lemma 4.3. *Let $\varphi, \varphi' \in \mathcal{C}(Y)$ be two 1-simplices with the same source and target. The following are equivalent:*

(a) φ, φ' lie in the same path component of $\mathrm{NN}(Y)$ with fixed endpoints,
(b) $[\varphi'] = [\varphi]$ in the homotopy category $\mathrm{h}\mathcal{C}(Y)$;
(c) φ' can be decomposed into the sequence $\varphi_1 = id$, $\varphi_2 = \varphi$;
(d) φ' can be decomposed into the sequence $\varphi_1 = \varphi$, $\varphi_2 = id$;

here id is the degenerate 1-simplex (the identity element).

Proof. The equivalence of (b) through (d) can be found in [Lur09, §1.2.3]. The equivalence of (a) and (d) follows from §2.1.5. □

Mapping cones also interact nicely with decompositions:

Lemma 4.4. *Let $\mathcal{C}$ be an ∞-category and let $F : \mathcal{C} \to \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]])$ be an ∞-functor. Suppose that σ is the composition of $\sigma_1, \dots, \sigma_n$ and the homology of the mapping cones of the chain maps $F(\sigma_i)$ is x-torsion for each i. Then the homology of the mapping cone of $F(\sigma)$ is also x-torsion.*

Proof. It suffices to prove the case $n = 2$, by induction. Each of the three vertices of σ is mapped by F to a chain complex over $\mathbf{k}[[x]]$; let us call these C_0, C_1, C_2. The three faces are mapped to chain maps f, g, h. The equation for the ∞-functor:

$$h - gf = kd + dk$$

where $k : C_0[1] \to C_2$ is the map associated to the 2-simplex. Throughout we use gradings modulo two.

Introduce:

$$\Gamma = C_2 \oplus C_0[1] \quad \Delta = C_1 \oplus C_0[1] \oplus C_2 \oplus C_1[1]$$

with morphisms:

$$d_\Gamma = \begin{bmatrix} d & h \\ 0 & d \end{bmatrix} \quad d_\Delta = \begin{bmatrix} d & f & 0 & 1 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & g \\ 0 & 0 & 0 & d \end{bmatrix} \quad p = \begin{bmatrix} g & k & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad i = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & k \\ 0 & f \end{bmatrix},$$

Some care is needed to explain the sign conventions, but the basic rule is that the "shifting" symbol [1] introduces $\pm$ signs; we leave this point to the reader and work modulo 2. We claim:

- $d_\Gamma^2 = d_\Delta^2 = 0$,
- p, i are chain maps.

- p, i are chain homotopy inverses of each other.

The non-obvious part is that ip is chain homotopic to 1; for this we have:

$$1 - ip = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & f & 0 & 1 \end{bmatrix} = \begin{bmatrix} d & f & 0 & 1 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & g \\ 0 & 0 & 0 & d \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d & f & 0 & 1 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & g \\ 0 & 0 & 0 & d \end{bmatrix}$$

Thus we have constructed a complex Δ which is chain homotopy equivalent to Γ (the cone of h). On the other hand, Δ has a two term filtration whose associated graded complex is isomorphic to the direct sum of the cone of f and the cone of g. Thus if the cones of f and g are x-torsion, then so is the cone of h. □

Our next lemma gives an explicit formula representing the composition in the case of $\mathcal{C}(Y)$:

Lemma 4.5. *Let $\varphi_1, \dots, \varphi_n$ be a composable sequence. Writing $\varphi_i = \varphi_{i,s,t}$ as an element in $\mathrm{NN}(Y)$, the 1-simplex:*

$$(\varphi_{n,s,t} \circ \varphi_{n,0,t}^{-1}) \circ \cdots \circ (\varphi_{1,s,t} \circ \varphi_{1,0,t}^{-1}) \circ \varphi_{1,0,t} \tag{42}$$

represents the composition of $\varphi_1, \dots, \varphi_n$.

Proof. It suffices to prove the case $n = 2$, as the rest follows by induction. A two-simplex is the information of:

- $\Phi^{012} : [0, \infty) \to \mathrm{NN}(Y)$,
- $\Phi^{01}, \Phi^{12}, \Phi^{02} \in \mathrm{NN}(Y)$,

satisfying certain interrelations; see Definition 2.3. In particular, we are seeking a map:

$$\Phi^{012} : [0, \infty) \to \mathrm{NN}(Y)$$

such that

- $\Phi^{012}(0) = (s \mapsto \varphi_{2,s,t} \circ \varphi_{2,0,t}^{-1} \circ \varphi_{1,s,t})$,
- $\Phi^{012}(x) = \varphi_2 \#_{\tau(x)} \varphi_1$ for x large enough.

Finding such Φ^{012} is a standard affair of interpolating between timewise composition and concatenation. This can be achieved in a way compatible with the axioms in Definition 2.3. □

4.2.2. **Isomorphisms in the homotopy category.** We characterize those 1-simplices which project to isomorphisms in the homotopy category.

Lemma 4.6. *The following are equivalent conditions on a 1-simplex φ:*

(a) $[\varphi]$ is an isomorphism in the homotopy category;
(b) $\varphi_{s,t}$ satisfies $\varphi_{s,1} = \varphi_{0,1}$ for all $s \in \mathbb{R}$.

Proof. That (b) implies (a) follows from the fact that $\varphi_{1-s,t}$ remains a well-defined 1-simplex, and that the composition:

$$\varphi_{1-s,t} \circ \varphi_{1,t}^{-1} \circ \varphi_{s,t}$$

is homotopic through non-negative paths to the identity morphism based at $\varphi_{0,t}$. The implication that (a) implies (b) is slightly more subtle, but follows from the same reasoning used in §2.1.6 appealing to the non-existence of C^1 small loops of contactomorphisms. □

4.2.3. **Time reparametrization trick.** In this subsection, we will use the explicit formula for the composition (42) to decompose morphisms into simpler pieces. Let us introduce the following special class of 1-simplices, which have certain benefits vis-à-vis the construction of extenders.

Definition 4.7. *A 1-simplex $\sigma_{s,t}$ is said to be of type* (M) *provided that there is a positive isotopy ψ_s and a contact isotopy φ_τ such that:*[43]

$$\sigma_{s,t} = \psi_{s\beta(2t-1)} \circ \varphi_{\beta(2t)} = \begin{cases} \varphi_{\beta(2t)} \text{ for } t \leq 1/2 \\ \psi_{s\beta(2t-1)} \circ \varphi_1 \text{ for } t \geq 1/2 \end{cases}$$

If, in addition, ψ_s is autonomous, we say that $\sigma_{s,t}$ is of type (AM)*. We implicitly assume that endpoints do not lie on the discriminant, in order for $\sigma_{s,t}$ to be considered as a 1-simplex in $\mathcal{C}(Y)$.*

Not every morphism σ can be decomposed into morphisms of type (M) or type (AM), because of the assumption that ψ must be positive. However, if we restrict to those σ which are positive then:

Lemma 4.8. *Any positive morphism σ can be decomposed into a sequence $\sigma_1, \sigma_2, \sigma_3$ where σ_1 and σ_3 are isomorphisms and σ_2 is of type (M).*

Proof. Let us say that two 1-simplices are equivalent $\simeq$ if they differ by right or left multiplication by isomorphisms. Then we claim:

- $\sigma_{s,t} \simeq \sigma_{s,\beta(2t)}$,
- $\sigma_{s,\beta(2t)} \simeq \sigma_{s,\beta(2t-1)}\sigma_{0,\beta(2t-1)}^{-1}\sigma_{0,\beta(2t)}$,
- $\sigma_{s,\beta(2t-1)}\sigma_{0,\beta(2t-1)}^{-1}\sigma_{0,\beta(2t)} \simeq \sigma_{s\beta(2t-1),1}\sigma_{0,1}^{-1}\sigma_{0,\beta(2t)}$,

the crucial observation is that these homotopies are relative the sets $t = 1$, $t = 0$, which is what ensures the equivalence up to right or left muliplcation by isomorphisms (using Lemma 4.6). The final morphism is of type (M), provided we set $\psi_s = \sigma_{s,1} \circ \sigma_{0,1}^{-1}$ and $\varphi_\tau = \sigma_{0,\tau}$ to satisfy Definition 4.7. □

[43]In the following β is a standard cut-off function; one can take, e.g., $\beta(x) = f(x)$ for $x \in [0,1]$ and $\beta'(x) = 0$ for $x \notin [0,1]$, where f is as in Figure 1

4.2.4. **Extender ansatz.** Suppose that $\sigma_{s,t}$ is of type (M) for φ_τ and ψ_s as in Definition 4.7. Then the generator of $\sigma_{s,t}$ on the symplectization SY is:

- $\partial_t \sigma_{s,t} = X'_{s,t} \circ \sigma_{s,t}$ and $\lambda(X'_{s,t}) = H'_{s,t}$;

here the "prime" notation signifies that these are Liouville equivariant[44] for all s, t; we will deform $H'_{s,t}$ in the subsequent discussion to construct an extender $H_{s,t}$ satisfying the axioms in §4.1. We observe:

$$H'_{s,t} = 2\beta'(2t)F_{\beta(2t)} + 2s\beta'(2t-1)S_{s\beta(2t-1)},$$

where S_s generates ψ_s and F_τ generates φ_τ.

Define the *extender ansatz* by the formula:

$$H_{s,t} = 2\beta'(2t)F_{\beta(2t)} + 2s\beta'(2t-1)\gamma(S_{s\beta(2t-1)}) \tag{43}$$

where γ is the convex cut-off function illustrated in Figure 7, which is required to vanish in a neighborhood of 0.

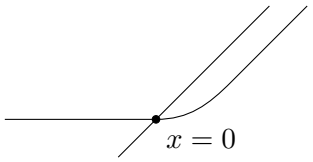


FIGURE 7. Convex cut-off function $\gamma(x)$ used in the extender ansatz equals 0 in a neighborhood of $x \leq 0$ and equals $x - \epsilon$ for $x \geq 1$. Assume that $\gamma''(x) > 0$ if $\gamma'(x) \in (0, 1)$.

Lemma 4.9. *The isotopy $\Psi_{s,t}$ generated by* (43) *satisfies (E1) through (E5), but maybe not axiom (E6) (in other words, we do not show all the orbits of the extender ansatz have the same action).*

Proof. Recall that $K_{s,t}$ is determined from $H_{s,t}$ by the initial value problem:

$$\begin{cases} \partial_t K_{s,t} + \omega(X_{H_{s,t}}, X_{K_{s,t}}) = \partial_s H_{s,t}, \\ K_{s,0} = 0; \end{cases} \tag{44}$$

see, e.g., [Can23, pp. 16]. It follows that $K_{s,t} = 0$ on the negative end where $H_{s,t} = H_{0,t}$. Thus (E1) holds. The two axioms (E2) and (E3) are clear.

We compute:

$$\partial_s H_{s,t} = 2\beta'(2t-1)[\gamma(S_{s\beta(2t-1)}) + s\gamma'(S_{s\beta(2t-1)})D_{s\beta(2t-1)}\beta(2t-1)],$$

where $D_s = \partial_s S_s$. After some manipulation, this yields:

$$\partial_s H_{s,t} = \partial_t(\beta(2t-1)\gamma(S_{s\beta(2t-1)})),$$

[44]Recall that here we mean these objects commute with the Liouville flow on the symplectization SY; Liouville equivariant Hamiltonian vector fields are uniquely determined by the ideal restrictions to contact vector fields of Y.

and thus we can set:

$$K_{s,t} = \beta(2t-1)\gamma(S_{s\beta(2t-1)})$$

to solve the IVP (44), as $\omega(X_{K_{s,t}}, X_{H_{s,t}}) = 0$ (this is clear for $t \leq 1/2$ as $K_{s,t}$ vanishes there, and it holds for $t \geq 1/2$ since then both vector fields are proportional to the vector field X_{S_s}). Then, by inspection it follows that $K_{s,1} = \gamma(S_s)$, i.e., (E4) holds.

The remaining axiom (E5) is concerned with the actions of the orbits of $X_{H_{1,t}}$. If $H_{s,t}$ is given by (43), then we compute:

$$\int_0^{1/2} H_{1,t}(y(t))dt - y^*\lambda = 0,$$

provided that $y(t) = \Psi_{1,t}(y(0))$. This is because $H_{1,t}$ is 1-homogeneous with respect to the Liouville flow, for $t \leq 1/2$. Moreover:

$$\int_{1/2}^1 H_{1,t}(y(t))dt - y^*\lambda = \int_0^1 \gamma(S_s) - S_s\gamma'(S_s)ds < 0. \tag{45}$$

where the equality follows from the change of variables for integrals, and the inequality follows from the fact that $\gamma(S_s) > 0$ must hold somewhere along the orbit $y(t)$, otherwise φ_t would lie on the discriminant; convexity then yields the inequality $\gamma(S_s) - S_s\gamma'(S_s) < 0$.

As a final remark, we note that compactness of the orbit set follows from the assumption that $\varphi_1, \psi_1\varphi_1$ do not lie on the discriminant. This completes the proof. □

4.2.5. **The orbit set in the autonomous case.** Let $H_{s,t}$ be as in (43). The final axiom (E6) concerns a global property of the orbit set of the system generated by $H_{1,t}$, namely, that all orbits have the same action. This property is not automatic, but there is a straightforward criterion in the case the input data was of type (AM), i.e., in the case ψ_σ autonomous.

Lemma 4.10. *Suppose that there is a unique $\sigma_0 \in (0,1)$ such that $\psi_{\sigma_0}\varphi_1$ lies on the discriminant, and let $H_{s,t}$ be the extender ansatz given in* (43) *for data $\psi_\sigma, \varphi_\tau$ of type (AM). Then $H_{s,t}$ satisfies the final axiom (E6).*

Proof. The argument is similar to many arguments in symplectic homology theory and its cousin Rabinowitz–Floer homology, in particular, the arguments used to analyze translated points. The key idea is that S is a positive one-homogeneous Hamiltonian and can be used as a coordinate on the symplectization. We observe:

$$H_{s,t} = 2\beta'(2t)F_{\beta(2t)} + 2s\beta'(2t-1)\gamma(S),$$

which can be solved explicitly:

$$\Psi_{s,t} = \psi_{s\beta(2t-1)\gamma'(S)} \circ \varphi_{\beta(2t)},$$

here $\psi_{s\beta(2t-1)\gamma'(S)}$ is the diffeomorphism *preserving the level sets of* S obtained by flowing by $2s\beta'(2t-1)\gamma'(S)X_S$ over the time interval $t \in [0,1]$. Therefore $\Psi_{1,1}$ has a fixed point at x if and only if:

$$\gamma'(S(x)) = \sigma_0. \tag{46}$$

Since γ is convex, and strictly convex when $\gamma'(S) = \sigma_0$, it holds that $S(x)$ is uniquely determined by σ and (46). As in (45), the action at such a fixed point is equal to $\gamma(S(x)) - S(x)\gamma'(S(x))$. It follows there is a unique action value, i.e., (E6) follows. □

4.2.6. **Completing the proof of Theorem 4.2.** Let us describe our strategy for achieving (E6) in general. Fix a positive 1-simplex σ; the first step is to use Lemma 4.8 to decompose σ into $\sigma_1, \sigma_2, \sigma_3$ where σ_1, σ_3 are isomorphisms in $\mathcal{C}(Y)$ and σ_2 is of type (M). For the purposes of proving Theorem 4.2, we can replace σ by σ_2, and just suppose that σ was (M) from the start.

Let us therefore suppose that σ is type (M) for inputs ψ_s, φ_τ, and without loss of generality, let us assume the length of the Moore path is 1.

We prove the following statement concerning the generic intersections with the discriminant:

Lemma 4.11. *Suppose that φ_1 and $\psi_1\varphi_1$ do not have discriminant points. For a generic perturbation of ψ_s relative its endpoints, we can ensure that:*

$$D = \{(y,s) : \psi_s \circ \varphi_1 \text{ has } y \text{ as a discriminant point}\}$$

is a finite set, and the projection map $(y,s) \in D \mapsto s \in (0,1)$ is injective.

Moreover, we can assume that:

$$\mathrm{im}(\mathrm{d}\psi_s \circ \mathrm{d}\varphi_1 - 1) \text{ is transverse to } T_s \tag{47}$$

at all fixed points of $\psi_s \circ \varphi_1$ in SY, where T_s is the vector field generating the positive isotopy ψ_s (the ideal restriction of T_s is an s-dependent Reeb flow on Y).

The proof is given in §4.3. Let us briefly comment on the conclusions:

- discriminant points, when they occur, are isolated in $Y \times (0,1)$;
- condition (47) says that the subspace $\mathrm{im}(\mathrm{d}\psi_s \circ \mathrm{d}\varphi_1 - 1)$ in TSY has codimension 1 at any lift to SY of a discriminant point in Y.

Because positivity of ψ_s is an open condition, we can assume that ψ_s and φ_1 satisfy the conclusions of Lemma 4.11 by a small perturbation of σ.

The third step is to "chop up" the s-interval so as to isolate the discriminant points; this is illustrated in Figure 8. To be clear, "chopping up" refers to the process of decomposing $[0,1] = [s_0, s_1] \cup [s_1, s_2] \cup \ldots [s_{n-1}, s_n]$ with $0 = s_0 < \cdots < s_n = 1$ so that:

- $\psi_{s_j}\varphi_1$ does not have discriminant points for $j = 0, \ldots, n$.

Then one considers the 1-simplices $\sigma^j_{s,t}$ given by:

$$\sigma_{j,s,t} = \sigma_{s_{j-1}+s(s_j-s_{j-1}),t} = \psi_{(s_{j-1}+s(s_j-s_{j-1}))\beta(2t-1)} \circ \varphi_{\beta(2t)}. \tag{48}$$

It is clear that σ can be decomposed into $\sigma_1, \ldots, \sigma_n$. Observe that:

$$\psi_{(s(s_j-s_{j-1})\beta(2t-1)+s_{j-1}\beta(2t-\tau))} \circ \varphi_{\beta(2t)}.$$

is a homotopy relative the boundary of the square, as $\tau \in [0,1]$, which agrees with (48) when $\tau = 1$, and which is of type (M) when $\tau = 0$ with data:

(c1) $\varphi^j_t = \psi_{s_{j-1}t}\varphi_t$,
(c2) $\psi^j_s = \psi_{s(s_j-s_{j-1})+s_{j-1}} \circ \psi^{-1}_{s_{j-1}}$.

Thus we can replace σ by the type (M) data given by (c1) and (c2). By choosing the partition fine enough, we can suppose that $\sigma_{s,1}$ has at most one transverse intersection with the discriminant.

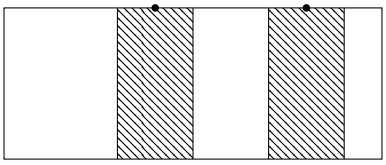


FIGURE 8. Chopping up a 1-simplex into a concatenation so that each piece has either 0 or 1 intersection with the discriminant; the shaded parts have 1 intersection with the discriminant, located at the midpoint.

The idea now is to pick the partition $s_0 < \cdots < s_n$ so that $s_j - s_{j-1}$ becomes very small. Using this trick, we first determine what happens if the case $\sigma_{s,1}$ has no intersections with the discriminant:

Lemma 4.12. *Suppose that $\psi_s\varphi_1$ does not have discriminant points, for any value of $s \in [0,1]$. For a sufficiently fine partition $0 = s_0 < \cdots < s_n = 1$, the extender ansatz from §4.2.4 associated to the data given by (c1) and (c2) satisfies (E6); moreover it has no 1-periodic orbits at all.*

Relatedly, if $t \mapsto \psi_{st}\varphi_t$ has no discriminant orbits in $\kappa \subset \pi_0(\Lambda Y)$, then the extender ansatz for (c1) and (c2) has no 1-periodic orbits in κ for sufficiently fine partition.

Proof. Unpacking the definitions, we need to show that, for $s_j - s_{j-1}$ small enough, the system generated by:

$$H_{1,t} = 2\beta'(2t)F^j_{\beta(2t)} + 2\beta'(2t-1)\gamma(S^j_{\beta(2t-1)}) \tag{49}$$

has no 1-periodic orbits, where:

- F^j_t generates $\psi_{s_{j-1}t}\varphi_t$, and,
- S^j_s generates $\psi_{s(s_j-s_{j-1})+s_{j-1}} \circ \psi^{-1}_{s_{j-1}}$.

In particular, S_s^j is proportional to $s_j - s_{j-1}$ and hence converges in the C^∞ topology to the zero function, as the partition gap tends to zero. The argument then follows from a standard compactness argument: since F_t^j generates $\psi_{s_{j-1}t}\varphi_t$, the system generated by (49) has a time-1 map which is close to $\psi_{s_{j-1}}\varphi_1$; since the latter has no orbits (uniformly in $s_{j-1} \in [0,1]$), and this is an open condition on the generating Hamiltonian, we conclude that (49) defines a trivial extender, provided the partition is chosen small enough.

The same argument applies verbatim to the modified statement when the free homotopy class κ is included. □

Lemma 4.13. *Suppose $\psi_s\varphi_1$ has exactly one discriminant point at $s = s_0$, and the intersection is transverse in the sense of* (47)*. After localizing to a sufficiently small interval around $s = s_0$, the extender ansatz satisfies (E6).*

Proof. We will prove that, after a sufficiently small localization, the extender ansatz has a single 1-periodic orbit. Evidently, this implies there is a single action value.

Let us consider a sequence of localizations to the interval $[s_0 - \epsilon_n, s_0 + \epsilon_n]$ of the extender ansatz:

$$H_{1,t}^n = 2\beta'(2t)F_{\beta(2t)}^n + 2\beta'(2t-1)\gamma(S_{\beta(2t-1)}^n),$$

where F_t^n generates $\psi_{(s_0-\epsilon_n)t}\varphi_t$, and S_s^n generates $\psi_{s2\epsilon_n+s_0-\epsilon_n} \circ \psi_{s_0-\epsilon_n}^{-1}$.

Following the same argument used in the proof of Lemma 4.12, $H_{1,t}^n$ generates a system $\xi_{n,t}$ whose time-1 map $\xi_{n,1}$ is C^∞ close to $\psi_{s_0}\varphi_1$. In particular, if $\xi_{n,1}$ has a fixed point z_n, then z_n must converge to some point on the Liouville flow line $Z_{\mathbb{R}}(y)$ through the unique discriminant point y of $\psi_{s_0}\varphi_1$.

The time-1 map of $\xi_{n,t}$ can be written as a composition of two maps:

- $\psi_{s_0-\epsilon_n}\varphi_1$,
- the isotopy generated by $V_{n,t} = 2\epsilon_n\gamma'(S_t^n)T_{2\epsilon_n t+s_0-\epsilon_n}$.

Since $S_t^n \approx 2\epsilon_n r$, where r is the Hamiltonian generating T_{s_0}, we conjugate with the Liouville flow $Z_{-\log(2\epsilon_n)}$ so that $2\epsilon_n r$ converges to r; then the second part is approximately the flow by $2\epsilon_n\gamma'(r)T_{s_0}$, with error given by terms which are of order ϵ_n^2.

Similarly, up to terms of order ϵ_n^2, the first part can be replaced by $\psi_{s_0}\varphi_1$ followed by the flow of $-\epsilon_n T_{s_0}$ for time 1. Thus, in this analysis to first order, we have that $\xi_{n,1}$ is approximately:

- $\psi_{s_0}\varphi_1$, followed by,
- the time 1 flow of $(2\gamma'(r) - 1)\epsilon_n T_{s_0}$.

There is a unique fixed point located at $\gamma'(r) = 1/2$ on the Liouville flow line $Z_{\mathbb{R}}(y)$. This is non-degenerate, by our transversality assumptions. This non-degeneracy ensures that the analysis up to first order in ϵ_n is stable

under small perturbations of order ϵ_n^2; a more precise argument uses the Banach fixed point theorem to prove the existence of a unique fixed point. Thus $\xi_{n,1}$ has a unique fixed point for n sufficiently large, as desired. □

4.3. **On the generic intersections with the discriminant.** The discriminant may be highly singular, but it has a canonical resolution as a smooth (Fréchet) manifold, as follows. Consider the graph map:

$$\Gamma : \mathrm{Cont}(Y) \times SY \to SY \times SY$$

sending (ϕ, y) to $(y, \phi(y))$; this map is submersive (by the abundance of contact Hamiltonians), and the inverse image of the diagonal yields a Fréchet manifold $\Gamma^{-1}(\Delta)$. The projection (forgetting the SY factor) yields a continuous surjection:

$$\Gamma^{-1}(\Delta)/\mathbb{R} \to \text{the discriminant} \tag{50}$$

which is the aforementioned resolution; here the quotient corresponds to the $\mathbb{R}$-action by the Liouville flow on SY. This suggests the following characterization[45] of when a finite-dimensional family of contactomorphisms should be called transverse to the discriminant:

Definition 4.14. *Let P be a finite dimensional smooth manifold. We say that a smooth map $f : P \to \mathrm{Cont}(Y)$ is* transverse to the discriminant *if:*

$$\Gamma_f : (p, y) \in P \times SY \mapsto \Gamma(f(p), y) \in SY \times SY$$

is transverse to the diagonal Δ. One can also speak of maps which are transverse along some closed subset $A \subset P$.

Proposition 4.15. *Let P be a smooth, finite dimensional, compact manifold, and $f : P \to \mathrm{Cont}(Y)$ a smooth map. Suppose f is transverse to the discriminant along some closed, possibly empty, subset $A \subset P$. Then f is homotopic relative A to a smooth map which is everywhere transverse to the discriminant and arbitrarily C^∞ close to f.*

Proof. This follows from the abundance of contact Hamiltonians and the Sard–Smale theorem. The details are left to the reader. □

By counting dimensions, one can also bound the number of discriminant points which appear at any given moment in generic 1-parameter families of contactomorphisms:

Proposition 4.16. *Suppose that P is a one-dimensional manifold and consider a smooth map $f : P \to \mathrm{Cont}(Y)$ which is transverse to the discriminant. After a generic perturbation, there is at most one discriminant point*[46] *which occurs at any given point p.*

[45] A toy model for discriminant points are critical points of a smooth function with critical value zero.

[46] In fact, if P has dimension d, then there are at most $d-1$ many discriminant points which occur above any given point $p \in P$, for generic f. We will not use this fact.

Proof. The subset $D = \Gamma_f^{-1}(\Delta)/\mathbb{R}$ is a zero dimensional submanifold of the total space $P \times Y$. Thus, for any given $p \in P$, there are a finite number of discriminant points, say $y_1, \ldots, y_N$. One perturbs f in a neighborhood of each point y_i in order to slightly move its basepoint p. The technical reason this is possible is that Sard Smale argument establishes the universal moduli space of all pairs (f, p, y) such that y is a discriminant point of $f(p)$ has a submersive projection $(f, p, y) \mapsto p$, (again by the abundance of contact Hamiltonians). □

We relate this notion of transversality to Lemma 4.11.

Proposition 4.17. *Suppose that $s \in [0,1] \mapsto \psi_s \circ \varphi_1$ is transverse to the discriminant, where ψ_s is positive isotopy. Then, for each s such that $\psi_s \circ \varphi_1$ lies on the discriminant, it holds that* $\mathrm{im}(\mathrm{d}\psi_s \circ \mathrm{d}\varphi_1 - 1)$ *is transverse to T_s in SY, where T_s is the generator of ψ_s.*

Proof. By the above discussion, transversality ensures:

$$(s, y) \in [0,1] \times SY \mapsto (y, \psi_s(\varphi_1(y)))$$

is transverse to the diagonal. Differentiating with respect to y and s at a fixed point yields the subspace of tangent vectors of the form

$$v \oplus (\mathrm{d}\psi_s \circ \mathrm{d}\varphi_1(v) + T_s(y)).$$

If this is transverse to the diagonal, then we can solve:

$$u \oplus u + v \oplus (\mathrm{d}\psi_s \circ \mathrm{d}\varphi_1(v) + T_s(y)) = 0 \oplus w$$

for any w. The only option is $u = -v$, so:

$$-v + \mathrm{d}\psi_s \circ \mathrm{d}\varphi_1(v) + T_s(y) = w$$

can be solved for any w, for some v. This yields the desired result. □

Combining Propositions 4.15, 4.16, and 4.17 yields Lemma 4.11. □

4.4. **The local Floer cohomology of an extender.** In this section we develop the local Floer cohomology of an extender.

4.4.1. **Admissible perturbations.** Let $\psi_{s,t}$ be an extender, as in §4.1. Fix a Liouville equivariant ω-tame almost complex structure J on SY.

An *ϵ-admissible perturbation* for $\psi_{s,t}, J$ is a family $\delta_{s,t}$ of Hamiltonian diffeomorphisms such that:

- $\delta_{s,0} = \delta_{0,t} = id$, for all s, t,
- $\delta_{s,t}$ is compactly supported, uniformly in s, t,
- $\psi_{1,t}\delta_{1,t}$ is a non-degenerate Hamiltonian system, for which all Floer differential cylinders are cut transversally,
- the Hamiltonian generator of $t \mapsto \delta_{s,t}$, and its first derivative with respect to s, are bounded in absolute value by ϵ.

As in the statement of Theorem 4.1, we let $\kappa \subset \pi_0(\Lambda Y)$ be a collection of free homotopy classes of loops.

Lemma 4.18. *If ϵ is small enough, then the Floer differential on:*

$$\mathrm{CF}(\psi_{1,t}\delta_{1,t}, J, \kappa)$$

squares to zero, and the chain homotopy type of the resulting complex is independent of the choice of J or the ϵ-admissible perturbation $\delta_{s,t}$.

Proof. The Floer differential is defined as usual (as a sum of the index 1 Floer differential cylinders, recording the input and output orbits, and the orientation line, as an endomorphism). To prove it squares to zero, we need only prove the required compactness result:

- *any sequence of Floer cylinders u_n remains in a compact subset of the symplectization SY.*

The other necessary a priori estimates on the derivatives of u_n, say, follow from the standard machinery once the above C^0 bound is proved.

To analyze this compactness problem, we introduce the symplectization coordinate r. We assume that $\psi_{1,t}$ is Liouville equivariant, and $\delta_{s,t} = 0$, on the ends $r \leq 0$ and $r \geq r_0$, for some $r_0 > 0$, (positioning the left end at $r \leq 0$ is without any loss of generality, as we can translate by the Liouville flow).

Then, using the mean-value property for energy density as in [RS01, Appendix B], we conclude that the C^0 sizes of $\partial_s u_n$ and $\partial_s u_n - X_t(u_n)$ must be small; here X_t is the Hamiltonian vector field of $\psi_{1,t}\delta_{1,t}$. This is because the energy of Floer cylinders can be made arbitrarily small (by shrinking ϵ).

In particular, if u_n fails to remain bounded, then one can find numbers s_n so that subcylinders $u_n([s_n - 1, s_n + 1] \times \mathbb{R}/\mathbb{Z})$ are mapped into the region $r \leq 0$ (negative end), or the region $r \geq r_0$ (positive end). In either case, we obtain a contradiction (for ϵ small enough), because in these ends the Hamiltonian system $\psi_{1,t}\delta_{1,t} = \psi_{1,t}$ has no orbits, so these subcylinders occupy a minimum quantum of energy $\hbar$ (by a standard argument, e.g., [Sal97] or [BC24, Proposition 2.2.]). The contradiction is ensured by picking ϵ small enough.

A similar argument works for proving a priori C^0-bounds on continuation cylinders relating two choices $J, \delta_{s,t}$ and $J', \delta'_{s,t}$, if both are ϵ-admissible and ϵ is small enough. In this fashion we prove that the resulting chain complexes are chain homotopy equivalent (the quasi-isomorphisms are continuation maps, as in [HS95, Sal97]). □

By virtue of the preceding result, the following is well-defined:

Definition 4.19. *For any extender $\psi_{s,t}$ and any $\kappa \subset \pi_0(\Lambda Y)$, we define $\mathrm{CF}_{\mathrm{loc}}(\psi_{s,t}, \kappa)$ to be the chain homotopy type of $\mathrm{CF}(\psi_{1,t}\delta_{1,t}, J, \kappa)$ for any J and any ϵ-admissible $\delta_{s,t}$, for ϵ small enough.*

4.4.2. **Insertion of an extender into the filling.** Let us present the G-filling W of Y as a compact aspherical domain Ω with contact type boundary, with the positive half of $S\partial W$ attached as the convex end. This presentation specifies a radial coordinate r so that $\partial\Omega = \{r = 1\}$. We fix the geometric set-up as shown in Figure 9.

Pick an extender $\psi_{s,t}$ whose ideal restriction is the 1-simplex $\varphi_{s,t}$, and pick an ϵ-admissible perturbation $\delta_{s,t}$.

Fix regular Borel data $\psi_{\eta,0,t}$ whose ideal restriction is $\varphi_{0,t}$. We can suppose that the t-generator $X_{\eta,0,t}$ of $\psi_{\eta,0,t}$ is Liouville equivariant outside of Ω. We can also suppose (by appropriate choice of the initial radial coordinate) that all orbits of $X_{v,0,t}$ remain entirely in Ω, when v is any critical point on EG.

Introduce $X_{s,t}$ as the t-generator of the perturbed extender $\psi_{s,t}\delta_{s,t}$; we suppose that $X_{s,t} = X_{0,t}$ holds on $r \leq 1$, $X_{s,t} = X_{1,t}$ holds on $r \geq r_0$, and that $\delta_{s,t} = id$ holds for r outside $[1, r_0]$. This $X_{s,t}$ lifts from SY to $S\partial W$, as a family of G-invariant vector fields.

This set-up ensures that the piecewise definition:

$$X_{\eta,s,t} = \begin{cases} X_{\eta,0,t} \text{ inside } \Omega, \\ \mathrm{d}Z_L \circ X_{s,t} \circ Z_{-L} \text{ for } r \geq 1, \end{cases} \tag{51}$$

is smooth and integrates to a path of Borel data whose ideal restriction is $\varphi_{s,t}$ (throughout we always use the same J in our Borel data). In other words, we conjugate the extender $\psi_{s,t}$ and the ϵ-admissible perturbation by Liouville flow. This has the effect of exponentially scaling the actions of the orbits arising from the extender; importantly, orbits with negative action will have a *very* negative action when we increase L.

Claim 4.20. *A generic choice of $X_{\eta,0,t}$ and $\delta_{s,t}$ ensures that* (51) *is a regular path of Borel data.*

Proof. The engine for ensuring regularity is the usual Sard-Smale argument of [FHS95, MS12].

Any continuation cylinder (or Floer differential cylinder) which intersects Ω can be assumed to be regular by the generic choice of $X_{\eta,0,t}$. The other cylinders are contained entirely in the symplectization end; since the action is free on this part, the cylinders project to a cylinder in SY for data determined by $X_{s,t}$. One ensures regularity for the projected cylinders by generic variation of $\delta_{s,t}$ (and the Sard-Smale theorem). The regularity of the original cylinders (before projection) follows since the projection map is a covering map (variations of the cylinder before projection correspond to variations of the cylinder after projection). □

To summarize, we have used the perturbed extender to build an explicit regular path of Borel data whose ideal restriction is the 1-simplex under consideration. The construction depends on the extender $\psi_{s,t}$, the initial Borel datum $X_{\eta,0,t}$, the parameter L, and the choice of perturbation $\delta_{s,t}$.

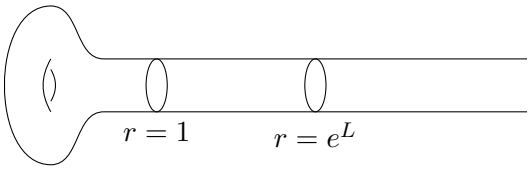


FIGURE 9. Regions in the G-filling W of Y. The length parameter L should be considered as quite large.

4.4.3. **Proof of Theorem 4.1.** In this section we prove that the cone of the morphism $\mathrm{CF}_{\mathrm{eq}}(\varphi_{0,t}) \to \mathrm{CF}_{\mathrm{eq}}(\varphi_{1,t})$ associated to the 1-simplex $\varphi_{s,t}$ arising as the ideal restriction of the extender $\psi_{s,t}$ lies in the local Floer cohomology chain homotopy class $\mathrm{CF}_{\mathrm{loc}}(\psi_{s,t}, \kappa)$ from Definition 4.19, where κ is the collection of W-contractible orbits.

We fix the geometric set-up from §4.4.2, yielding the regular path of Borel data $X_{\eta,s,t}$. We will now analyze the cone of the continuation map associated to this regular path of Borel data.

The equivariant chain complex for the Borel data determined by $X_{\eta,1,t}$ is generated as a $\mathbf{k}[[x]]$-module by summands $\mathfrak{o}(\gamma) \otimes \mathfrak{o}(v)$ where v is a critical point of the pseudogradient on BG lying above the pole $[1 : 0 : \cdots]$, and γ is a contractible orbit of $X_{v,1,t}$, as we have explained already in §2.2.4, §2.3, §2.4.11. These summands come in two types:

(a) summands where γ is contained in Ω, and is an orbit of $X_{0,1,t}$,
(b) summands where γ passes through $r \in [1, r_0]$, and projects to a W-contractible orbit for the vector field $X_{1,t}$ contributing to the local Floer cohomology of the extender $\psi_{s,t}$ and the ϵ-admissible perturbation $\delta_{s,t}$.

On the other hand, the equivariant chain complex for the Borel data determined by $X_{\eta,0,t}$ is generated as a $\mathbf{k}[[x]]$-module by only those summands of type (a). The summands of type (a) for $\mathrm{CF}_{\mathrm{eq}}(X_{\eta,0,t})$ and $\mathrm{CF}_{\mathrm{eq}}(X_{\eta,1,t})$ are literally the same. In other words, there is a decomposition:

$$\mathrm{CF}_{\mathrm{eq}}(X_{\eta,1,t}) = \mathrm{CF}_{\mathrm{eq}}(X_{\eta,0,t}) \oplus E, \tag{52}$$

where E is the $\mathbf{k}[[x]]$-module generated by the summands of type (b).

Claim 4.21. *With respect to the direct sum decomposition* (52)*, it holds that:*

$$d_{eq} = \begin{bmatrix} d_{eq} & \Delta \\ 0 & d_E \end{bmatrix} \qquad \mathfrak{c} = \begin{bmatrix} id \\ 0 \end{bmatrix},$$

where d_{eq} is the differential on $\mathrm{CF}_{\mathrm{eq}}(X_{\eta,1,t})$ and $\mathfrak{c}$ is the continuation map. Moreover, the term d_E only counts cylinders which remain outside of Ω.

In the deduction, we allow the operations of increasing the length parameter L in §4.4.2, and picking the perturbation term $\delta_{s,t}$ smaller, if necessary.

Proof. By (E5), and the operation of increasing L which scales actions exponentially, we may assume that all of the orbits in summand (b) have actions which are much more negative than the actions of the orbits in summand (a). Even taking into account the curvature of Hamiltonian connections arising in the definition of the equivariant differential, we can then preclude the existence of cylinders with input in (a) and output in (b), in the definition of d_{eq}. This proves the first part.

Furthermore, by subsequently picking $\delta_{s,t}$ small enough, and using (E6), we may suppose that all orbits contributing to summand (b) have nearby actions; then, by the same energy estimate used in Lemma 4.18 we conclude that all cylinders contributing to d_E remain in the complement of Ω.

For the conclusion about the continuation map $\mathfrak{c}$, we need to be careful that this "increasing L" operation does not simultaneously increase the curvature of the Hamiltonian connection used to define $\mathfrak{c}$ — indeed, since the curvature is non-zero in the region where the extender is supported, there is a risk that we also scale the curvature exponentially when we conjugate by the Liouville flow. However, this is where we use assumption (E4); since $K_{s,1}$ is non-negative, it follows, from the same arguments used in the cited references in Remark 2.13 (which we used to get the a priori energy estimates on the moduli spaces of §2.2.3), that the curvature of the Hamiltonian connection used to define $\mathfrak{c}$ is non-positive everywhere on the region $r \geq 1$, and so exponential scaling will not introduce arbitrarily positive curvature.

To conclude that the matrix entry in $\mathfrak{c}$ acting on the summands of type (a) is the identity, we use the standard fact (used in all Floer theoretic works concerning continuation maps) that continuation cylinders for s-independent continuation data which contribute to $\mathfrak{c}$ are only the stationary cylinders. □

Corollary 4.22. *Assume the set-up, conclusion, and notation of Claim 4.21. The cone of the continuation map* $\mathfrak{c}$ *is in the chain homotopy class of* (E, d_E).

Proof. This is standard homological algebra; the projection map:

$$C \oplus E \oplus C[1] \to E$$

is the desired chain homotopy equivalence, when the domain is given the cone differential (with sign conventions due to supergradings left to the reader):

$$\begin{bmatrix} d_C & \Delta & id \\ 0 & d_E & 0 \\ 0 & 0 & d_C \end{bmatrix}.$$

The inverse map sends $e \in E$ to $0 \oplus e \oplus \Delta e$. □

To complete the proof of Theorem 4.1, we prove:

Proposition 4.23. *Assume the set-up, conclusion, and notation of Claim 4.21. The complex* (E, d_E) *is in the chain homotopy class of* $\mathrm{CF}_{\mathrm{loc}}(\psi_{s,t}, \kappa)$.

Proof. This should be understood as the well-known "Cartan isomorphism" in equivariant cohomology, which asserts that the G-equivariant cohomology of a space with a free G-action is isomorphic to the ordinary cohomology of the quotient space. Versions of the Cartan isomorphism in equivariant Floer theory have appeared before, see, e.g., [SC25].

For the Borel equivariant cohomology of spaces, the idea is the following, if a space M admits a free action, then:

$$M \times_G EG \to M/G,$$

sending $[(m, \eta)]$ to $[m]$ is a fibration with contractible fibers, and hence is a homotopy equivalence. The task of the present claim is to translate this into Floer theoretic language.

Let us identify E with the direct sum of generators:

$$E = \bigoplus \mathfrak{o}(\zeta) \otimes \mathfrak{o}(\eta) \otimes \mathbf{k} \simeq \mathrm{CF}(X_{1,t}) \otimes \mathrm{CM}(EG) \tag{53}$$

where η is a critical point of the pseudogradient on EG and ζ is a W-contractible orbit in SY. This is a small sleight of hand: this complex is identified with the usual complex by the identification:

$$\mathfrak{o}(\zeta) \otimes \mathfrak{o}(x^k v_{i,g}) = x^k \mathfrak{o}(g\gamma_\zeta) \otimes \mathfrak{o}(v_i),$$

where we pick some distinguished lift γ_ζ to $S\partial W$ of each W-contractible orbit ζ; here $v_{i,g}$ is the lift of v_i as indicated in Figure 3. That this identification is well-defined uses the freeness of the action on $S\partial W$.

Because the data is equivariant in the region $r \geq 1$, the trajectories (π, u) contributing to d_E project to Floer cylinders in SY and come in two types:

- π has index 1 and u has index 0,
- π has index 0 and u has index 1.

It follows that, with respect to the decomposition (53), the differential is a tensor product differential on $\mathrm{CF}(X_{1,t}) \otimes \mathrm{CM}(EG)$.

Let 1 denote the sum of the local minima in $\mathrm{CM}(EG)$ (in the notation of Figure 3, this element 1 is the sum $v_{0,g}$ as g ranges over all elements of G), and consider the chain map:

$$a \in \mathrm{CF}(X_{1,t}) \mapsto a \otimes 1 \in \mathrm{CF}(X_{1,t}) \otimes \mathrm{CM}(EG).$$

The contractibility of EG and standard Morse theoretic arguments imply that this is a chain homotopy equivalence, as desired. □

4.4.4. **Proof of Theorem 1.21.** The first part of the theorem follows from Lemma 4.12 and Theorem 4.1; briefly, one decomposes the 1-simplex into a composition of 1-simplices associated to extenders which have zero local Floer cohomology in the free homotopy class of W-contractible orbits.

The second part follows from a soft argument involving approximating paths by zig-zags of positive and negative paths; if we assume the original path

remains in the complement of the W-contractible discriminant, then we can pick the approximation so that each segment of the zig-zag is also in the complement of the W-contractible discriminant. □

5. **Tying up loose ends**

In this section, we prove various technical lemmas needed to complete the proofs of the main applications stated in §1.2. Throughout we assume that Y admits a G-filling W, with $G = \mathbb{Z}/p\mathbb{Z}$, as in §1.6, and that the G-action has at least one fixed point q_0. At this stage, we have completed the proof of our main structural theorems Theorems 1.14, 1.16, and 1.21.

5.1. **The unit element.** We prove a few lemmas about unit elements. Let us briefly recall the construction; in $\mathrm{CM}_{\mathrm{eq}}(X_\eta)$, for any Morse–Borel datum X_η on W, there is a well-defined cycle 1 by summing up all local minima. Then, for any 1-simplex $\varphi_{s,t}$ in the PSS category $\mathcal{P}(Y)$ starting at id and ending at $\varphi_{1,t} \in \mathcal{C}(Y)$, one has a chain map:

$$\mathrm{PSS} : \mathrm{CM}_{\mathrm{eq}}(X_\eta) \to \mathrm{CF}_{\mathrm{eq}}(\varphi_{1,t}).$$

This gives a class $1(\varphi_{s,t}) \in \mathrm{HF}_{\mathrm{eq}}(\varphi_{1,t})$, and pushing forward by the colimit map $\mathrm{HF}_{\mathrm{eq}}(\varphi_{1,t}) \to \mathrm{SH}_{\mathrm{eq}}(W)$ gives a class $1(\varphi_{s,t}) \in \mathrm{SH}_{\mathrm{eq}}$.

5.1.1. **The unit element is well-defined.** We prove Lemma 1.20 stating that the unit element $1(\varphi_{s,t}) \in \mathrm{SH}_{\mathrm{eq}}(W)$ is well-defined independently of the 1-simplex $\varphi_{s,t}$ chosen.

Proof of Lemma 1.20. The key idea is the following: for any 1-simplex $\varphi_{s,t}$ in $\mathcal{P}(Y)$ starting at id, there is a 2-simplex whose 01 face is $\varphi_{s,t}$ and whose 02 face is a Reeb flow R_{ast} for all sufficiently large speeds $a > 0$.

Indeed, if $R_{ast}\varphi_{s,t}^{-1}\varphi_{1,t}$ is positive (which it certainly is for a large enough), then we can simply use Lemma 4.5.

The result then follows, since the axioms of an ∞-functor:

$$\mathcal{P}(Y) \to \mathrm{N}_{\mathrm{dg}}\mathrm{Ch}(\mathbf{k}[[x]])$$

imply $1(R_{ast}) = \mathfrak{c}(1(\varphi_{s,t}))$, where $\mathfrak{c}$ is the chain map associated to the $[1,2]$ edge. Passing to the colimit, we conclude $1(\varphi_{s,t}) = 1(R_{ast})$, for all sufficiently large a, for any 1-simplex $\varphi_{s,t}$, and the desired result follows. □

5.1.2. **The unit element is not eternal.** We prove the assertion:

Lemma 5.1. *If* $\mathrm{SH}_{\mathrm{eq}}(W) \neq 0$*, the unit* 1 *does not lie in the image of:*

$$\mathrm{HF}_{\mathrm{eq}}(R_{-\epsilon t}) \to \mathrm{SH}_{\mathrm{eq}}(W)$$

for any negative Reeb flow $R_{-\epsilon t}$*.*

Proof. This is related to the results of [Rit13, Rit14, CHK23, Can24b, DUZ25] on non-existence of eternal classes in the symplectic cohomology of Liouville manifolds.

Let us fix an equivariant Morse–Borel datum X on W as in §3.1.6, and consider the chain complexes:

- $\mathrm{CM}_{\mathrm{eq}}(X)$,
- $\mathrm{CM}_{\mathrm{neq}}(X) = \mathrm{CM}_{\mathrm{eq}}(X) \otimes_{\mathbf{k}[[x]]} \mathbf{k}$ (*the non-equivariant quotient*).

Since $\mathrm{HM}_{\mathrm{eq}}(X)$ is a finitely generated $\mathbf{k}[[x]]$-module, it fits into a canonical short exact sequence $0 \to T \to \mathrm{HM}_{\mathrm{eq}}(X) \to F \to 0$, where T is a torsion $\mathbf{k}[[x]]$-module and F is a free $\mathbf{k}[[x]]$-module (by classification of modules over a principal ideal domain).

Claim 5.2. *The projection of the unit element to the free quotient F cannot be written as xa for any element a.*

Proof of Claim 5.2. This follows from a similar argument to the one used in the localization statement §3.2.1; one can reduce the general case to the case $W = \mathrm{pt}$, by "localizing" at a local minimum q_0 on the fixed point submanifold. Here we need to again appeal to the fact that the characteristic of the coefficient field $\mathbf{k}$ matches the order of the prime cyclic group G, to conclude that the projection of the unit to the free quotient cannot be written as xa, since in this case:

$$\mathrm{HM}^*_{\mathrm{eq}}(\mathrm{pt}) \simeq \mathbf{k}[[x]]1 \oplus \mathbf{k}[[x]]\theta, \tag{54}$$

where θ is an element of degree 1 (here we assume that $p \geq 3$, for simplicity, the result is similar and easier in the case $p = 2$). □

Using Theorem 1.18, if 1 lies in the image of $\mathrm{HF}_{\mathrm{eq}}(R_{-\epsilon t}) \to \mathrm{SH}_{\mathrm{eq}}$, then there would be some element $b \in \mathrm{HM}_{\mathrm{eq}}(W)$ so that:

(a) $1 - b$ is mapped (via PSS and continuation) to zero in $\mathrm{SH}_{\mathrm{eq}}$,
(b) b is represented by a cycle lying in the image of the Morse continuation map $\mathrm{CM}_{\mathrm{eq}}(-X) \to \mathrm{CM}_{\mathrm{eq}}(X)$.

Next we use the product structure on the equivariant cohomology $\mathrm{HM}_{\mathrm{eq}}(W)$; we do not need to assume that this product structure respects the $\mathbf{k}[[x]]$-module structure, and so we can use the soft construction in §5.2 (below); in any case, we only need the product structure in Morse theory for the next steps.

Observe that $(1 - b)^{kp} = 1 - b^{kp}$ for any number k (where p is the prime characteristic), and so that (a) and (b) still hold, with b replaced by b^{kp}.

By taking k large enough, we can then assume b^{kp} is mapped to zero in the non-equivariant group $\mathrm{HM}_{\mathrm{neq}}(W)$ (the homology of the complex $\mathrm{CM}_{\mathrm{neq}}(X)$).

This is because every cycle lying in the image of $\mathrm{CM}_{\mathrm{neq}}(-X)$ is nilpotent[47], for degree reasons; this is same argument as [CHK23, §2.3.3].

In particular, it follows by the long-exact sequence:

$$\mathrm{HM}_{\mathrm{eq}}(W) \to \mathrm{HM}_{\mathrm{eq}}(W) \to \mathrm{HM}_{\mathrm{neq}}(W),$$

that $b^{kp} = xa$ for some $a \in \mathrm{HM}_{\mathrm{eq}}(W)$. Thus $1 - xa$ is mapped to zero in $\mathrm{SH}_{\mathrm{eq}}(W)$, via PSS and continuation.

The final step is to pass to the free quotient using:

Lemma 5.3. *The free quotient of* $\mathrm{HM}_{\mathrm{eq}}(W) \simeq \mathrm{HF}_{\mathrm{eq}}(R_{\epsilon t})$ *is mapped injectively into the free quotient of* $\mathrm{SH}_{\mathrm{eq}}(W)$ *(the quotient by the torsion submodule).*

Proof of Lemma 5.3. Suppose that $\zeta \in \mathrm{HF}_{\mathrm{eq}}(R_{\epsilon t})$ is not torsion; consider the long exact sequence of the cone:

$$\cdots \to \mathrm{Cone} \to \mathrm{HF}_{\mathrm{eq}}(R_{\epsilon t}) \to \mathrm{HF}_{\mathrm{eq}}(R_{st}) \to \mathrm{Cone} \to \mathrm{HF}_{\mathrm{eq}}(R_{\epsilon t}) \to \dots.$$

Suppose that ζ becomes torsion after continuation to s. Then $x^k\zeta$ is sent to zero for some power k large enough. Thus $x^k\zeta$ lies in the image of the map from Cone to $\mathrm{HF}_{\mathrm{eq}}(R_{\epsilon t})$. But the $x^{k+\ell}\zeta = 0$ for some further power ℓ, because all cones are torsion (by Theorem 1.14). This contradicts the hypothesis that ζ is not torsion, and proves the result. □

Then, combining 5.3 and 5.2, we conclude $1 - xa$ must inject into the free quotient of $\mathrm{SH}_{\mathrm{eq}}(W)$, contradicting the earlier deduction that $1 - xa$ was sent to zero. This completes the proof of the lemma. □

5.2. **Pair of pants product.** In this section, we describe the construction of the pair-of-pants product on equivariant Floer cohomology, which is required to establish the sub-additivity property (R5) in §5.3.5. Due to the technical nature of this construction, the similarity with the main results of [Can24b], and the existence of equivariant product structures in [GMP23], we do not give a detailed construction.

The structural theorem we have is:

Theorem 5.4. *Assume the setup and conclusion of Theorem 1.14. Then there exists a natural transformation* Π *between the two functors:*

- $\varphi_{0,t}, \varphi_{1,t} \in \mathrm{h}\mathcal{C} \times \mathrm{h}\mathcal{C} \mapsto \mathrm{HF}_{\mathrm{eq}}(\varphi_{0,t}) \otimes \mathrm{HF}_{\mathrm{eq}}(\varphi_{1,t})$,
- *constant functor* $\mathrm{h}\mathcal{C} \times \mathrm{h}\mathcal{C} \to \mathrm{SH}_{\mathrm{eq}}$,

and $\Pi_{\varphi_{0,t},\varphi_{1,t}}$ *factors through the natural map* $\mathrm{HF}_{\mathrm{eq}}(\varphi_{0,t}\varphi_{1,t}) \to \mathrm{SH}_{\mathrm{eq}}$.

Furthermore, the induced product:

$$\Pi : \mathrm{SH}_{\mathrm{eq}} \otimes \mathrm{SH}_{\mathrm{eq}} \to \mathrm{SH}_{\mathrm{eq}}$$

[47]It is important here that ω vanishes on holomorphic spheres; see [Can24b, §1.7]. Otherwise quantum corrections can ruin this argument.

is unital with 1 *acting as the unit element.*

The generalization from "non-equivariant" to "equivariant" is akin to the generalization of "non-family Floer cohomology" to "family Floer cohomology" in the sense of [Hut08]. In this context, we refer the reader to the discussion in [BC25, §3.6.3] and [GMP23] which construct products structure on versions of family Floer cohomology.

Let $\Sigma = \mathbb{C} \setminus \{0, 1\}$ be the pair-of-pants surface equipped with two positive cylindrical inputs C_0, C_1 and one negative output C_∞. For a triple of contact isotopies $\varphi_{0,t}, \varphi_{1,t}, \varphi_{\infty,t}$ such that $\varphi_{0,t} \circ \varphi_{1,t} \leq \varphi_{\infty,t}$, we will consider a suitable space of Hamiltonian connections on $SY \times \Sigma$ whose ideal restrictions match the respective isotopies at the cylindrical ends. These Hamiltonian connections are constructed exactly as in [Can24b].

These lift to G-invariant Hamiltonian connections on $S\partial W \times \Sigma$. The subtle part is how to extend these to the compact part of W in a way which is compatible with Borel data $\varphi_{i,\eta,t}$ extending $\varphi_{i,t}$.

One considers families of Hamiltonian connections $\mathfrak{H}_\eta$ on $\Sigma \times W$, parametrized by points $\eta \in EG$, which agree with the connections determined by Borel data $\varphi_{i,\eta,t}$ in the ends, and whose ideal restriction are the G-invariant connections described above. Given a function $f : \Sigma \mapsto EG$, it makes sense to speak of the Hamiltonian connection obtained by setting $\eta = f(z)$ (in a similar manner to how we set $\eta = \pi(s)$ when defining the equivariant operations on cylinders).

Just as the equivariant operations in §2.2.4, etc, were defined using flow lines of a pseudogradient on BG, the product we are describing will be defined in terms of *Morse flow trees* on BG. There are various approaches to this problem. For our purposes, we suppose there is a space $\mathcal{T}$ of trees $\tau : T \to BG$ and which agree with flow lines of our pseudogradients on the legs of the flow tree. Let us denote by $\bar{\tau}$ the lift of a flow tree to EG. Here T is the underlying space of a flow tree (i.e., three copies of $[0, \infty)$ connected at the vertex).

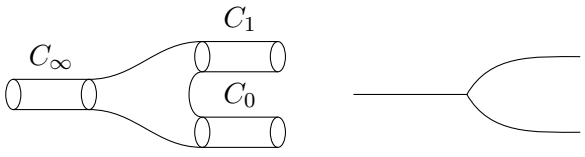


FIGURE 10. Pair of pants surface and the flow tree T.

The moduli space used to define the product consists of rigid pairs $(\bar{\tau}, u)$, where $\tau \in \mathcal{T}$, and $u : \Sigma \to W$ is a solution to the Floer equation associated to the Hamiltonian connection $\mathfrak{H}_\eta$ obtained by setting $\eta = \bar{\tau}(p(z))$ where $p : \Sigma \to T$ is an appropriate map sending the pair of pants surface onto the trivalent graph. This moduli space carries a G-action, and we count the

rigid G-orbits. The asymptotics are interpreted using the same language of "distinguished lifts" common to all of our equivariant operations.

To make a long story short, this induces a chain map which descends to a product on homology:

$$* : \mathrm{HF}_{\mathrm{eq}}(\varphi_{0,t}) \otimes \mathrm{HF}_{\mathrm{eq}}(\varphi_{1,t}) \to \mathrm{HF}_{\mathrm{eq}}(\varphi_{\infty,t}).$$

This a priori depends on the choice of Hamiltonian connections $\mathfrak{H}_\eta$ used (unlike the setting of Borel data, it is not clear whether the space of Hamiltonian connections on SY with non-positive curvature behaves as a contractible space). However, the same arguments used in [Can24b] show that the map obtained by post-composing $*$ with the colimit map yield a well-defined product $\Pi_{\varphi_{0,t},\varphi_{1,t}}$ as in Theorem 5.4.

By standard arguments, this operation satisfies two properties:

(1) The product commutes with the continuation maps.
(2) The element $1 \in \mathrm{SH}_{\mathrm{eq}}(W)$ from in §5.1 is the unit for this product.

These ensure that if the colimit maps for $\varphi_{0,t}, \varphi_{1,t}$ hit the unit, then the colimit map for $\varphi_{\infty,t}$ also hits the unit. This, as in [Can24b], is the mechanism used to verify the sub-additivity of the spectral invariants (R5).

Remark 5.5. What we *do not* verify is that the resulting product structure respect the $\mathbf{k}[[x]]$-module structure. This seems to be a subtle point. We do believe the product can be made to be $\mathbf{k}[[x]]$-linear on the level of homology groups. Whether this can be done on chain level is left as question for future research. If one proves the product structure is $\mathbf{k}[[x]]$-linear, then the invariants of type μ will be super-additive. We emphasize that this "failure of $\mathbf{k}[[x]]$ linearity" does not affect any of the stated results in this paper (since we do not claim that μ is super-additive).

5.3. **The axioms for the spectral invariants.** We establish the axioms (R1) through (R5) for the spectral invariants c_R defined in (8) in §1.6.

5.3.1. **Spectrality.** Property (R1) follows from Theorem 1.21 and the fact the unit is not eternal §5.1.2. Indeed, the unit not being eternal implies $c_R(\varphi_t) \in \mathbb{R}$. Having established this finitness, if s is not in the spectrum, then s cannot be the infimal value for which $\mathrm{HF}(\varphi_t^{-1} \circ R_{st}) \to \mathrm{SH}_{\mathrm{eq}}(W)$ hits the unit element, since the continuation maps for slightly lower values of s are isomorphisms by Theorem 1.21.

Moreover, since the statement of Theorem 1.21 includes the refinement by the W-contractible discriminant, we conclude that

5.3.2. **Monotonicity.** Property (R2) follows from the fact that, if $\varphi_{0,t} \leq \varphi_{1,t}$, then there is a 1-simplex $R_{st} \circ \varphi_{1,t}^{-1} \to R_{st} \circ \varphi_{0,t}^{-1}$, for any value of s.

5.3.3. **Continuity from above.** Property (R3) follows by definition when computing $c_R(\varphi_t)$ when φ_t lies on the discriminant. Otherwise it follows from Theorem 1.21.

5.3.4. **Normalization.** Property (R4) follows from Lemma 5.1 in §5.1.2.

5.3.5. **Sub-additivity.** Property (R5) follows from the existence of the pair of pants product in §5.2; once the product structure is set-up, the argument is exactly the same as one used to prove [Can24b, Theorem 4] and [DUZ25, Theorem 2.11].

5.4. **The axioms for the integer-valued measurements.** In this section, we establish the axioms (G1) through (G5) for the measurement μ in §1.6. We also prove the implication (10) concerning the vanishing of $\mathrm{SH}_{\mathrm{neq}}(W)$, and Proposition 1.12 relating the values of μ on linear symplectic isotopies with the Conley-Zehnder index.

5.4.1. **On the non-equivariant symplectic cohomology.** In this section we prove implication (10). Observe that for a Borel data $(\psi_{\eta,t}, J)$ it holds that:

$$\mathrm{CF}_{\mathrm{neq}}(\psi_{\eta,t}, J) = \begin{cases} \mathrm{CF}(\psi_{v_{0,e}}) \otimes \mathfrak{o}(v_0) \oplus \mathrm{CF}(\psi_{v_{1,e}}) \otimes \mathfrak{o}(v_1) & \text{if } p \geq 3, \\ \mathrm{CF}(\psi_{v_{0,+}}) & \text{if } p = 2, \end{cases} \tag{55}$$

where the differential only counts the flow lines which lie above the pole $[1 : 0 : \cdots]$ in projective space (either $\mathbb{C}P^\infty$ or $\mathbb{R}P^\infty$, depending on $p \geq 3$). See §2.4.11 and §3.1.3 for details on the chain complex.

It follows fairly tautologically that $\mathrm{SH}_{\mathrm{neq}}(W) = \mathrm{SH}(W; \mathbf{k})$ in the case $p = 2$, so we henceforth assume $p \geq 3$.

Recall $\mathrm{SH}_{\mathrm{neq}}(W)$ is the colimit of $\mathrm{HF}_{\mathrm{neq}}(\psi_{\eta,t}, J)$, and so we need to prove this colimit vanishes under the assumption that the "ordinary" symplectic cohomology $\mathrm{SH}(W; \mathbf{k})$ vanishes.

With respect to the direct sum decomposition in (55) we can write the differential on $\mathrm{CF}_{\mathrm{neq}}(\psi_{\eta,t}, J)$ as a lower triangular matrix, where the diagonal entries are the ordinary Floer differentials.

Now, for any element $e \in \mathrm{SH}_{\mathrm{neq}}(W)$, we can find some $\psi_{\eta,t}$ so that e lies in the image of the continuation map from $\mathrm{HF}_{\mathrm{neq}}(\psi_{\eta,t}, J)$. This element e then splits into a summand $e_0 + e_1$. Since the element e_1 is a cycle, and the ordinary symplectic cohomology vanishes, we may assume (by replacing $\psi_{\eta,t}$ by something closer to the colimit) that e_0 is an exact cycle, say $e_0 = df_0$. Then e is cohomologous to $e_1' = e_1 + \Delta f_0$, where Δ is the off-diagonal term in the differential. But now e_1' is a cycle. By passing further into the colimit *and using the fact that the continuation maps are also lower triangular*, we may suppose e_1' is an exact cycle. But thus the entire cycle e is exact (up to passing closer to the colimit, in the sense that we apply

some continuation map). Since e was an arbitrary element of the colimit, we conclude $\mathrm{SH}_{\mathrm{neq}}(W) = 0$, as desired.

5.4.2. **Integer valuedness.** Since $\mu(\varphi_t)$ is defined as a supremum of the set of integers d for which $x^{-d}1$ lies in the image of the colimit map, to prove $\mu(\varphi_t)$ is finite, it is sufficient (to obtain the integer valuedness of μ) to prove that this set of integers is non-empty and bounded from above.

That the set of integers is bounded from above follows from the structure theorem for finitely generated modules over the ring $\mathbf{k}[[x]]$. We argue by contradiction, and suppose $\mathrm{HF}_{\mathrm{eq}}(\varphi_t) \to \mathrm{SH}_{\mathrm{eq}}(W)$ can hit $x^{-d}1$ for arbitrarily large values of d. Since x acts invertibly on $\mathrm{SH}_{\mathrm{eq}}(W)$, the x-torsion part of $\mathrm{HF}_{\mathrm{eq}}(\varphi_t)$ is mapped to zero under the colimit map. On the other hand, since the cones of continuation maps are x-torsion, the free part of $\mathrm{HF}_{\mathrm{eq}}(\varphi_t)$ is mapped injectively into $\mathrm{SH}_{\mathrm{eq}}(\varphi_t)$. By our assumption, there exists a_0, a_d in the free part such that:

- $a_0 \mapsto 1$
- $a_d \mapsto x^{-d}1$.

But then $a_0 = x^d a_d$ by the aforementioned injectivity. In particular, a_0 can be divided by x^d for arbitrarily large d. In a finitely generated $\mathbf{k}[[x]]$-module, this can only happen if $a_0 = 0$, which implies $1 \in \mathrm{SH}_{\mathrm{eq}}(W)$ vanishes, contradicting Lemma 5.1 that the unit was not eternal.

To prove the set of integers is non-empty (i.e., μ is not $-\infty$), we need to prove that *some* power of the unit is hit. However, since the cones of continuation maps are torsion, the map $\mathrm{HF}_{\mathrm{eq}}(R_{-at}) \to \mathrm{HF}_{\mathrm{eq}}(R_{\epsilon t})$ hits $x^k 1$ for large enough k, for any $a > 0$. Thus $\mu(R_{-at})$ is at least $-k$. By monotonicity (see §5.4.3) it follows that $\mu(\varphi_t)$ is at least $-k$ if φ_t is greater than R_{-at}. Since $a > 0$ was arbitrary, we conclude the desired result.

5.4.3. **Monotonicity.** Property (G1) follows from the fact that, if $\varphi_{0,t} \leq \varphi_{1,t}$, then image of the map $\mathrm{HF}_{\mathrm{eq}}(\varphi_{0,t}) \to \mathrm{SH}_{\mathrm{eq}}(W)$ is contained in the image of the map $\mathrm{HF}_{\mathrm{eq}}(\varphi_{1,t}) \to \mathrm{SH}_{\mathrm{eq}}(W)$.

In particular, if $\mathrm{HF}_{\mathrm{eq}}(\varphi_{0,t}) \to \mathrm{SH}_{\mathrm{eq}}(W)$ hits $x^{-d}1$, then the same holds for $\varphi_{1,t}$; thus $\mu(\varphi_{1,t}) \geq \mu(\varphi_{0,t})$.

5.4.4. **Continuity from above.** For (G2) we use the same argument as 5.3.3.

5.4.5. **Normalization.** (G3) follows from Lemma 5.1 and §5.4.2.

5.4.6. **Non-triviality.** Every element of the colimit, including $x^{-d}1$, lies in the image of $\mathrm{HF}_{\mathrm{eq}}(\varphi_t) \to \mathrm{SH}_{\mathrm{eq}}(W)$ for some zero-simplex φ_t. Thus (G4) holds, i.e., μ attains arbitrarily large values.

5.4.7. **Discriminant.** Property (G5) follows from Theorem 1.21. Moreover, we have the refinement from Theorem 1.11 where the discriminant is replaced by the W-contractible discriminant (Definition 1.9), since the statement of Theorem 1.21 includes this refinement.

5.4.8. **Agreement with the Conley-Zehnder index.** In this section we prove Proposition 1.12 that $\mu(\varphi_t) = \mathrm{CZ}(\varphi_t) - n$ whenever φ_t is a linear symplectic isotopy whose time-1 map does not have 1 as an eigenvalue. In this setting, we focus only on the case $G = \mathbf{k} = \mathbb{Z}/2\mathbb{Z}$.

The result follows easily from the fact that φ_t admits a canonical extension to $\mathbb{C}^n$ as equivariant Borel data with a single generator $\gamma(\varphi_t)$ (the constant orbit located at the origin). The underlying chain complex is:

$$\mathbf{k}[[x]]\gamma(\varphi_t),$$

and all differentials vanish.

Now consider s large enough that the φ_t admits a continuation to R_{st}, and let $\mathfrak{c}$ be the continuation map. Similarly let $\mathfrak{c}$ denote the continuation map from $R_{\epsilon t}$ to R_{st} for $\epsilon \in (0,1)$. It follows that:

- $\mathfrak{c}(\gamma(\varphi_t)) = x^{k(\varphi_t)}\gamma(R_{st})$,
- $\mathfrak{c}(\gamma(R_{\epsilon t})) = x^{k(R_{\epsilon t})}\gamma(R_{st}) = 1$.

Note that we exclude the case that $\mathfrak{c} = 0$ by the structural theorems we have proved above (e.g., otherwise μ would not be a finite integer).

The integers $k(\varphi_t)$ and $k(R_{\epsilon t})$ can be computed using the Fredholm index formula for Cauchy-Riemann operators, and we have:

- $\mathrm{CZ}(\varphi_t) - \mathrm{CZ}(R_{st}) + k(\varphi_t) = 0$,
- $\mathrm{CZ}(R_{\epsilon t}) - \mathrm{CZ}(R_{st}) + k(R_{\epsilon t}) = 0$.

Subtracting, we conclude:

$$\mathrm{CZ}(\varphi_t) - n = k(R_{\epsilon t}) - k(\varphi_t).$$

Because $x^{k(\varphi_t)}\gamma(R_{st}) = x^{k(\varphi_t)-k(R_{\epsilon t})}1$, we conclude $\mu(\varphi_t) = k(R_{\epsilon t}) - k(\varphi_t)$, and the desired result follows. □

5.5. **Acknowledgements.** First and foremost, the authors wish to acknowledge and thank Joseph Helfer for a crucial suggestion leading to our Definition 2.3. Additionally, the authors wish to thank Igor Uljarević for invaluable discussions about contact Floer homology. The first named author wishes to thank Laurent Côté, and Julio Sampietro Christ for enlightening discussions surrounding equivariant cohomology. The third named author wishes to thank Fabio Gironella and Zhengyi Zhou for interesting discussions surrounding the paper [GZ21]. The authors also thank the organizers of the Simons Center program: *Contact geometry, general relativity, and thermodynamics*.

The first named author was partially supported by funding from the Fondation Courtois, the ANR grant CoSy, and the Université de Montréal DMS.

The second named author was partially supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship, Award No. DMS-2503414.

The third named author is partially supported by National Key R&D Program of China No. 2023YFA1010500, NSFC No. 12301081, and NSFC No. 12361141812.

References

[AA26] S. Allais and P-A. Arlove, *Spectral selectors and contact orderability*, Comm. Cont. Math. **28** (2026), no. 6, 1–40, doi:10.1142/S0219199725500713.

[AAS24] S. Allais, P-A. Arlove, and S. Sandon, *Spectral selectors on lens spaces and applications to the geometry of the group of contactomorphisms*, 2024.

[AB84] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), no. 1, 1–28, doi:10.1016/0040-9383(84)90021-1. MR 721448

[Abo15] Mohammed Abouzaid, *Symplectic cohomology and Viterbo's theorem*, Free loop spaces in geometry and topology, IRMA Lect. Math. Theor. Phys., vol. 24, Eur. Math. Soc., Zürich, 2015, pp. 271–485. MR 3444367

[AF12] P. Albers and U. Frauenfelder, *A variational approach to Givental's nonlinear Maslov index*, Geom. Funct. Anal. **22** (2012), 1033–1050.

[AFM15] P. Albers, U. Fuchs, and W. J. Merry, *Orderability and the weinstein conjecture*, Compositio Math. **151** (2015), 2251–2272.

[AFM17] ———, *Positive loops and L^∞-contact systolic inequalities*, Sel. Math. New Ser. **23** (2017), 2491–2521.

[AK17] P. Albers and J. Kang, *Vanishing of Rabinowitz Floer homology on negative line bundles*, Math. Z. **285** (2017), 493–517.

[AK23] ———, *Rabinowitz Floer homology of negative line bundles and Floer Gysin sequence*, Adv. Math. **431** (2023), 1–130.

[Alu09] P. Aluffi, *Algebra: Chapter 0*, Graduate Studies in Mathematics, vol. 104, AMS, 2009, doi:10.1090/gsm/104.

[AM13] P. Albers and W. J. Merry, *Translated points and Rabinowitz Floer homology*, Journal of Fixed Point Theory and Applications **13** (2013), 201–214.

[AM18] ———, *Orderability, contact non-squeezing, and Rabinowitz Floer homology*, J. Symp. Geom. **16** (2018), no. 6, 1481–1547.

[AS10] M. Abouzaid and P. Seidel, *An open string analogue of Viterbo functoriality*, Geom. Topol. **14** (2010), 627–718, doi:10.2140/gt.2010.14.627.

[ASZ16] P. Albers, E. Shelukhin, and F. Zapolsky, *Spectral invariants for contactomorphisms of prequantization bundles and applications*, `https://youtu.be/DWel-3BOQrl`, 2016.

[BC24] F. Brocić and D. Cant, *Bordism classes of loops and Floer's equation in cotangent bundles*, J. Fixed Point Theory Appl. **26** (2024), 1–29, doi:10.1007/s11784-024-01114-x.

[BC25] ———, *Parametric gromov width for liouville domains*, arXiv:2504.15207, 04 2025, doi:10.48550/arXiv.2504.15207.

[BEH+03] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, *Compactness results in Symplectic Field Theory*, Geometry and Topology **7** (2003), 799–888.

[BKK24] J. Bae, J. Kang, and S. Kim, *Rabinowitz Floer homology for prequantization bundles and Floer Gysin sequence*, Math. Ann. **390** (2024), 5299–5381, doi:10.1007/s00208-024-02878-w.

[Bor60] Armand Borel, *Seminar on transformation groups*, Annals of Mathematical Studies, no. 46, Princeton University Press, 1960, Contributions by G. Bredon, E. E. Floyd, D. Montgomery, and R. Palais.

[BZ15] M. S. Borman and F. Zapolsky, *Quasimorphisms on contactomorphism groups and contact rigidity*, Geom. Topol. **19** (2015), 365–411.

[Can22] D. Cant, *A dimension formula for relative symplectic field theory*, Stanford University PhD Thesis. Available at: `https://dylancant.ca/thesis.pdf`, 7 2022.

[Can23] ———, *Shelukhin's Hofer distance and a symplectic cohomology barcode for contactomorphisms*, `https://dylancant.ca/sh-barcode.pdf`, 2023.

[Can24a] ———, *Contactomorphisms of the sphere without translated points*, J. Symplectic Geom. **22** (2024), no. 1, 1–10.

[Can24b] ———, *Remarks on eternal classes in symplectic cohomology*, arXiv:2410.03914, October 2024, Accepted for publication in J. Fixed Point Theory Appl.

[Caz24] G. Cazassus, *Equivariant Lagrangian Floer homology via cotangent bundles of EG_N*, J. Topol. **17** (2024), no. 1, 1–61, doi:10.1112/topo.12328.

[CF09] K. Cieliebak and U. Frauenfelder, *A Floer homology for exact contact embeddings*, Pacific J. Math. **239** (2009), no. 2, 251–316.

[CFHW96] K. Cieliebak, A. Floer, H. Hofer, and K. Wysocki, *Applications of symplectic homology II: Stability of the action spectrum*, Math. Z. **223** (1996), 27–45.

[CFO10] K. Cieliebak, U. Frauenfelder, and A. Oancea, *Rabinowitz floer homology and symplectic homology*, Ann. Scient. Éc. Norm. Sup. **4** (2010), no. 43, 957–1015.

[Che96] Yu. V. Chekanov, *Critical points of quasi-functions and generating families of Legendrian manifolds*, Functional Analysis and its Applications **30** (1996), no. 2, 118–128.

[CHK23] D. Cant, J. Hedicke, and E. Kilgore, *Extensible positive loops and vanishing of symplectic cohomology*, arXiv:2311.18267, December 2023, Accepted for publication in Alg. Geom. Topol.

[CN16] V. Chernov and S. Nemirovski, *Universal orderability of Legendrian isotopy classes*, J. Symplectic Geom. **14** (2016), no. 1, 149–170.

[CS15] V. Colin and S. Sandon, *The discriminant and oscillation lengths for contact and legendrian isotopies*, J. Eur. Math. Soc. **17** (2015), 1657–1685.

[CS16] R. Casals and O. Spáčil, *Chern–Weil theory and the group of strict contactomorphisms*, J. Topol. Anal. **8** (2016), no. 1, 59–87, doi:10.1142/S1793525316500035.

[DUZ23] D Djordjević, I Uljarević, and J Zhang, *Quantitative characterization in contact Hamiltonian dynamics - I*, arXiv:2309.00527, 2023, pp. 1–48.

[DUZ25] Danijel Djordjević, Igor Uljarević, and Jun Zhang, *Quantitative contact hamiltonian dynamics*, 2025.

[EKP06] Y. Eliashberg, S. S. Kim, and L. Polterovich, *Geometry of contact transformations and domains: orderability versus squeezing*, Geom. Topol. **10** (2006), no. 3, 1635–1747, doi:10.2140/gt.2006.10.1635.

[EP00] Y. Eliashberg and L. Polterovich, *Partially ordered groups and geometry of contact transformations*, Geom. Funct. Anal. **10** (2000), 1448–1476.

[FH93] A. Floer and H. Hofer, *Coherent orientations for periodic orbit problems in symplectic geometry*, Math. Z. **212** (1993), no. 1, 13–38, doi:10.1007/BF02571639.

[FHS95] A. Floer, H. Hofer, and D. Salamon, *Transversality in elliptic Morse theory for the symplectic action*, Duke Math. J. **80** (1995), no. 1, 251–292, doi:10.1215/S0012-7094-95-08010-7.

[Flo89] A. Floer, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989), no. 4, 575–611.

[FPR18] M. Fraser, L. Polterovich, and D. Rosen, *On Sandon-type metrics for contactomorphism groups*, Ann. Math. Québec **42** (2018), 191–214.

[FS07] U. Frauenfelder and F. Schlenk, *Hamiltonian dynamics on convex symplectic manifolds*, Isr. J. Math. **159** (2007), 1–56.

[GG10] V. L. Ginzburg and B. Z. Gürel, *Local Floer homology and the action gap*, J. Symplectic Geom. **8** (2010), no. 3, 323–357.

[Giv90] A.B. Givental, *Nonlinear generalization of the Maslov index*, Theory of singularities and its applications, Adv. Sov. Math., vol. 1, AMS, 1990, pp. 71–103.

[Giv91] ———, *The nonlinear Maslov index*, Geometry of low-dimensional manifolds, Lecture Note Series, London Math. Soc., 1991, pp. 35–43.

[GKPS21] Gustavo Granja, Yael Karshon, Milena Pabiniak, and Sheila Sandon, *Givental's non-linear Maslov index on Lens spaces*, Int. Math. Res. Not. **2021** (2021), no. 23, 18225–18299, doi:10.1093/imrn/rnz350.

[GMP23] E. Gonzalez, C.Y. Mak, and D. Pomerleano, *Coulomb branch algebras via symplectic cohomology*, arXiv:2305.04387, 5 2023.

[GPS20] S. Ganatra, J. Pardon, and V. Shende, *Covariantls functorial wrapped Floer theory on Liouville sectors*, Pub. Math. de l'IHÉS **131** (2020), 73–200, doi:10.1007/s10240-019-00112-x.

[GZ21] F. Gironella and Z. Zhou, *Exact orbifold fillings of contact manifolds*, arXiv:2108.12247, 2021.

[Hed24] J. Hedicke, *Lorentzian distance functions in contact geometry*, J. Topol. Anal. **16** (2024), no. 2, 205–225, doi:10.1142/S179352532250008X.

[HLS20] K. Hendricks, R. Lipshitz, and S. Sarkar, *A simplicial construction of G-equivariant Floer homology*, Proc. London Math. Soc. **121** (2020), 1798–1866, doi:10.1112/plms.12385.

[HS95] H. Hofer and D. Salamon, *Floer homology and Novikov rings*, The Floer Memorial Volume, Progress in Mathematics, vol. 133, Birkhäuser, Basel, 1995, pp. 483–524, doi:10.1007/978-3-0348-9217-9.

[HS24] J. Hedicke and E. Shelukhin, *Non-orderability and the contact Hofer norm*, arXiv:2411.19887, 2024, Accepted for publication in Duke Math. J.

[Hut08] M. Hutchings, *Floer homology of families I*, Alg. Geom. Topol. **8** (2008), 435–492, doi:10.2140/agt.2008.8.435.

[KvK16] M. Kwon and O. van Koert, *Brieskorn manifolds in contact topology*, Bull. Lond. Math. Soc. **48** (2016), no. 11, 173–241.

[Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009, doi:10.1515/9781400830558.

[Lur17] ———, *Higher algebra*, `https://www.math.harvard.edu/~lurie/papers/HA.pdf`, 9 2017.

[Lur26] ———, *Kerodon*, `https://kerodon.net`, 2026.

[McD91] D. McDuff, *Symplectic manifolds with contact type boundaries*, Invent. math. **103** (1991), 651–671.

[Mcl12] M. Mclean, *Local Floer homology and infinitely many simple Reeb orbits*, Alg. Geom. Topol. **12** (2012), 1901–1923, doi:10.2140/agt.2012.12.1901.

[Mil68] J. Milnor, *Singular points of complex hypersurfaces*, Princeton University Press and the University of Tokyo Press, 1968.

[MO70] J. Milnor and P. Orlik, *Isolated singularities defined by weighted homogeneous polynomials*, Topology **9** (1970), 385–393.

[MS12] D. McDuff and D. Salamon, *j-holomorphic curves and symplectic topology*, 2nd ed., American Mathematical Society, Colloquium Publications, 2012.

[Nak23] L. Nakamura, *a new metric on the contactomorphism group of orderable contact manifolds*, arXiv:2307:10905, 2023, p. 25.

[Oh05] Y-G. Oh, *Spectral invariants, analysis of the Floer moduli space, and geometry of the Hamiltonian diffeomorphism group*, Duke Math. J. **126** (2005), no. 1, 199–295.

[Oh21] ———, *Geometry and analysis of contact instantons and entanglement of Legendrian links I*, arXiv:2111.02597, 2021.

[Par16] J. Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves*, Geometry & Topology **20** (2016), 779–1034, doi:10.2140/gt.2016.20.779.

[PRSZ20] L. Polterovich, D. Rosen, K. Samvelyan, and J. Zhang, *Topological persistence in geometry and analysis – revision*, `https://sites.google.com/site/polterov/topological-persistence-in-geometry-and-analysis-revision`, 2020, Revision.

[PSS96] S. Piunikhin, D. Salamon, and M. Schwarz, *Symplectic Floer-Donaldson theory and quantum cohomology*, Contact and symplectic geometry, Publications of the Newton Institute, vol. 8, Cambridge University Press, 1996, pp. 171–200.

[Rit13] A. Ritter, *Topological quantum field theory structure on symplectic cohomology*, J. Topol. **6** (2013), 391–489, doi:10.1112/jtopol/jts038.

[Rit14] A. F. Ritter, *Floer theory for negative line bundles via Gromov-Witten invariants*, Adv. Math. **262** (2014), 1035–1106.

[Rit16] A. Ritter, *Circle actions, quantum cohomology, and the Fukaya category of Fano toric varieties*, Geom. Topol. **20** (2016), 1941–2052.

[RS01] J. W. Robbin and D. A. Salamon, *Asymptotic behaviour of holomorphic strips*, Annales de l'Institut Henri Poincaré - Analyse non linéaire **18** (2001), no. 5, 573–612.

[Sal97] D. Salamon, *Lectures on Floer homology*, IAS/Park City Graduate Summer School on Symplectic Geometry and Topology, 1997.

[San11a] S. Sandon, *Contact homology, capacity and non-squeezing in $\mathbb{R}^{2n} \times S^1$ via generating functions*, Ann. Inst. Fourier **61** (2011), no. 1, 145–185.

[San11b] ———, *Equivariant homology for generating functions and orderability of lens spaces*, J. Symplectic Geom. **9** (2011), no. 2, 123–146.

[San12] ———, *On iterated translated points for contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times \mathrm{S}^1$*, Int. J. Math. **23** (2012), no. 2, 1–14.

[San13] ———, *A Morse estimate for translated points of contactomorphisms of spheres and projective spaces*, Geom. Dedicata **165** (2013), 95–110.

[SC25] J. Sampietro Christ, *Equivariant Floer homology is isomorphic to reduced Floer homology*, arXiv:2505.02647, 5 2025.

[Sch00] M. Schwarz, *On the action spectrum for closed symplectically aspherical manifolds*, Pacific J. Math. **193** (2000), 419–461.

[Sei08a] P. Seidel, *A biased view of symplectic cohomology*, Current developments in mathematics, 2006, Int. Press, Somerville, MA, 2008, pp. 211–253, doi:10.4310/CDM.2006.V2006.N1.A4.

[Sei08b] ———, *Fukaya categories and Picard-Lefshetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, 2008, doi:10.4171/063.

[Sei15] ———, *The equivariant pair-of-pants product in fixed point floer cohomology*, Geom. Funct. Anal. **25** (2015), 942–1007.

[She17] E. Shelukhin, *The Hofer norm of a contactomorphism*, J. Symplectic Geom. **15** (2017), no. 4, 1173–1208, doi:10.4310/JSG.2017.v15.n4.a7.

[She22] ———, *On the Hofer-Zehnder conjecture*, Ann. of Math. **193** (2022), no. 3, 775–839.

[SS10] P. Seidel and I. Smith, *Localization for involutions in Floer cohomology*, Geom. Funct. Anal. **20** (2010), 1464–1501.

[SZ21] E. Shelukhin and J. Zhao, *The $\mathbb{Z}/p\mathbb{Z}$-equivariant product-isomorphism in fixed point Floer homology*, J. Symplectic Geom. **19** (2021), no. 5, 1101–1188.

[Ter21] B. Tervil, *Translated points for contactomorphisms of prequantization spaces over monotone symplectic toric manifolds*, J. Symplectic Geom. **19** (2021), no. 6, 1421–1493, doi:10.4310/JSG.2021.v19.n6.a3.

[Ulj23] I. Uljarević, *Selective symplectic homology with applications to contact non-squeezing*, Compositio Math. **159** (2023), 2458–2482, doi:10.1112/S0010437X23007480.

[UZ22] I. Uljarević and J. Zhang, *Hamiltonian perturbations in contact Floer homology*, J. Fixed Point Theory Appl. **24** (2022), no. 71, 1–20.

[Var21] U. Varolgunes, *Mayer-Vietoris property for relative symplectic cohomology*, Geom. Topol. **25** (2021), 547–642, doi:10.2140/gt.2021.25.547.

[Vit92] C. Viterbo, *Symplectic topology as the geometry of generating functions*, Math. Ann. **292** (1992), no. 1, 685–710.

[Vit99] ———, *Functors and computations in Floer homology with applications I*, Geom. Funct. Anal. **9** (1999), 985–1033, doi:10.1007/s000390050106.

[Zap20] F. Zapolsky, *Quasi-morphisms on contactomorphism groups and Grassmanians of 2-planes*, Geom. Dedicata **207** (2020), 287–309.

[Zha19] J. Zhao, *Period symplectic cohomologies*, J. Symplectic Geom. **17** (2019), no. 5, 1513–1578.

Email address: dylan@dylancant.ca

Département de mathématiques et de statistique, Université de Montreal, Pavillon André-Aisenstadt, 2920 Chemin de la Tour, Montreal, Quebec, H3T 1J4, Canada

Email address: eric.kilgore@usc.edu

Department of Mathematics, University of Southern California, 3620 S. Vermont Ave, KAP 104, Los Angeles, CA 90089

Email address: jzhang4518@ustc.edu.cn

The Institute of Geometry and Physics, University of Science and Technology of China, 96 Jinzhai Road, Hefei, Anhui, 230026, China