THE SPECTRAL DIAMETER OF A SYMPLECTIC ELLIPSOID

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ABSTRACT. The spectral diameter of a symplectic ball is shown to be equal to its capacity; this result upgrades the known bound by a factor of two and yields a simple formula for the spectral diameter of a symplectic ellipsoid. We also study the relationship between the spectral diameter and packings by two balls.

1. *Introduction*

1.1. *Spectral diameter as a capacity.* A well-known construction in Floer theory associates a *spectral invariant* to a compactly supported Hamiltonian system φ_t on a convex-at-infinity symplectic manifold W. The sum of the spectral invariants of φ_t and its inverse is called the *spectral norm* $\gamma(\varphi_t)$; we review the definition in §2.1.7. For an open set $U \subset W$ one can therefore consider its *spectral diameter*:

$$\gamma(U) = \sup\{\gamma(\varphi_t) : \varphi_t \text{ is supported in } U\}.$$

Such a quantity is a symplectic capacity for U in the sense of, e.g., [CHLS07], and has been considered in [Sch00, FS07, Mai22]; this capacity should not be confused with the well-known spectral capacity $c(U)$ introduced by [Vit92]; for further discussion see §2.1.8. Our main result is the exact formula for the spectral diameter of a symplectic ellipsoid in $W = \mathbb{C}^n$:

Theorem 1.1. *The spectral diameter of the ellipsoid:*

$$E(a_1, \dots, a_n) = \{z : \textstyle\sum \pi a_i^{-1} |z_i|^2 < 1\},$$

with $a_1 \leq \cdots \leq a_n$, is equal to:

$$\gamma(E(a_1, \dots, a_n)) = \begin{cases} a_n & \text{if } a_n \in [a_1, 2a_1], \\ 2a_1 & \text{if } a_n \in [2a_1, \infty); \end{cases} \tag{1}$$

in particular, $\gamma(B(1)) = 1$ and $\gamma(Z(1)) = 2$.

To the authors' knowledge, the equality $\gamma(B(1)) = 1$ has so far not appeared in the literature. The inequality $\gamma(Z(1)) \leq 2$ follows from a displacement energy bound and has been observed before; see §1.2 and §2.2.

1.2. *Outline of argument.* The argument proving Theorem 1.1 is divided into three main steps:

(1) *The case of a cylinder*: $\gamma(E(1,\infty,\dots)) \leq 2$. This step is proved using a well-known upper bound on the spectral diameter in terms of the displacement energy; see §2.2 and §2.3.
(2) *The case of a long ellipsoid*: $\gamma(E(1,\dots,2)) \geq 2$. This lower bound uses the standard moment map $\mathbb{R}^{2n} \to [0,\infty)^n$ and toric geometry. Briefly, one shows that $E(1,\dots,2)$ contains two balls of capacity 1, and then explicitly constructs systems supported in these two balls to obtain the stated lower bound; see §2.5 and §2.6.
(3) *The case of a ball*: $\gamma(B(1)) = 1$. This step is the most delicate; the argument relies on the Hamiltonian circle action on $\mathbb{C}^n$ which rotates all the coordinates, and analyzing the effect on the action filtration of Floer homology groups. The proof is given in §2.9.

Assuming (1), (2), and (3), we prove the theorem. It is well-known that:

$$\gamma(\sqrt{a}U) = a\gamma(U)$$

for open sets $U \subset \mathbb{R}^{2n}$; see, e.g., [CHLS07, §2]. Therefore, using (1) and (2), if $a_n \in [2a_1,\infty)$, we have:

$$\gamma(E(a_1,\dots,a_n)) = a_1\gamma(E(1,\dots,a_n/a_1)) = 2a_1,$$

because $E(1,\dots,2) \subset E(1,\dots,a_n/a_1) \subset E(1,\infty,\dots)$. On the other hand, if $a_n \in [a_1,2a_1]$, then a similar argument yields:

$$\gamma(E(a_1,\dots,a_n)) = \frac{a_n}{2}\gamma(E(2a_1/a_n,\dots,2)) \geq \frac{a_n}{2}\gamma(E(1,\dots,2)) = a_n,$$

while (3) implies:

$$\gamma(E(a_1,\dots,a_n)) \leq \gamma(E(a_n,\dots,a_n)) = a_n,$$

so $\gamma(E(a_1,\dots,a_n)) = a_n$ for $a_n \in [a_1,2a_1]$. This is what we wanted to show.

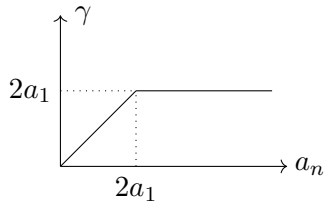


FIGURE 1. Spectral diameter of the ellipsoid $E(a_1,\dots,a_n)$.

1.3. *Convex toric domains.* The result of Theorem 1.1 in fact proves that in $\mathbb{R}^4$ the spectral diameter is a 2-*normalized capacity*; see [GR25, Definition 1.6] (in other dimensions the spectral diameter is not 2-normalized). It was pointed out to the authors by Jean Gutt that all 2-normalized capacities coincide on *convex toric domains* in $\mathbb{R}^4$; see [GR25, Theorem 1.9]. The *second Gutt-Hutchings capacity* c_2^{GH} is also a 2-normalized capacity; see [GH18, Lemma 2.1], and therefore:

Corollary 1.2. *It holds that* $\gamma(\Omega) = c_2^{\mathrm{GH}}(\Omega)$ *for any four-dimensional convex toric domain* Ω.

Remark. The value of $c_2^{\mathrm{GH}}(\Omega)$ for convex toric domains $\Omega \subset \mathbb{R}^{2n}$ is computed explicitly in [GH18, Theorem 1.6].

1.4. *The spectral diameter and packings by two balls.* Part of the argument used in the proof of Theorem 1.1 involves the additivity of the spectral diameter with respect to packings by two balls. We state this result as it is of independent interest:

Theorem 1.3. *Let* U *be an open set in an aspherical and convex-at-infinity symplectic manifold* W. *Suppose there exists a symplectic embedding:*

$$B(a_1) \sqcup B(a_2) \to U;$$

then $a_1 + a_2 \leq \gamma(U)$.

Such additivity does not hold for packings by three or more balls; for example, the cylinder in $\mathbb{C}^n$ contains infinitely many disjoint balls with a given capacity but has a bounded spectral diameter.

Theorem 1.3 also enables one to conclude:

Corollary 1.4. *The spectral diameter of a polydisk:*

$$P(a_1, \ldots, a_n) = D(a_1) \times \cdots \times D(a_n),$$

with $a_1 \leq \cdots \leq a_n$, *is equal to* $2a_1$ *if* $n \geq 2$; *if* $n = 1$ *it is equal to* a_1.

Proof. When $n \geq 2$ the polydisk admits an embedding of two disjoint balls of capacity a_1, see §2.5. Hence, Theorem 1.3 implies that the spectral diameter is at least $2a_1$. The displacement energy of the polydisk is a_1; see §2.3. The statement then follows from Proposition 2.2. The case $n = 1$ is obvious. □

The spectral diameter obstructs symplectic embeddings of $P(a, a)$ into $B(a')$ unless $2a \leq a'$; when $n = 2$, this embedding obstruction recovers the result proved in [EH90]. In higher dimensions the spectral diameter obstruction is weaker than the one obtained in [EH90].

1.5. *A two-ball capacity.* Consider the capacity:

$$c_{2B}(U) := \sup\{a+b : \text{ there exists a symplectic } B(a) \sqcup B(b) \subset U\}.$$

Our method proves that $c_{2B}(U) = \gamma(U)$ whenever U is a symplectic ellipsoid or polydisk. This begs the question: *what is the largest class $\mathcal{C}$ of domains U for which $c_{2B}(U) = \gamma(U)$?* The construction of [Her98] shows that the class $\mathcal{C}$ does not contain certain starshaped domains; it is based on the fact that arbitrarily small neighborhoods of the torus $\partial D(r) \times \cdots \times \partial D(r)$ have a large spectral diameter (proportional to r) but with a small c_{2B} capacity.

The equality $c_{2B}(B(1)) = \gamma(B(1)) = 1$ implies Gromov's two-ball theorem [Gro85, 0.3.B]. Gromov's method yields $c_{2B}(\mathbb{C}P^n) = 1$; this fact can also be proved using spectral diameter. Indeed, we will prove the following variant of Theorem 1.3 which is applicable to $\mathbb{C}P^n$:

Proposition 1.5. *Suppose that (M^{2n}, ω) is a closed symplectic manifold and:*

$$\omega(u) \neq 0 \implies |c_1(u)| \geq n+1$$

for all $u \in \pi_2(M)$. If there is a symplectic embedding $B(a) \sqcup B(b) \to U$, where $U \subset M$ is an open set, then $\gamma(U)$ is at least $a+b$.

As a corollary we conclude $c_{2B}(\mathbb{C}P^n) \leq \gamma(\mathbb{C}P^n)$. When combined with the result[1] of [EP03] that $\gamma(\mathbb{C}P^n) \leq 1$ (see [KS21, Theorem H]) this establishes the two-ball theorem for $\mathbb{C}P^n$.

1.6. *On the spectral displacement energy.* The *spectral displacement energy* of a precompact open set $U \subset W$ is the value:

$$e_\gamma(U) = \inf\{\gamma(\psi_t) : \psi_1(U) \cap U = \emptyset\};$$

such a quantity was considered in [Vit92, §4]. Combining the displacement energy bound of §2.2 and the existence of a certain compactly supported Hamiltonian system displacing $B(1)$ from itself it is possible to prove the following:

Theorem 1.6. *The spectral displacement energy $e_\gamma(B(1))$ is equal to 1.*

The construction of the system is recalled in §2.3. This equality shows one does not obtain the spectral diameter $\gamma(B(1)) = 1$ directly from the displacement energy bound from §2.2. Indeed, the displacement energy bound yields:

$$\gamma(B(1)) \leq 2e_\gamma(B(1)) = 2,$$

which is suboptimal in view of Theorem 1.1.

[1]The result that $\gamma(\mathbb{C}P^n) \leq 1$ is not explicitly stated in [EP03], but it follows from their result that the spectral invariant of the unit on $\mathbb{C}P^n$ is a quasimorphism (a homomorphism up to a bounded error), together with a precise determination of this bounded error. The same argument we give in §2.11 recovers this bound.

1.7. *Spectral diameter of special balls in projective space.* Consider $\mathbb{C}P^n$ with the Fubini-Study symplectic form, normalized so that the class of $\mathbb{C}P^1$ has symplectic area 1. In this case, it is well-known that the symplectic structure on $\mathbb{C}P^n$ is determined as the symplectic reduction of $\partial B(1) \subset \mathbb{C}^{n+1}$.

Let $a \in (0,1)$. The function $S_a = \pi|z_0|^2 - a$ is well-defined on $\mathbb{C}P^n$ via the quotient map $\partial B(1) \to \mathbb{C}P^n$, and generates a 1-periodic Hamiltonian circle action on $\mathbb{C}P^n$.

The set $\{S_a \geq 0\}$ is symplectomorphic to a ball of capacity $1-a$, a fact whose verification is left to the reader. Let us call a ball in $\mathbb{C}P^n$ *special* if it is obtained by applying a Hamiltonian diffeomorphism to $\{S_a \geq 0\}$, for some $a \in (0,1)$. Beyond dimensions $n = 1, 2$, it does not seem to be known whether the image of every embedding of a closed ball into $\mathbb{C}P^n$ is special.

Using methods similar to our proof of Theorem 1.1, we prove:

Theorem 1.7. *The spectral diameter of a special ball is equal to its capacity.*

Here the spectral diameter is computed using the spectral invariants within $\mathbb{C}P^n$ and uses the coefficient field $\mathbb{Z}/2$.

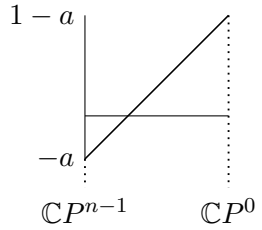


FIGURE 2. Schematic illustration of S_a, graphed as a function of $\pi|z_0|^2$. The point $\mathbb{C}P^0$ is the maximum of S_a and corresponds to the line $z_1 = \cdots = z_n = 0$. The divisor $\mathbb{C}P^{n-1}$ is the Morse-Bott minimum of S_a and represents all lines passing through the hyperplane $z_0 = 0$.

A special ball can be parametrized by an embedding $i : B(a) \to \mathbb{C}P^n$. Thus any compactly supported Hamiltonian system ψ_t on $B(a)$ can be "pushed forward" to $\mathbb{C}P^n$ by the formula $i\psi_t i^{-1}$, extended to the complement of the ball as the identity system. It is natural to wonder whether:

$$\gamma_{\mathbb{C}^n}(\psi_t) = \gamma_{\mathbb{C}P^n}(i\psi_t i^{-1}) \tag{2}$$

holds for every system ψ_t. As we show in §2.10, the equality (2) fails in general (our example requires a to be close to 1). This is noteworthy as it shows we cannot simply appeal to the known bound on the spectral diameter of $\mathbb{C}P^n$ from [EP03] to deduce $\gamma_{\mathbb{C}^n}(\psi_t) \leq 1$ for all ψ_t supported in $B(1)$.

Interestingly enough, if (2) holds for even an arbitrarily small special ball, then Theorem 1.7 can be used to recover Theorem 1.1 for balls. Establishing

sufficient conditions to ensure (2) in the presence of symplectic spheres seems to be a non-trivial task, even for small balls, and we save further study of (2) for future research.

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2. *Proofs*

2.1. *Floer homology in convex-at-infinity manifolds.* This section is concerned with a recollection of various Floer theoretic objects used in this paper. Throughout we work in the setting of a semipositive and convex-at-infinity symplectic manifold W. We refer the reader to our other paper [AAC23] for details on this class of symplectic manifolds. Let us merely comment here that *convex-at-infinity* means that W has a non-compact end modelled on the positive half of a symplectization SY, and hence it makes sense to speak of the Liouville flow in the convex end. The contact manifold Y is called the ideal boundary of W.

The Hamiltonian isotopies φ_t we will consider are called *contact-at-infinity*, which means they commute with the Liouville flow in the convex end. It is well-known (see, e.g., [BC24]) that such isotopies have an ideal restriction to a contact isotopy of the ideal boundary Y. For the proof of our main result, it is sufficient to work with systems whose ideal restriction is a Reeb flow with some speed; see §2.1.5. Such systems are well-studied; see, e.g., [AAC23] and the references therein.

2.1.1. *Cappings.* For each fixed point x of the time-one map ψ_1, a *representative capping* is a smooth map $u : [-1, 1] \times \mathbb{R}/\mathbb{Z} \to W$ so that $u(-1, t) = x$ and $u(1, t) = \psi_t(x)$. Representative cappings are considered up to equivalence: the difference of two representative cappings forms a sphere and if this sphere has zero symplectic area then the representatives are declared to be equivalent. An equivalence class of representatives will be referred to as a *capping*. Capped orbits are denoted as pairs (x, u).

Requiring that $u(-1, t) = x$ has the following advantage:

Lemma 2.1. *If u is a capping of x then $\bar{u}(s, t) = \psi_t^{-1}(u(-s, t))$ is a capping of x for the system ψ_t^{-1}, and the action of $\bar{u}$ is minus the action of u.* □

2.1.2. *Action.* To each capping one can associate an action:

$$\mathcal{A}(\psi_t; x, u) = \int H_t(\psi_t(x))\mathrm{d}t - \int u^*\omega,$$

where H_t is the normalized generator of ψ_t, i.e.,

$$\omega(X_t, -) = -dH_t,\ \partial_t \psi_t = X_t \circ \psi_t.$$

For simplicity we suppose that W is connected, and we consider two classes of normalization in this paper:

(1) If W is open, and Y_0 is a chosen connected component of the ideal boundary of W, then a Hamiltonian function H_t is normalized if it is one-homogeneous in the non-compact end corresponding to Y_0; see [AAC23] for the definitions of *ideal boundary* and *one-homogeneous*.
(2) If W is closed, then a Hamiltonian function H_t is normalized if the integral of $H_t \omega^n$ over W vanishes for each t.

In (1) different choices of Y_0 give different normalizations. The crucial properties are that the set of normalized Hamiltonians is a vector subspace and a constant normalized function is zero.

2.1.3. *The Floer homology vector space.* Let ψ_t be a contact-at-infinity system and suppose that ψ_1 has non-degenerate fixed points.

Define $\mathrm{CF}(\psi_t)$ to be the $\mathbb{Z}/2$-vector space of semi-infinite sums generated by capped orbits (x, u) of ψ_t, requiring that $\mathcal{A}(\psi_t; x, u) \geq L$ holds for only finitely many terms in the sum, for each L.

2.1.4. *The Floer homology differential.* The Floer differential is defined as usual in fixed point Floer homology; see, e.g., [DS93, Sei15]. The relevant moduli space $\mathcal{M}(\psi_t, J_t)$ is the space of twisted holomorphic curves:

$$\begin{cases} w : \mathbb{C} \to W, \\ \partial_s w + J_t(w)\partial_t w = 0, \\ \psi_1(w(s, t+1)) = w(s, t). \end{cases}$$

In order for the cylinder $u(s,t) = \psi_t(w(s,t))$ to solve a smooth PDE, we require that J_t is ψ_1-*twisted-periodic*, i.e., $J_{t+1}(w) = \mathrm{d}\psi_1^{-1} J_t(\psi_1(w))\mathrm{d}\psi_1$.

Moreover, to achieve a priori estimates for bubbles which may appear, it is important to assume that J_t is ω-tame for each t; this means that $\omega(v, J_t v)$ is a positive quadratic form in each tangent space. Additionally, in the convex end of W, we require that J_t is invariant with respect to the Liouville flow. Finally, in the presence of holomorphic spheres, one needs to use semipositivity to ensure compactness; this requires additional genericity assumptions on J_t. This point is explained in detail in, e.g., [AAC23] (and the references therein). Since our main applications concern the aspherical manifold $\mathbb{C}^n$, we will not comment further on this point.

By counting rigid-up-to-translation elements in $\mathcal{M}(\psi_t, J_t)$, with $w(-\infty)$ considered as input and $w(+\infty)$ considered as output one obtains a map:

$$d_{\psi_t, J_t} : \mathrm{CF}(\psi_t) \to \mathrm{CF}(\psi_t).$$

The cylinder u determines the capping of the output in terms of the capping of the input. With these homological conventions, the Floer differential decreases action. The homology of $(\mathrm{CF}(\psi_t), d_{\psi_t, J_t})$ is denoted[2] by $\mathrm{HF}(\psi_t)$.

2.1.4.a. It follows that J_{-t} is ψ_1^{-1} twisted periodic which yields an inversion identification $\iota : \mathcal{M}(\psi_t, J_t) \to \mathcal{M}(\psi_t^{-1}, J_{-t})$ given by $\iota(w)(s,t) = w(-s,-t)$.

It is for the sake of this simple "inversion formula" that we opt to work from the fixed point Floer homology perspective.

2.1.5. *Reeb flows.* This section is only relevant when W is open. We describe the set-up on $\mathbb{C}^n$, although everything holds verbatim on a general convex-at-infinity manifold W if one replaces $\pi|z|^2$ by a suitable function r.

Let $R_{\delta,t}$ be the Hamiltonian flow generated by:

$$\mu_\delta(\pi|z|^2 - 1) + 1$$

where μ_δ is a convex cut-off function so that:

(1) $\mu_\delta(x)$ is the constant $\delta/2$ for $x \leq 0$,
(2) $\mu_\delta(x) = x$ for $x \geq \delta$,
(3) $\mu'_\delta(x) > 0$ for $x > 0$.

It is important that $R_{\delta,t}$ agrees with the flow of $\pi|z|^2$ for $\pi|z|^2 > 1 + \delta$, i.e., the ideal restriction of $R_{\delta,t}$ is the standard one-periodic Reeb flow.

2.1.6. *Spectral invariants.* For any class $a \in \mathrm{HF}(\psi_t)$, let:

$$c(\psi_t; a) := \inf\{\sup_i \mathcal{A}(\psi_t; x_i, u_i) : \text{the cycle } \textstyle\sum_i (x_i, u_i) \text{ represents } a\};$$

loosely speaking, $c(\psi_t; a)$ is a homological min-max over all representative cycles; see, e.g., [BP94, Sch00, Oh05, FGS05, FS07, Ush08].

When ψ_t is contact-at-infinity and its ideal restriction is a negative Reeb flow there is a distinguished class $1 \in \mathrm{HF}(\psi_t)$ which plays the role of the unit element for the pair-of-pants product; see §2.1.9. This is initially defined for non-degenerate systems. One extends the definition of $\mathrm{HF}(\psi_t)$ and the unit element to all systems whose ideal restriction is a non-positive Reeb flow as an inverse limit over continuation maps. The spectral invariant $c(\psi_t; 1)$ extends to the limit; we refer the reader to [AAC23] for details on continuation maps and this extension.

2.1.7. *The spectral norm.* The *spectral norm* of a compactly supported system ψ_t is defined by:

$$\gamma(\psi_t) := c(\psi_t; 1) + c(\psi_t^{-1}; 1) = \lim_{s \to 0} c(R_{-st}\psi_t; 1) + c(R_{-st}\psi_t^{-1}; 1), \tag{3}$$

where R_t is one of the Reeb flows constructed in §2.1.5 (the parameters going into the definition of R do not matter in the limit $s \to 0$). It is crucial

[2]To be pedantic, $\mathrm{HF}(\psi_t)$ should be defined as a limit of the homologies of $(\mathrm{CF}(\psi_t), d_{\psi_t, J_t})$ as J_t varies over all admissible complex structures.

that ψ_t is compactly supported, as the ideal restrictions of ψ_t and ψ_t^{-1} are then both non-positive.

A similar definition of a spectral norm in the case of $\mathbb{C}^n$ appears in [Vit92] using the framework of generating functions. The paper which introduced the spectral norm using the framework of Floer theory in aspherical manifolds is [Sch00], and the extension to convex-at-infinity W is due to [FS07].

2.1.8. *Spectral diameter.* As explained in the introduction, one obtains a *spectral diameter*[3] $\gamma(U)$, for any open set $U \subset W$, as the supremum of $\gamma(\psi_t)$ over systems ψ_t with compact support in U. The spectral diameter as a capacity is considered in [Sch00, FS07]; an earlier capacity appears in [Vit92], namely the *spectral capacity*:

$$c(U) := \sup\{c(H_t; 1) : H_t \text{ has compact support in } U\},$$

which is known to be a normalized capacity on $\mathbb{C}^n$. See §2.1.10 for our conventions used to define $c(H_t; 1)$ on closed manifolds.

2.1.9. *Sub-additivity for spectral invariants and pair-of-pants product.* Suppose that φ_t, ψ_t are two contact-at-infinity systems. The pair-of-pants product operation is a map:

$$*_{\mathrm{PP}} : \mathrm{HF}(\varphi_t) \otimes \mathrm{HF}(\psi_t) \to \mathrm{HF}(\varphi_t\psi_t),$$

defined by counting solutions to Floer's equation over a pair-of-pants surface, as in, e.g., [Sch95, Sch00, Sei15, KS21, AAC23]. The precise structure of Floer's equation involves the choice of a Hamiltonian connection over the pair-of-pants, and using a connection with zero curvature produces sharp energy estimates in terms of the actions of asymptotics. These energy estimates imply the sub-additivity of the spectral invariants:

$$c(\varphi_t\psi_t; a *_{\mathrm{PP}} b) \leq c(\varphi_t; a) + c(\psi_t; b).$$

For further discussion we refer the reader to [AAC23, §2.4] which describes in detail the pair-of-pants product in convex-at-infinity symplectic manifolds.

The unit element is defined by the PSS construction applied to the fundamental class (see [FS07] for a detailed exposition in the convex-at-infinity setting). It is well-known that (in the convex-at-infinity setting) a priori estimates can only be ensured for PSS cylinders provided the output system has a "negative" ideal restriction. For this reason, we will only define the unit element as an element of $\mathrm{HF}(\varphi_t)$ if the ideal restriction of φ_t is a Reeb flow with a negative speed. In this case, for any other contact-at-infinity system ψ_t and any class $b \in \mathrm{HF}(\psi_t)$, the element $1 *_{\mathrm{PP}} b$ is the image of b under the continuation map $\mathrm{HF}(\psi_t) \to \mathrm{HF}(\varphi_t\psi_t)$; see [KS21].

[3]If one defines the distance $d(\varphi_t, \psi_t) = \gamma(\varphi_t^{-1}\psi_t)$ then $\gamma(U)$ is the diameter of the set of all systems supported in U.

2.1.10. *Non-normalized Hamiltonians.* It is sometimes convenient to generalize the definition to allow non-normalized Hamiltonian functions via the rule:

$$c(H_t + f(t); a) = c(\psi_t; a) + \int_0^1 f(t)dt,$$

where H_t is the normalized generator for ψ_t, and $f(t)$ is a time-dependent shift. If we use the symbol $c(\psi_t; a)$, then we require using the normalized generator. On the other hand, if we use the symbol $c(H_t; a)$, then we allow H_t to be non-normalized.

2.2. *The displacement energy bound.* In this section we recall the displacement energy bound on the spectral norm:

Proposition 2.2. *Let W be a rational, semipositive, and convex-at-infinity symplectic manifold, and let ψ_t, φ_t be two compactly supported Hamiltonian systems such that ψ_1 displaces the support of φ_t. Then $\gamma(\varphi_t) \leq 2\gamma(\psi_t)$. Moreover, if W is open, then $c(\varphi_t; 1) \leq \gamma(\psi_t)$.*

Proof. See [Gin05] which proves the result in the both the open and the closed case, assuming W is aspherical; see also [Vit92, HZ94, Sch00, Oh05, Ush10b]. The argument extends easily from aspherical to rational and semipositive, all that is required is that spectral invariants are well-defined and are sub-additive (semipositivity ensures their well-definedness) and valued in the action spectrum (hence the rationality assumption; see [Oh05]). □

2.3. *An upper bound to the capacity of a cylinder.* In this section we bound the spectral displacement energy of the cylinder following closely the ideas in [Pol01, §2.4].

A disc $B(1) \subset \mathbb{C}$ is symplectomorphic to a square with sides of length 1; in particular, $Z(1)$ is symplectomorphic to $R = (0,1)^2 \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$. Therefore, it is enough to show that $\gamma(R) \leq 2$. Consider the Hamiltonian system:

$$\psi_t(x_1, y_1, \ldots, x_n, y_n) = (x_1, y_1 + t, \ldots, x_n, y_n),$$

generated by $H(x, y) = x_1$. Here, we identify $z_j = x_j + iy_j$ for all j. Then:

$$\psi_1(R) \cap R = \emptyset,$$

and the displacement occurs in $(0,1) \times (0,2) \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$. Of course ψ_t is neither compactly supported nor normalized.

Let φ_t be a Hamiltonian system supported inside $(0,1)^2 \times K \subset R$, for a compact set $K \subset \mathbb{C}^{n-1}$. Consider a Hamiltonian system ψ'_t generated by a function H' obtained by cutting-off H outside an arbitrarily small neighbourhood of $(0,1) \times (0,2) \times K$; we can ensure that

$$\max H' - \min H' \leq 1 + \delta$$

for some small δ. This construction yields a system whose time-one map ψ_1' displaces the support of φ_t. Moreover, by §2.2:

$$\gamma(\varphi_t) \leq 2\gamma(\psi_t') \leq 2(\max H' - \min H') = 2 + 2\delta.$$

Here we use the well-known estimate that the spectral norm is less than the Hofer norm; see, e.g., [KS21]. Since φ_t is arbitrary and δ can be chosen arbitrarily small, we conclude the desired result $\gamma(Z(1)) = \gamma(R) \leq 2$.

2.4. *Displacement energy of a ball.* In this section we show that the spectral displacement energy $e_\gamma(B(1))$ of $B(1)$ is equal to 1; this is the statement of Theorem 1.6.

We begin by showing $e_\gamma(B(1)) \geq 1$. Let ψ_t be a Hamiltonian system such that ψ_1 displaces $B(1)$ and, hence, the support of any Hamiltonian system φ_t supported therein. Proposition 2.2 implies $c(\varphi_t; 1) \leq \gamma(\psi_t)$.

It is known that one can find a bump function supported in $B(1)$ generating a Hamiltonian system φ_t whose spectral invariant $c(\varphi_t; 1)$ is arbitrarily close to 1; indeed, this follows from the construction in §2.6. In particular, one concludes that $\gamma(\psi_t) \geq 1$; this proves the desired inequality.

We now show that $e_\gamma(B(1)) \leq 1$; to that effect, it is enough to find a Hamiltonian system ψ_t' whose time-1 map displaces $B(1)$ and $\gamma(\psi_t') \leq 1$. The system constructed in §2.3 satisfies the desired properties. This concludes the proof of Theorem 1.6.

2.5. *Ball packings of toric domains.* This section explains the toric approach to ball packings following [Sch05]. This is used to show that:

(1) the ellipsoid $E(a_1, \ldots, a_n)$ with $a_n \geq 2a_1$, and
(2) the polydisk $P(a_1, \ldots, a_n)$,

each contain two disjoint balls whose capacities are arbitrarily close to a_1. Then Theorem 1.3 bounds their spectral diameter from below by $2a_1$. On the other hand, the displacement energy bound of §2.2 and the construction in §2.3 bounds their spectral diameter from above by $2a_1$. Such arguments played a role in the proof of Theorem 1.1 and the results in §1.5.

Define the moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}$ by $\mu(z) = (\pi|z_1|^2, \ldots, \pi|z_n|^2)$. Given a domain $D \subset \mathbb{C}^n$, its image $\mu(D)$ will be referred to as its *toric image*. It is easy to check that the toric image of an ellipsoid is a simplex while the toric image of a polydisk is a rectangular parallelepiped. A domain D is called *toric* provided $D = \mu^{-1}(\mu(D))$; a moment's thought will reveal the both ellipsoids and polydisks are toric domains.

The standard technique used to construct a ball packing of a toric domain is to decompose its toric image into simplices, each of which are supposed

to represent embedded symplectic balls. The key is the following symplectomorphism Φ between $\mathbb{R}^n_{>0} \times T^n$ and $\mathbb{C}^n \setminus \{z_1 \cdots z_n = 0\}$:

$$(a_1, \ldots, a_n, \theta_1, \ldots, \theta_n) \mapsto \frac{1}{\sqrt{\pi}}(\sqrt{a_1}e^{2\pi i\theta_1}, \ldots, \sqrt{a_n}e^{2\pi i\theta_n}),$$

where $(a_1, \ldots, a_n)$ are the coordinates on $\mathbb{R}^n_{>0}$, $(\theta_1, \ldots, \theta_n)$ are the coordinates on $T^n = \mathbb{R}^n/\mathbb{Z}^n$; the symplectic form on the domain is $\sum \mathrm{d}a_i \wedge \mathrm{d}\theta_i$.

It is a rather deep fact that:

Lemma 2.3. *If $\Delta(a) \subset \mathbb{R}^n_{>0}$ is the open simplex consisting of open convex combinations of $0, ae_1, \ldots, ae_n$, then the Gromov width of $\mu^{-1}(\Delta(a))$ is a.*

Proof. For the proof see, e.g., [Sch05, §3.1]. This is not immediate; indeed:

$$\mu^{-1}(\Delta(a)) = B(a) \setminus \{z_1 \ldots z_n = 0\},$$

so any embedding of a ball must miss the removed parts. However, one can still embed balls with capacity arbitrarily close to a. □

Introduce the group $G = \mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ of special affine transformations. Then G acts on $\mathbb{R}^n$ in a natural way, and the action extends to an action on $\mathbb{R}^n \times T^n$ by canonical transformations (in particular, the action is via symplectomorphisms). As a consequence, we conclude the following corollary of Lemma 2.3: *if the toric image of a toric domain Ω contains a disjoint union $\Delta(a) \sqcup g(\Delta(a))$ where $g \in G$, then Ω contains two disjoint symplectic balls whose capacities are each arbitrarily close to a.*

Thus to prove that the ellipsoid $E(a_1, \ldots, a_n)$ with $a_n \geq 2a_1$, and the polydisk $P(a_1, \ldots, a_n)$ each contain two disjoint balls of capacity arbitrarily close to a_1, it suffices to cut its toric image into subsimplices; this can be done and is shown in Figure 3.

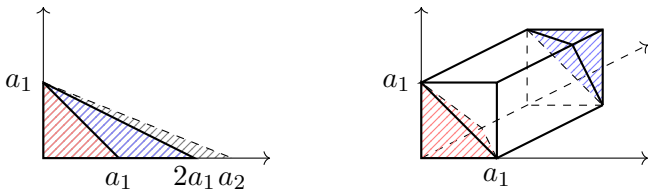


FIGURE 3. Decomposing the toric image of the ellipsoid (shown in dimension 2) and the polydisk (shown in dimension $n = 3$) into standard simplices. In the picture on the right we have set $a_1 = a_2$.

2.6. *Spectral diameter and packings of two balls.* We present the proof of Theorem 1.3 which provides a lower bound on the spectral diameter of domains in aspherical manifolds based on their packings by two balls. As in the statement of the theorem, let $U \subset W$ be an open domain in W, and let:

$$B(a) \sqcup B(b) \to U$$

be an embedding of a disjoint union of Darboux balls. We will show that $\gamma(U) \geq a+b$ by explicitly constructing systems supported in these two balls. Other works have estimate spectral norms of systems supported in a disjoint union of balls, e.g., [Sey15, HLRS16, Ish16, Tan22, GT23]; these results are much more general and apply to any systems. Our result only requires a lower bound and can be deduced by elementary means.

The reason for assuming that W is aspherical is so that the action is associated directly to the orbits (there is a single capping) and the spectral invariants lie in a compact nowhere dense action spectrum.

Consider the following Hamiltonian function:

$$H_{a,\eta,\epsilon}(z) = \mu_{\epsilon\eta}(\eta(a - \pi|z|^2)) - \epsilon\eta/2$$

where $0 < \eta < 1$ is any number, ϵ is smaller than a, and μ_δ is the cut-off function in §2.1.5. Morally, one should think of the pointwise limit as $\epsilon \to 0$:

$$H_{a,\eta,0} = \max\{\eta(a - \pi|z|^2), 0\},$$

which is simply a radial bump function of height ηa supported in the ball of capacity a.

It is important that this system has:

(1) a 1-periodic orbit at $z = 0$ whose action is $\eta(a - \epsilon/2)$,
(2) a family of 1-periodic orbits when $\pi|z|^2 \geq a$ whose actions are 0,

and these are all the 1-periodic orbits (incidentally these orbits are constant).

Let $K_{a,b,\eta,\mu}$ be the system on U obtained by implanting $H_{a,\eta,\epsilon}$ in the image of the ball $B(a)$ and the system $-H_{b,\mu,\epsilon}$ on the image of the ball $B(b)$. Here ϵ is required to be smaller than a and b.

Let ψ_t be the generated system. One concludes:

(a) the spectrum of ψ_t is the set $\{0, \eta(a - \epsilon/2), \mu(\epsilon/2 - b)\} + C$, and,
(b) $\{0, \eta(\epsilon/2 - a), \mu(b - \epsilon/2)\} - C$ is the spectrum of ψ_t^{-1}.

Here C is a constant shift needed to normalize $K_{a,b,\eta,\mu}$, and is only necessary when W is closed. Hence, for the possible values for the spectral norm of ψ_t lie in the following set:

$$\{\eta(a - \epsilon/2), \mu(b - \epsilon/2), \eta a + \mu b - \epsilon(\eta + \mu)\}.$$

We claim that, if both η, μ are positive, then the spectral norm is given by the third number. We prove this by showing the first two cases lead to a

contradiction. The key idea is to exploit continuity and spectrality of the spectral invariants, and non-degeneracy of the spectral norm.

In the first case we have $c(\psi_t) = \eta(a-\epsilon/2)+C$ and $c(\psi_t^{-1}) = -C$; by taking $\eta \to 0$ and using continuity of the spectral invariants we obtain a system that is not the identity (since $\mu > 0$) and has zero spectral norm which is a contradiction as the spectral norm is non-degenerate.

The second case, when $c(\psi_t) = C$ and $c(\psi_t^{-1}) = \mu(b-\epsilon/2) - C$, is similarly ruled out. Hence the only possibility is that $\gamma(\psi_t) = \eta a + \mu b - \epsilon(\eta+\mu)$. By taking ϵ to 0 and η, μ to 1 we conclude $\gamma(U) \geq a + b$, as desired.

Remark. The fact that $\gamma(\psi_t) = \eta a + \mu b - \epsilon(\eta+\mu)$ can also be argued using the maximum formula for the spectral invariant of the unit (see [HLRS16, GT23]):

$$\begin{aligned}\gamma(\psi_t) &= c(H_{a,\eta,\epsilon} - H_{b,\mu,\epsilon}; 1) + c(-H_{a,\eta,\epsilon} + H_{b,\mu,\epsilon}; 1) \\ &= \max\{c(H_{a,\eta,\epsilon}; 1), c(-H_{b,\mu,\epsilon}; 1)\} + \max\{c(-H_{a,\eta,\epsilon}; 1), c(H_{b,\mu,\epsilon}; 1)\} \\ &= \max\{(a-\epsilon/2)\eta, 0\} + \max\{0, (b-\epsilon/2)\mu\} = \eta a + \mu b - \epsilon(\eta+\mu).\end{aligned}$$

The key step in the argument (beyond appealing to the maximum formula) is that $c(H_{a,\eta,\epsilon}; 1)$ is equal to $(a-\epsilon/2)\eta$ and that $c(-H_{b,\mu,\epsilon}; 1) \leq 0$ (and a similar deduction with the signs reversed). The fact that $c(-H_{b,\mu,\epsilon}; 1)$ is non-positive follows from the monotonicity properties of spectral invariants; the fact that $c(H_{a,\eta,\epsilon}; 1)$ is equal to $(a-\epsilon/2)\eta$ follows from spectrality and the non-degeneracy of the spectral norm.

2.6.1. *Beyond the aspherical case.* The analogue of Theorem 1.3 in the non-aspherical case is more subtle, and the argument in §2.6 requires some modification because the action functional becomes multivalued. In this section we refine the argument by taking into account the indices of the orbits, and prove Proposition 1.5 announced in the introduction.

Proof of Proposition 1.5. Consider the Hamiltonian $K_{a,b,\eta,\mu}$ as above, and take a small Morse perturbation K' of $K_{a,b,\eta,\mu}$. For η, μ less than 1, the only orbits of K' are its critical points. Using the constant cappings, the maximum has action approximately $a\eta$, the minimum has action approximately $-b\mu$, and all others critical points have actions close to zero. We note as well that all orbits of sK' are non-degenerate for all $s \in (0, 1]$, and hence the Conley-Zehnder indices of these orbits are related to their Morse indices[4] in such a way that any non-constant recapping will not contribute to the spectral invariant of the unit. In this fashion one concludes, by following

[4]With the authors conventions, $\mathrm{CZ}(x) = n - \mathrm{MI}(x)$ if the system is generated by a Morse function F, x is a critical point of F, and x is a non-degenerate orbit for the system sF for all $s \in (0, 1]$. One uses proves the desired relation for s close to zero, following standard arguments, and then uses the continuity of CZ under deformations of orbits which remain non-degenerate.

the argument in §2.6, that $\gamma(K') \approx a\eta + b\mu$; taking the limit $\eta, \mu \to 1$ yields the desired result. □

2.7. *Duality and inversion in rational symplectic manifolds.* We suppose throughout this section that (W, ω) is *rational*, i.e., $\omega(\pi_2(M))$ is a discrete subgroup of $\mathbb{R}$; the minimal positive generator of this subgroup is denoted by ρ and is called the *rationality constant* of (W, ω).

The goal is to relate the spectral invariants of ψ_t^{-1} with the spectral invariants of ψ_t. For related discussion we refer the reader to [EP03, Ush10a, LZ18]. We will show:

Lemma 2.4. *Let (W, ω) be rational, and let ψ_t be a contact-at-infinity Hamiltonian system with non-degenerate fixed points. For any class $b \in \mathrm{HF}(\psi_t^{-1})$, we have:*

$$c(\psi_t^{-1}; b) = -\inf\{c(\psi_t; a) : \langle a, b\rangle = 1\};$$

where $\langle -, -\rangle$ denotes the duality pairing $\mathrm{HF}(\psi_t) \otimes \mathrm{HF}(\psi_t^{-1}) \to \mathbb{Z}/2$ defined on generators by $\langle (y, v), (x, u)\rangle = 1$ if and only if $x = y$ and $u = \bar{v}$; see §2.7.1 for more details.

Proof. The proof uses the duality isomorphism in §2.7.1 for rational symplectic manifolds. For the rest of the argument we refer the reader to the proof of [EP03, Lemma 2.2]. □

2.7.1. *The duality isomorphism.* Let (y, v) be a capped orbit of ψ_t, and consider the capped orbit of ψ_t^{-1} given by $(y, \bar{v})$ given by $\bar{v}(s, t) = \psi_t^{-1}(v(-s, t))$ as in §2.1.1. Let us denote by:

$$\langle -, -\rangle : \mathrm{CF}(\psi_t) \otimes \mathrm{CF}(\psi_t^{-1}) \to \mathbb{Z}/2,$$

the pairing defined by:

$$\langle (y, v), (x, u)\rangle = \begin{cases} 1 & x = y \text{ and } u = \bar{v}, \\ 0 & \text{otherwise,} \end{cases}$$

where the equality between u and $\bar{v}$ holds in the space of symplectic cappings, as described in §2.1.1.

We first observe that, if $\sum(y_j, v_j)$ and $\sum(x_i, u_i)$ are semi-infinite sums of the correct type to define Floer chains, then only finitely many terms in each can contribute to their pairing, because:

$$\langle (y_j, v_j), (x_i, u_i)\rangle \neq 0 \text{ if and only if } \mathcal{A}(\psi_t; y_j, v_j) + \mathcal{A}(\psi_t^{-1}; x_i, u_i) = 0,$$

and only finitely many terms in each sum are equal to the negative of a term in the other sum.

It follows from the identification of §2.1.4.a that this pairing satisfies:

$$\langle d(y, v), (x, u)\rangle + \langle (y, v), d(x, u)\rangle = 0,$$

and therefore the map:

$$\alpha \in \mathrm{CF}_{\leq -A}(\psi_t) \mapsto \langle \alpha, - \rangle \in \mathrm{Hom}(\mathrm{CF}_{\geq A}(\psi_t^{-1}), \mathbb{Z}/2) \tag{4}$$

is a chain map, where $\mathrm{CF}_{\geq A} = \mathrm{CF}/\mathrm{CF}_{<A}$. When (W, ω) is rational it is not hard to see that (4) is an isomorphism on chain level; briefly, the reason is that any element of $\mathrm{Hom}(\mathrm{CF}_{\geq A}(\psi_t^{-1}), \mathbb{Z}/2)$ can be regarded as an infinite sum of the form $\sum_j \langle (y_j, v_j), - \rangle$ with $\mathcal{A}(\psi_t; y_j, v_j) \leq -A$ (any such infinite sum is well-defined as an element of the dual space). The rationality assumption ensures that there are only finitely many capped orbits of ψ_t with action in a given compact interval, and hence $\sum (y_j, v_j)$ is well-defined as an element of $\mathrm{CF}_{\leq -A}(\psi_t)$. This proves that (4) is surjective; the proof of injectivity is easier and is left to the reader.

2.8. *Naturality transformations.* Given a contact-at-infinity system ϕ_t with $\phi_1 = \mathrm{id}$ whose orbits $\phi_t(x)$ are contractible, one can associate a chain level *naturality transformation*

$$\mathfrak{n} : \mathrm{CF}(\psi_t) \to \mathrm{CF}(\phi_t \psi_t)$$

for any other contact-at-infinity system ψ_t. While not relevant to the arguments in our paper, we note that there is a close relationship between naturality transformations and the *Seidel representation*; see [Sei97, Theorem 1]. This perspective is applied to spectral invariants in [KS21].

Bearing in mind that W is assumed to be connected, the operation depends on an auxiliary choice of capping of one of the orbits $\phi_t(x)$, namely, a cylinder $u : [-1, 1] \times \mathbb{R}/\mathbb{Z} \to W$ so that $u(-1, t) = x$ and $u(1, t) = \phi_t(x)$. For any other choice of point $y \in W$, one can take any path $\eta(s)$ from x to y and define u_y to be the capping of y relative ϕ_t obtained by concatenating:

(1) the inverse of η (from y to x),
(2) the capping u of x,
(3) the cylinder $\phi_t(\eta)$ from $\phi_t(x)$ to $\phi_t(y)$.

Since ϕ_t has zero flux, u_y is independent of η up to equivalence.

The transformation $\mathfrak{n}$ sends the capped orbit (y, v) in $\mathrm{CF}(\psi_t)$ to the concatenation $(y, u_y \# \phi_t(v))$. It is immediate from our definition of the Floer homology differential in terms of the time-1 map that $\mathfrak{n}$ is a chain map; see, e.g., [KS21, pp. 3308] for a similar construction. The map on homology can shift grading and depend on the auxiliary choice of the cappings.

One computes that $\mathfrak{n}$ shifts action values according to the action of the capped orbit (y, u_y) of the system ϕ_t. Since (y, u_y) is a critical point for the action functional (on the covering space of cappings), this action shift is independent of y. As in [KS21, Proposition 31], it follows that:

Lemma 2.5. *If $a \in \mathrm{HF}(\psi_t)$, then $c(\phi_t \psi_t; \mathfrak{n}(a)) = c(\psi_t; a) + \mathcal{A}(\phi_t; x, u)$.* □

2.8.1. *A particular naturality transformation.* In this section we consider the particular loop ϕ_t on $W = \mathbb{C}^n$ generated by $H = \pi|z|^2$. This Hamiltonian function is normalized and it generates the $\mathbb{R}/\mathbb{Z}$-action $\phi_t(z) = e^{2\pi i t}z$. Therefore ϕ_t induces a naturality transformation on Floer homology:

$$\mathfrak{n} : \mathrm{HF}(\psi_t) \to \mathrm{HF}(\phi_t\psi_t)$$

for any contact-at-infinity system ψ_t on $\mathbb{C}^n$.

Since $\mathbb{C}^n$ is symplectically aspherical, there is a unique choice of auxiliary capping (x, u); let us therefore take $x = 0$ and u to be the constant capping. It follows that $\mathfrak{n}$ is action preserving. To be precise:

Lemma 2.6. *Let $a \in \mathrm{HF}(\psi_t)$ where ψ_t is a contact-at-infinity system on $\mathbb{C}^n$. Then $c(\psi_t; a) = c(\phi_t\psi_t; \mathfrak{n}(a))$.* □

Consider $\mathrm{HF}(\psi_t R_{-\epsilon t})$ for $\epsilon \in (0, 1)$ and where ψ_t is compactly supported. It is well-known that this group is 1-dimensional over $\mathbb{Z}/2$. Indeed, the dimension is independent of the choice of ψ_t and one can find a representative so that $\psi_t R_{-\epsilon t}$ has a single non-degenerate orbit located at $z = 0$. It must therefore hold that the unit element is the sole non-zero element.

Let us then define $\mathrm{pt} \in \mathrm{HF}(\phi_t\psi_t R_{-\epsilon t})$ to be the image of 1 under $\mathfrak{n}$. We call this element the "point class" since $\phi_t\psi_t R_{-\epsilon t}$ has a positive slope at infinity (namely $1 - \epsilon$), and so the generator should be thought of as the minimum of a Morse function (rather than the unit element which should be thought of as the maximum of a Morse function). Since $\mathfrak{n}$ is an isomorphism, the point class generates $\mathrm{HF}(\phi_t\psi_t R_{-\epsilon t})$. This point class plays a role in §2.9.

To summarize, if the ideal restriction of φ_t equals R_{st} for some $s \in (0, 1)$, we use the naturality map $\mathfrak{n} : \mathrm{HF}(\phi_t^{-1}\varphi_t) \to \mathrm{HF}(\varphi_t)$ to define $\mathfrak{n}(1) = \mathrm{pt} \in \mathrm{HF}(\varphi_t)$.

2.9. *The spectral diameter of a ball.* This section proves that the spectral diameter of $B(a)$ equals a. The lower bound is well-known and follows from a simplified version of the construction in §2.6. It is then enough to prove that $\gamma(\psi_t) \leq a$ for all systems ψ_t compactly supported in $B(a)$. Without loss of generality, let us set $a = 1$, and fix a system ψ_t supported in $B(1)$.

The proof splits into five claims. The argument involves the naturality transformation $\mathfrak{n}$ from §2.8.1 which sends the unit class 1 to the class pt. It is important that 1 generates $\mathrm{HF}(R_{-st}\psi_t)$ and pt generates $\mathrm{HF}(R_{st}\psi_t)$ for any $s \in (0, 1)$ and any compactly supported ψ_t; i.e., there are two relevant Floer homologies and each is one-dimensional.

Claim 2.7. *Let $s \in (0, 1)$ and for $\delta > 0$ let $R_s = R_{\delta,s}$ be as in §2.1.5; then:*

(i) $\gamma(\psi_t) = 2s(1 + \delta/2) + c(R_{-st}\psi_t; 1) + c(R_{-st}\psi_t^{-1}; 1)$,
(ii) $c(R_{-st}\psi_t^{-1}; 1) = -c(\psi_t R_{st}; \mathrm{pt})$.

Let ϕ_t be the loop generated by $H = \pi|z|^2$, and suppose $s > 1/2$; then:

(iii) $c(R_{st}\psi_t;\mathrm{pt}) + c(\phi_t R_{-2st};1) \geq c(\phi_t R_{-st}\psi_t;\mathrm{pt})$,
(iv) $c(\phi_t R_{-st}\psi_t;\mathrm{pt}) = c(R_{-st}\psi_t,1)$, *and*
(v) $c(\phi_t R_{-2st};1) \leq (1-2s)(1+\delta/2)$.

Before we prove the claim, we use it to determine the spectral diameter of the ball $B(1)$. We estimate:

$$\begin{aligned}
\gamma(\psi_t) &= 2s(1+\delta/2) + c(R_{-st}\psi_t;1) + c(R_{-st}\psi_t^{-1};1)\\
&= 2s(1+\delta/2) + c(R_{-st}\psi_t;1) - c(\psi_t R_{st};\mathrm{pt})\\
&\leq 2s(1+\delta/2) - c(\phi_t R_{-st}\psi_t;\mathrm{pt}) + c(\phi_t R_{-2st};1) + c(R_{-st}\psi_t;1)\\
&= 2s(1+\delta/2) + c(\phi_t R_{-2st};1)\\
&\leq 2s(1+\delta/2) + (1-2s)(1+\delta/2) = 1+\delta/2.
\end{aligned}$$

The jth (in)equality uses the jth item in Claim 2.7. Since δ can be chosen arbitrarily small, we conclude $\gamma(\psi_t) \leq 1$. We now prove the five claims.

Proof of Claim 2.7. We begin by recalling the definition of the spectral norm:

$$\gamma(\psi_t) := \lim_{\epsilon\to 0} c(R_{-\epsilon t}\psi_t;1) + c(R_{-\epsilon t}\psi_t^{-1};1).$$

Therefore to prove (i) it is enough to prove the following identity for all systems ψ_t:

$$c(R_{-st}\psi_t;1) + (s-\epsilon)(1+\delta/2) = c(R_{-\epsilon t}\psi_t;1). \tag{5}$$

Let K_t generate ψ_t so that $R_{-\sigma t}\psi_t$ is generated by:

$$K_t^\sigma = K_t - \sigma(\mu_\delta(\pi|z|^2-1)+1), \tag{6}$$

where we use the fact that ψ_t and R_s commute with each other. The cut-off function μ_δ is defined in §2.1.5.

Observe that the 1-periodic orbits of the system $R_{-\sigma t}\psi_t$ are independent of $\sigma \in (0,1)$, and all remain in the ball $B(1)$; indeed, outside of the ball we have $R_{-\sigma t}\psi_t = R_{-\sigma t}$ which rotates with a non-zero speed because of our construction in §2.1.5. The action of the orbits depends on σ, as is clear from equation(6). Therefore a continuity argument (and the nowhere density of the action spectrum) implies that the spectral invariant must change according to:

$$\frac{\partial}{\partial\sigma} c(R_{-\sigma t}\psi_t;1) = -(1+\mu_\delta(\pi|z|^2-1)) = -(1+\delta/2),$$

This implies equation (5), and completes the proof of (i).

The second item (ii) follows from the duality formula proved in Lemma 2.4 and the fact that the only non-zero class in $\mathrm{HF}(\psi_t R_{st})$ is the point class pt.

The third item (iii) is an immediate consequence of the subadditivity of the spectral invariants proved in §2.1.9.

Item (iv) follows from the fact that the naturality transformation defined by ϕ_t is action preserving and it sends the unit $1 \in \mathrm{HF}(R_{-st}\psi_t)$ to the point

class pt $\in \mathrm{HF}(\phi_t R_{-st}\psi_t)$ which is the only non-zero class in this group; see §2.8 for further discussion.

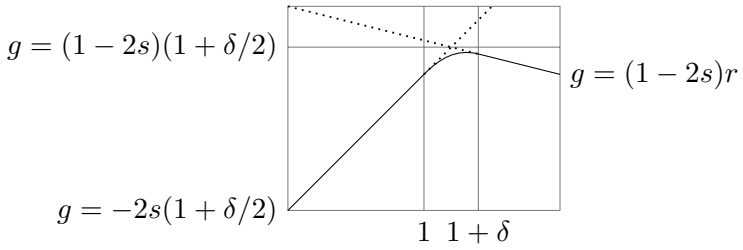


FIGURE 4. The graph of $g(r)$; its graph remains below the dashed lines $g = (1-2s)r$ and $g = r - 2s(1+\delta/2)$ and coincides with one of these lines when $r \notin (1, 1+\delta)$.

To establish (v) we argue as follows; first the Hamiltonian function G generating $\phi_t R_{-2st}$ is equal to:

$$G = \pi|z|^2 - 2s(\mu_\delta(\pi|z|^2 - 1) + 1) = g(\pi|z|^2);$$

see Figure 4 for the graph of $g(r)$.

The g-coordinate of the intersection of the two dashed lines in Figure 4 equals $g_* = (1-2s)(1+\delta/2)$, and by inspection of the shape of the graph we conclude that $G \leq g_*$. Since every orbit of $\phi_t R_{-2st}$ is either constant, assuming of course that $1/2 < s < 1$, or has the same action as the constant orbit located at $r = 0$, it follows that every orbit of $\phi_t R_{-2st}$ has action bounded from above by g_*; hence:

$$c(\phi_t R_{-2st}; 1) \leq g_* = (1-2s)(1+\delta/2),$$

which is what we wanted to show. □

2.10. *Comparison with the bound of Entov and Polterovich.* It is shown in [EP03] that the spectral diameter of $\mathbb{C}P^n$ is equal to the symplectic area of the class of a line, which we normalize to be 1. It also holds that $\mathbb{C}P^n$ contains a symplectically embedded copy of $B(1)$; let us fix the standard embedding $e : B(1) \to \mathbb{C}P^n$ whose image is the complement of the hyperplane divisor. Then, for any Hamiltonian system ψ_t with compact support in $B(1)$, we can implant ψ_t as a compactly supported system $e\psi_t e^{-1}$ in $\mathbb{C}P^n$, and know that $\gamma_{\mathbb{C}P^n}(e\psi_t e^{-1}) \leq 1$.

It is therefore natural to wonder whether $\gamma_{B(1)}(\psi_t) \leq \gamma_{\mathbb{C}P^n}(e\psi_t e^{-1})$ holds, as such a relation would imply our result from §2.9. However, this strategy does not work because:

Proposition 2.8. *There exist systems ψ_t supported in $B(1)$ so that:*

$$\gamma_{\mathbb{C}P^n}(e\psi_t e^{-1}) < \gamma_{B(1)}(\psi_t);$$

the difference between the two sides can be made arbitrarily close to 1.

Proof. Let ψ_t be the Hamiltonian system generated by a radial bump function $H = f(\pi|z|^2)$ where f is non-increasing, equals 1 on $[0, 1-\epsilon]$ and rapidly cuts off to zero so as to have compact support in $[0, 1)$.

It is well-known that $\gamma_{B(1)}(\psi_t) \geq 1 - \epsilon$; to prove this one observes:

(1) $c(-H; 1) \geq 0$, otherwise one could make the left hand side of:
$$c(-kH; 1) \leq kc(-H; 1)$$
arbitrarily negative, contradicting the known fact that the spectral capacity of $B(1)$ is bounded from above; see [Sey15, Lemma 13];
(2) $c(H; 1)$ is bigger than $1 - \epsilon$, since there exists a positive bump function K with supported in $B(1 - \epsilon)$ with $c(K; 1) = \max K = 1 - \epsilon$ and thus $1 - \epsilon \leq c(H; 1)$ by monotonicity of spectral invariants; the existence of such a bump function follows from the same argument given in §2.6.

On the other hand, by taking $\epsilon \to 0$, we arrange that $e\psi_t e^{-1}$ is supported in any chosen neighbourhood U of the divisor $\mathbb{C}P^{n-1}$, which is a single point when $n = 1$. The argument is finished by showing that:

$$\inf\{\gamma(U) : \mathbb{C}P^{n-1} \subset U\} = 0; \tag{7}$$

in words, the spectral diameter of small neighborhoods of the divisor is small. In the low-dimensional case $n = 1$, (7) follows from the displacement energy bound of §2.2, and the general case is established in [CZ24]; this completes the proof. □

2.11. *Spectral diameter of special balls in $\mathbb{C}P^n$.* The proof of Theorem 1.7 is based on the following two lemmas, in which we denote by the homology class in $M = \mathbb{C}P^n$ represented by the hyperplane by $\Gamma \in H_{2n-2}(M)$, the fundamental class by $[M]$ and the point class by [pt].

Lemma 2.9. *For any Hamiltonian system ψ_t on $M = \mathbb{C}P^n$, we have:*
$$\gamma(\psi_t) + c(\varphi_t\psi_t; [M]) - c(\varphi_t\psi_t; \Gamma) = 1;$$
where φ_t is the loop generated by S_a.

Introduce $U = \{S_a > 0\}$, i.e., U is the interior of the standard special ball of capacity $1 - a$. Then:

Lemma 2.10. *For any Hamiltonian system ψ_t supported in U, we have:*
$$c(\varphi_t\psi_t; [M]) - c(\varphi_t\psi_t; \Gamma) \geq a.$$
where φ_t is the loop generated by S_a.

From the above lemmas, it follows that the spectral diameter of U is at most $1 - a$. Combining this upper bound with the lower bound furnished by Proposition 1.5 yields Theorem 1.7.

In the proofs of both lemmas we allow ourselves to work with non-normalized Hamilltonians on $\mathbb{C}P^n$, using the conventions in §2.1.10.

2.11.1. *Proof of Lemma 2.9.* The proof is a simple application of the naturality transformation associated to the loop generated by S_a and the inverse loop generated by $\bar{S}_a$.

The naturality transformation associated to S_a sends the class represented by the maximum of a perfect Morse function to the class of the index $2n-2$ critical point. To see this, consider the perfect Morse function F obtained as a morsification of the Morse-Bott function $-\epsilon S_a$. One can arrange things so that $S_a + F$ is also a perfect Morse function with the same critical points.

The naturality transformation is defined on chain level by sending each underlying critical point to itself, and one observes that the maximum of F is sent to the index $2n-2$ critical point of $S_a + F$.

Finally, note that the orbits of $\tau(S_a + F) \approx \tau(1-\epsilon)S_a$ are constant for each $\tau \in [0,1]$, if the morsification is a small enough perturbation, and so the Morse indices of $S_a + F$ can be used to determine the Conley-Zehnder indices of the orbits. It follows that the index $2n-2$ critical point of $S_a + F$ is a chain level representative for Γ.

Consequently by Lemma 2.5:

$$c(H_t; [M]) = c(S_a \# H_t; \Gamma) + \text{const},$$

for every Hamiltonian H_t supported in U. The constant term is the action of a capped orbit of S_a and can be determined by sending $H_t \to 0$, in which case it must equal $c(0; [M]) - c(S_a; \Gamma)$. Since $S_a \approx (1-\epsilon)S_a$ can be perturbed to a perfect Morse function whose index $2n-2$ critical point has action approximately $-a$ (see Figure 2) and which represents Γ, it follows that $c(0; [M]) - c(S_a; \Gamma) = a$, and hence the above equation becomes:

$$c(H_t; [M]) = c(S_a \# H_t; \Gamma) + a. \tag{8}$$

The next stage of the argument is similar but is based instead on the naturality transformation generated by $\bar{S}_a = -S_a$. This naturality transformation sends the class of the maximum to the class of the minimum, and hence:

$$c(S_a \# H_t; [M]) = c(\bar{S}_a \# S_a \# H_t; [\text{pt}]) + \text{const} = c(H_t; [\text{pt}]) + \text{const}.$$

The constant term can again be determined by sending $H_t \to 0$, in which case it equals $c(S_a; [M])$ which equals $1 - a$, i.e., the critical value of the maximum as shown in Figure 2. Thus:

$$c(S_a \# H_t; [M]) = c(H_t; [\text{pt}]) + 1 - a. \tag{9}$$

To complete the proof add together (8) and (9), for H_t generating ψ_t. □

2.11.2. *Proof of Lemma 2.10.* Let $U = \{S_a > 0\}$, as in the statement, and suppose that H_t is a Hamiltonian function supported in U generating ψ_t.

For the purposes of the proof, we introduce two time-reparametrization operations on a Hamiltonian function G_t which do not affect spectral invariants:

$$G_t^* := \beta'(2t)G_{\beta(2t)} \text{ and } G_t^{**} := \beta'(2t-1)G_{\beta(2t-1)}.$$

Here $\beta : \mathbb{R} \to [0,1]$ is a standard smooth cut-off so $\beta(t) = 0$ for $t \leq 0$ and $\beta(t) = 1$ for $t \geq 1$; we require that $\beta'(t)$ is non-negative.

The significance of these operations is the following: G_t^* is supported where $t \in (0, 1/2)$ while G_t^{**} is supported where $t \in (1/2, 1)$.

That these operations do not affect spectral invariants is well-known, and boils down to the fact that G_t, G_t^*, G_t^{**} generate the same Hamiltonian isotopy up to time-reparametrization, and have identical space-time averages; see, e.g., [Sch00]. By the same arguments we must have

$$c(S_a \# H_t; \Pi) = c(S_a^* \# H_t^{**}; \Pi).$$

Now introduce the piecewise smooth function $K_a = \min\{S_a, 0\}$, as shown in Figure 5. Since K_a is pointwise less than S_a, it follows by the monotonicty of spectral invariants that, for any class Π, we have:

$$c(K_a^* \# H_t^{**}; \Pi) \leq c(S_a^* \# H_t^{**}; \Pi).$$

We note that K_a^* is not smooth, but nonetheless the spectral invariant $c(K_a^* \# H_t^{**}; \Pi)$ is well-defined via a limiting process, because of the Hofer continuity of spectral invariants.

The next stage of the argument is a deformation argument. Roughly speaking, the idea is to interpolate from K_a to 0 while keeping track of the indices and actions of the orbits during the process. To make this precise, we argue in a slightly ad hoc fashion.

Fix $0 < \epsilon < a$ and introduce the s-dependent family of functions:

$$T_s = \max\{K_a, sK_a - \epsilon\},$$

Note that on the set $\{K_a \geq -\epsilon\} = \{S_a \geq -\epsilon\}$, $T_s = T_1$. First smooth T_1 on this set so as to make $T_1^* \# H_t^{**}$ have non-degenerate orbits on $\{K_a \geq -\epsilon/2\}$.

For each s, smoothen T_s on $\{K_a \leq -\epsilon/2\}$ so that the only orbits outside in $\{K_a \leq -\epsilon/2\}$ are the Morse critical points of indices $\{0, 2, \ldots, 2n-2\}$ located near the divisor $\{K_a = -a\}$. Then, the orbits of $T_s^* \# H_t^{**}$ are described by the following list:

(1) orbits contained entirely in the region where $T_s^* \# H_t^{**} = T_1^* \# H_t^{**}$,
(2) orbits whose Floer homology grading is in

$$\{0, 2, \ldots, 2n-2\},$$

located near the divisor $\mathbb{C}P^{n-1}$.

Because the minimal Chern number of $\mathbb{C}P^n$ is $n+1$, it follows that the orbits of type (2) never appear in a linear combination representing $[M]$, which has degree $2n$. Consequently $c(T_s^*\#H_t^{**};[M])$ is independent of s, since it is valued in the s-independent nowhere dense spectrum of orbits of type (1). Note that $T_0^*\#H_t^{**}$ is ϵ-close to H_t^{**}, and thus we conclude that:

$$c(K_a^*\#H_t^{**};[M]) = c(T_1^*\#H_t^{**};[M]) = c(H_t;[M]) + O(\epsilon).$$

Since ϵ was arbitrary we conclude $c(K_a^*\#H_t^{**};[M]) = c(H_t;[M])$.

Combining the two steps, with $\Pi = [M]$, we conclude:

$$c(S_a\#H_t;[M]) = c(S_a^*\#H_t^{**};[M]) \geq c(K_a^*\#H_t^{**};[M]) = c(H_t;[M]). \tag{10}$$

Next, we recall (8), so that:

$$c(S_a\#H_t;[M]) - c(S_a\#H_t;\Gamma) = c(S_a\#H_t;[M]) - c(H_t;[M]) + a \geq a,$$

where we use (10) in the final step. This completes the proof. □

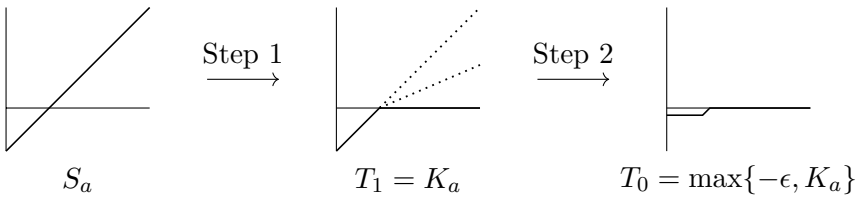


FIGURE 5. Deformation used in the proof of Lemma 2.10.

References

[AAC23] H. Alizadeh, M. S. Atallah, and D. Cant. Lagrangian intersections and the spectral norm in convex-at-infinity symplectic manifolds. arXiv:2312.14752, 2023.

[BC24] F. Brocić and D. Cant. Bordism classes of loops and Floer's equation in cotangent bundles. *J. Fixed Point Theory Appl.*, 26:1–29, 2024.

[BP94] M. Bialy and L. Polterovich. Geodesics of Hofer's metric on the group of Hamiltonian diffeomorphisms. *Duke Math. J.*, 76(1):273–292, 1994.

[CHLS07] K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk. Quantitative symplectic geometry. In B. Hasselblatt, editor, *Dynamics, Ergodic Theory and Geometry*, volume 54 of *MSRI Publications*, pages 1–44. Cambridge University Press, 2007.

[CZ24] D. Cant and J. Zhang. On the spectral capacity of submanifolds. arXiv:2409.14142, September 2024. Submitted.

[DS93] S. Dostoglou and D. Salamon. Instanton homology and symplectic fixed points. *London Math. Soc. Lecture Note Ser.*, 192:53–93, 1993.

[EH90] I. Ekeland and H. Hofer. Symplectic topology and Hamiltonian dynamics II. *Math. Z.*, 203:553–567, 1990.

[EP03] M. Entov and L. Polterovich. Calabi quasimorphism and quantum homology. *Int. Math. Res. Not.*, 2003:1635–1676, 2003.

[FGS05] U. Frauenfelder, V. Ginzburg, and F. Schlenk. Energy capacity inequalities via an action selector. In *Geometry, spectral theory, groups, and dynamics*, volume 387 of *Contemp. Math.*, pages 129–152. Amer. Math. Soc., 2005.

[FS07] U. Frauenfelder and F. Schlenk. Hamiltonian dynamics on convex symplectic manifolds. *Isr. J. Math.*, 159:1–56, 2007.

[GH18] J. Gutt and M. Hutchings. Symplectic capacities from positive S^1-equivariant symplectic homology. *Algebr. Geom. Topol.*, 18(6):3537–3600, 2018.

[Gin05] V. Ginzburg. The Weinstein conjecture and theorems of nearby and almost existence. In *The Breadth of Symplectic and Poisson Geometry: Festschrift in Honor of Alan Weinstein*, Progr. Math., pages 139–172. Birkhäuser, 2005.

[GR25] J. Gutt and V. GB Ramos. Characterizing symplectic capacities on ellipsoids. *Revista Matematica Iberoamericana*, 2025.

[Gro85] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.*, 82:307–347, 1985.

[GT23] Y. Ganor and S. Tanny. Floer theory of disjointly supported Hamiltonians on symplectically aspherical manifolds. *Alg. Geom. Topol.*, 23(2):645–732, 2023.

[Her98] D. Hermann. Non-equivalence of symplectic capacities for open sets with restricted contact type boundary. Université Paris XI, Orsay, 1998.

[HLRS16] V. Humilière, F. Le Roux, and S. Seyfaddini. Towards a dynamical interpretation of Hamiltonian spectral invariants on surfaces. *Geom. Topol.*, 20(4):2253–2334, 2016.

[HZ94] H. Hofer and E. Zehnder. *Symplectic Invariants and Hamiltonian Dynamics*. Birkhäuhser Advanced Texts, 1994.

[Ish16] S. Ishikawa. Spectral invariants of distance functions. *J. Topol. Anal.*, 08(4):655–676, 2016.

[KS21] A. Kislev and E. Shelukhin. Bounds on spectral norms and barcodes. *Geom. Topol.*, 25:3257–3350, 2021.

[LZ18] R. Leclercq and F. Zapolsky. Spectral invariants for monotone Lagrangians. *J. Topol. Anal.*, 10:627–700, 2018.

[Mai22] P-A. Mailhot. The spectral diameter of a Liouville domain. arXiv:2205.04618, 2022.

[Oh05] Y-G. Oh. Spectral invariants, analysis of the Floer moduli space, and geometry of the Hamiltonian diffeomorphism group. *Duke Math. J.*, 126(1):199–295, 2005.

[Pol01] L. Polterovich. *The geometry of the group of symplectic diffeomorphisms*. Birkhäuser, 2001.

[Sch95] M. Schwarz. Cohomology operations from S^1-cobordisms in Floer homology. `https://www.math.uni-leipzig.de/~schwarz/diss.pdf`, 1995.

[Sch00] M. Schwarz. On the action spectrum for closed symplectically aspherical manifolds. *Pacific J. Math.*, 193:419–461, 2000.

[Sch05] F. Schlenk. *Embedding problems in symplectic geometry*. Walter de Gruyter, 2005.

[Sei97] P. Seidel. π_1 of symplectic automorphism groups and invertibles in quantum homology rings. *Geom. funct. anal.*, 7:1046–1095, 1997.

[Sei15] P. Seidel. The equivariant pair-of-pants product in fixed point Floer cohomology. *Geom. Funct. Anal.*, 25:942–1007, 2015.

[Sey15] S. Seyfaddini. Spectral killers and Poisson bracket invariants. *J. Mod. Dyn.*, 9:51–66, 2015.

[Tan22] S. Tanny. A max inequality for spectral invariants of disjointly supported Hamiltonians. *J. Symp. Geom.*, 20(5):1159–1213, 2022.

[Ush08] M. Usher. Spectral numbers in Floer theories. *Compos. Math.*, 144:1581–1592, 2008.

[Ush10a] M. Usher. Duality in filtered Floer-Novikov complexes. *J. Topol. Anal.*, 2(2):233–258, 2010.

[Ush10b] M. Usher. The sharp energy-capacity inequality. *Comm. Cont. Math*, 12(3):457–473, 2010.

[Vit92] C. Viterbo. Symplectic topology as the geometry of generating functions. *Math. Ann.*, 292(1):685–710, 1992.

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