LENGTHS OF REEB CHORDS AND VITERBO RESTRICTION

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Abstract. Let Λ be a Legendrian in the contact boundary of a Liouville domain Ω. We explain how the non-existence of Reeb chords with endpoints on Λ of length up to a enables one to embed $D_\epsilon T^*\Lambda \times D(a)$ into Ω in an exact way. As in earlier work of Zhengyi Zhou, we use the Viterbo restriction map to deduce a contradiction in certain cases. In particular, we show that if M admits a submersion from a product of spheres (e.g., the n-torus), then all compact Legendrians $\Lambda \subset ST^*M$ admit a Reeb chord for every choice of contact form ST^*M. The obstruction we use in this case is based on the idea of inverting the degree-n classes in cohomology, and is similar to the notion of string point invertibility introduced by Egor Shelukhin.

1. Introduction

1.1. The notion of n-invertibility. Suppose that Ω^{2n} is a Liouville domain and consider a Legendrian submanifold $\Lambda \subset \partial\Omega$. The primary goal of this paper is to bound the length of the shortest non-constant Reeb chord of Λ from above. Here the Reeb flow is associated to the canonical contact form on $\partial\Omega$.

Let W denote the Liouville manifold obtained by completion of Ω. Our main results are phrased in terms of the symplectic cohomology $SH(W)$, and its additional structures:

(1) $PSS : H^*(W) \to SH(W)$,
(2) BV operator $\Delta : SH(W) \to SH(W)$,
(3) pair-of-pants product $* : SH(W) \otimes SH(W) \to SH(W)$.

The symplectic cohomology $SH(W)$ and its three additional structures are well-known; see, e.g., [Vit99, Vit96, Sei08, Abo15, Zha16, Zho21, Zho22a, Zho22b]. For further details on the conventions we use, see [BC25].

This leads us to our main definition:

Definition 1. *A Liouville manifold W is* n-invertible *if the only Δ-invariant ideal of $SH(W)$ which contains the subspace $PSS(H^n(W))$ is equal to $SH(W)$.*

Here we comment that n-invertible domains always have dimension $2n$, i.e., we adopt the somewhat common convention in symplectic geometry that n

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stands for "half of the dimension." The definition is closely related to that of *string point invertibility* introduced in [She22].

Let us set some terminology before proceeding to our main theorem.

- The domain Ω is completed to a *Liouville manifold* W, which means that $\Omega \subset W$ is embedded in such a way that the Liouville flow lines passing through the contact boundary Ω define the symplectization end SY of W (here $Y \simeq \partial\Omega$ is the ideal boundary).
- The *radial function* r is 1-homogeneous $dr(Z) = r$ and satisfies $r|_{\partial\Omega} = 1$. This function is smooth on the symplectization end SY and extends continuously to the rest of Ω.
- The *Reeb vector field* is defined to be $R = X_r$ (defined on SY).
- The *Floer cohomology group* $SH_c(\Omega)$ is defined as the "category theoretic limit" of Floer cohomologies of Hamiltonian systems whose generating vector field agree with cR outside of a compact set. We restrict c to be in the complement of the *spectrum* $Spec(\Omega)$ of periods of R. We assume throughout the body of the paper that Floer cohomologies are computed using coefficients in a field of characteristic two.

The spaces $SH_c(\Omega)$, for c in the complement of the spectrum, form a *persistence module* (in the sense of [PS16]); the colimit of this persistence module is easily seen to be the symplectic cohomology $SH(W)$.

The three structures $PSS, \Delta, *$ can be lifted to the persistence module as:

(1) natural maps $PSS : H^*(W) \to SH_c(\Omega)$ for $c > 0$,
(2) natural endomorphisms $\Delta : SH_c(\Omega) \to SH_c(\Omega)$,
(3) natural products $* : SH_{c_1}(\Omega) \otimes SH_{c_2}(\Omega) \to SH_{c_1+c_2}(\Omega)$.

We now attempt to "quantify" the notion of n-invertibility; first set:

$$\mathcal{I}_{0,c} = \operatorname{span}\left\{PSS(H^n(\Omega)) * SH_{c'}(\Omega) : c' < c\right\},$$

where $PSS(H^n(\Omega))$ takes values in $SH_{c-c'}(\Omega)$. Similarly define for $k \geq 1$:

$$\mathcal{I}_{k,c} = \mathcal{I}_{k-1,c} + \operatorname{span}\left\{\Delta_{c_1}(\mathcal{I}_{k-1,c_1}) * SH_{c_2} : c_i > 0 \text{ and } c_1 + c_2 = c\right\},$$

and let $\mathcal{I}_c \subset SH_c(\Omega)$ be the union of $\mathcal{I}_{k,c}$ over all k. Note that, since $SH_c(\Omega)$ is a finite dimensional vector space, the colimit $\mathcal{I}_c = \mathcal{I}_{c,k}$ is attained for some finite number k. This subspace $\mathcal{I}_c \subset SH_c(\Omega)$ is a natural subspace of the persistence module (i.e., continuation maps send $\mathcal{I}_{c'}$ into $\mathcal{I}_c$ for $c' < c$), and so the colimit of $\mathcal{I}_c$ is a subspace of the symplectic cohomology $\mathcal{I} \subset SH(W)$.

Lemma 1.1. *The ideal $\mathcal{I}$ is the smallest Δ-invariant ideal of $SH(W)$ which contains the subspace $PSS(H^n(W))$.*

Proof. Define ideals $\mathcal{I}_0 = PSS(H^n(W)) * SH(W)$, and:

$$\mathcal{I}_k = \mathcal{I}_{k-1} + \Delta(\mathcal{I}_{k-1}) * SH(W),$$

and then $\mathfrak{I}_k$ is easily seen to be the colimit of $\mathfrak{I}_{c,k}$ as $c \to \infty$, and since colimits commute with colimits, the monotone union $\mathfrak{I}_0 \cup \mathfrak{I}_1 \cup \dots$ is equal to the colimit $\mathfrak{I}$. From this description of $\mathfrak{I}$ we conclude the desired result. □

This leads to our second main definition in terms of the persistence refinement of the ideal $\mathfrak{I}$:

Definition 2. *The n-invertibility capacity of a Liouville domain Ω is:*

$$c_{\mathrm{ni}}(\Omega) = \inf \{c > 0 : PSS(1) \in \mathfrak{I}_c\}.$$

Then $c_{\mathrm{ni}}(\Omega) < \infty$ if and only if the completion W is n-invertible.

Our main result is:

Theorem 3. *Every Legendrian $\Lambda \subset \partial\Omega$ bounds a non-constant Reeb chord with period at most $c_{\mathrm{ni}}(\Omega)$, provided $c_{\mathrm{ni}}(\Omega) < \infty$.*

1.2. Examples of n-invertible manifolds.

1.2.1. *Domains with vanishing symplectic cohomology.* If $SH(W) = 0$, then the conclusion of Theorem 3 holds since the condition from Definition 1 is trivially satisfied. This recovers a result of Zhou [Zho22a].

1.2.2. *Cotangent bundles of products of spheres.*

Theorem 4. *For $n \geq 1$ we construct classes:*

- $A \in H_n(\Lambda S^n)$ *and,*
- $B \in H_{2n-1}(\Lambda S^n)$,

such that: $A * \Delta(B * [pt]) = [S^n]$. More generally, if $M = S^{n_1} \times \dots \times S^{n_k}$ with $n = n_1 + \dots + n_k$, there are classes:

$$A_i \in H_n(\Lambda M) \quad B_i \in H_{n+n_i-1}(\Lambda M) \text{ for } i = 1, \dots, k$$

such that the sequence C_i, $i = 0, \dots, k$, defined by $C_0 = [\mathrm{pt}]$:

$$C_i = A_i * \Delta(B_i * C_{i-1}) \text{ for } i = 1, \dots, k$$

satisfies $C_k = [S^{n_1} \times \dots \times S^{n_k}]$.

This proves that $T^*(S^{n_1} \times \dots \times S^{n_k})$ is n-invertible,[1] due to the morphism of BV-algebras established in [Vit99, Vit96, AS06, Abo15, BC25]:

$$H_*(\Lambda M) \to SH(T^*M)$$

sending $[pt]$ to $PSS(F)$ where F is the cohomology class represented by the cotangent fiber. Moreover, our argument also gives estimates on the n-invertibility capacity in terms of the lengths of the loops appearing in the classes $A, B_1, \dots, B_k$, where the length functional is the one determined by the domain $\Omega \subset T^*M$ considered in [BC25].

[1] For odd-dimensional spheres, one can appeal to formulas in [Men09, Theorem 16].

1.3. Rationality constants of non-exact Lagrangians. Let Ω be a Liouville domain. Recently, there has been interest [Zho22a, ACLS24] in bounding on the *rationality constant* of a non-exact Lagrangian inside Ω. Here we recall that a Lagrangian $L \subset \Omega$ determines a Liouville class $[\lambda|_L] \in H^1_{\mathrm{dR}}(L)$, and the *rationality constant* $\rho(L)$ is the infimal positive period of this class.

Our second main result is the following bound on the rationality constants of *aspherical*[2] Lagrangians inside of T^*T^n:

Theorem 5. *Let $\Omega \subset T^*T^n$ be the unit codisk bundle. Then there exists a constant $C > 0$ such that if $L \subset \Omega$ is a closed, non-exact, Lagrangian which is aspherical then $\rho(L) \leq C$.*

It is noteworthy that our result applies to T^*T^n, whereas previous results concerning aspherical Lagrangians L such as [Zho22a, Li23] do not apply when the ambient space is T^*T^n (but rather apply when the ambient space behaves like the cotangent bundle of a non-aspherical manifold).

Remark 1.2. *The constant C can be chosen to be any number $2c$ such that there exist classes:*

$$A_i^{\pm} \in SH_{c_{i,\pm}}(\Omega) \text{ for } c_{i,\pm} > 0 \text{ with } c_{i,+} + c_{i,-} \leq c$$

such that:

- *$A_i^{\pm}$ lies in the free homotopy class of the loop $t \mapsto x \pm te_i$,*
- *$A_i^+ * A_i^- = 1$ holds in $SH_c(\Omega)$.*

For related bounds, see [BC25].

1.4. Methods of proof. The first step used in the proof of Theorem 3 is a variation of the geometric construction similar to [Moh01] and [Ops25, Proposition 4.4]. The result can be considered as a version of the 1-jet neighborhood theorem for Legendrians.

Theorem 6. *Let Ω be a Liouville domain inducing a radial coordinate r on W, and let $\alpha = \lambda/r$ be the resulting contact form on the ideal boundary Y of the completion W. Let $\Lambda_t \subset Y$ be a Legendrian isotopy, and suppose that:*

- *for each smooth path $y(t) \in \Lambda_t$ it holds that $y^*\alpha > a\,dt$,*
- *$\Lambda_t \cap \Lambda_{t'} = \emptyset$ for each $t \neq t'$ in $[0, 1]$.*

Then there is an exact embedding:

$$D_\epsilon T^*\Lambda \times D(a) \to \Omega$$

taking values in the interior of Ω. Here $D_\epsilon T^*\Lambda$ is an arbitrarily small disk bundle in $T^*\Lambda$, $D(a)$ is the closed disk of area a, and $D_\epsilon T^*\Lambda \times D(a)$ is equipped with its standard Liouville structure.

[2] A manifold L is said to be aspherical if its universal cover is contractible.

It is clear that if Λ has no Reeb chords for the contact form α (as in the statement of Theorem 6) of periods up to a, then we can find an isotopy Λ_t satisfying the hypotheses of Theorem 6, and thereby conclude an exact embedding $D_\epsilon T^*\Lambda \times D(a) \to \Omega$.

The next step is to appeal to the well-known *Viterbo restriction* (or *Viterbo transfer*) map, whose properties are summarized in the following theorem:

Theorem 7. *If K, Ω are smooth Liouville domains and $\iota : K \to \Omega$ is an exact embedding taking values in the interior of Ω, then, for each pair of numbers $0 < c < b$ such that $c \notin Spec(\Omega)$ and $b \notin Spec(K)$, there is a linear map:*

$$\mathfrak{R} : SH_c(\Omega) \to SH_b(K).$$

Moreover, this map respects structures:

- $\mathfrak{R} \circ \Delta = \Delta \circ \mathfrak{R}$,
- $\mathfrak{R}(\zeta_1 * \zeta_2) = \mathfrak{R}(\zeta_1) * \mathfrak{R}(\zeta_2)$,
- $\mathfrak{R} \circ PSS = PSS \circ \mathfrak{R}$,

where $\mathfrak{R} : H^*(\Omega) \to H^*(K)$ is the pullback on cohomology in the last item.

Thus, given a Legendrian Λ which has no chords of length up to a, every smooth Liouville subdomain $K \subset D_\epsilon T^*\Lambda \times D(a)$ receives a map:

$$\mathfrak{R} : SH_c(\Omega) \to SH_b(K)$$

compatible with structures, as in Theorem 7. In particular, if a is larger than $c_{\mathrm{ni}}(\Omega)$, then we can pick $c_{\mathrm{ni}}(\Omega) < c < b < a$ so that:

$$1 = 0 \in SH_b(K).$$

This is because 1 lies in $\mathcal{I}_c(\Omega)$, and $\mathfrak{R}(\mathcal{I}_c(\Omega))$ vanishes in $SH_b(K)$ since:

$$\mathfrak{R} : H^n(\Omega) \to H^n(K)$$

vanishes (as there are no non-trivial degree n classes in $D_\epsilon T^*\Lambda \times D(a)$).

The next step uses the following filtered Künneth map result:

Theorem 8. *Let $Q \times D(a)$ be a Cartesian product of a Liouville domain with a disk $D(a)$. Then, for any number $0 < b < a$ not in $Spec(Q)$, there exists a Liouville subdomain of $K \subset Q \times D(a)$ and an isomorphism:*

$$\mathfrak{K} : SH_b(K) \to SH_b(Q)$$

such that $\mathfrak{K}(PSS(1)) = PSS(1)$, and such that $b \notin Spec(K)$. In particular, if the $PSS(1)$ is zero in $SH_b(K)$, then it is also zero in $SH_b(Q)$.

Combining all these steps, we conclude that if Λ admits no Reeb chords of length up to some number $a > c_{\mathrm{ni}}(\Omega)$, then the unit vanishes in $SH(T^*\Lambda)$. However, due to the comparison between string topology and symplectic cohomology, the unit cannot vanish in $SH(T^*\Lambda)$. This provides the desired contradiction, and proves Theorem 3, (modulo Theorems 6, 7, and 8). Moreover, the argument actually gives a stronger statement; we first introduce a definition:

Definition 1.3. *Let (Y, α) be a compact cooriented contact manifold with contact form α. If $\Lambda_s \subset Y$, $s \in [0, \infty)$, is a Legendrian isotopy, satisfying:*

- *for each smooth path $y(s) \in \Lambda_s$ it holds that $y^*\alpha \geq ds$,*

then we say Λ_s is uniformly fast relative α. *If Λ_s is uniformly fast relative some unspecified contact form, then we simply say it is* uniformly fast.

Theorem 9. *Let Ω be a Liouville domain inducing a radial coordinate r on W, and let $\alpha = \lambda/r$ be the resulting contact form on the ideal boundary Y of the completion W. Let $\Lambda_s \subset Y$, $s \in [0, \infty)$, be a Legendrian isotopy which is uniformly fast relative α, then there exists $0 \leq t_1 < t_2 \leq c_{\mathrm{ni}}(\Omega)$ such that $\Lambda_{t_1} \cap \Lambda_{t_2} \neq \emptyset$, provided $c_{\mathrm{ni}}(\Omega) < \infty$.*

Proof. This follows from the combination of Theorems 6, 7, and 8. □

Because the Reeb flow $\Lambda_s = R_s(\Lambda)$ is uniformly fast relative α, Theorem 9 implies $R_{t_1}(\Lambda)$ intersects $R_{t_2}(\Lambda)$ for $0 \leq t_1 < t_2 \leq c_{\mathrm{ni}}(\Omega)$, which implies Theorem 3 (since $\Lambda \cap R_{t_2 - t_1}(\Lambda) \neq \emptyset$).

Using Theorem 9 we prove the following in §2.3.

Theorem 10. *Suppose there is a submersion $E \to M$, where E is a compact manifold, and T^*E is n-invertible, e.g., E could be a product of spheres. Then any uniformly fast Legendrian isotopy $\Lambda_s \subset ST^*M$, $s \in [0, \infty)$ satisfies $\Lambda_{t_1} \cap \Lambda_{t_2} \neq \emptyset$ for some $0 \leq t_1 < t_2$. In particular, the Arnol'd chord conjecture holds for ST^*M.*

Turning now to Theorem 4, the method of proof is to use the framework of [BC25], which allows one to prove the desired relations in the BV algebra $H_*(\Lambda M)$ by direct manipulation of smooth maps taking values in M. The proof is entirely topological and all Floer theory is isolated in our earlier work [BC25].

Finally we discuss the proof of Theorem 5 concerning the rationality constants of aspherical Lagrangians L inside of T^*T^n. The key idea is again to use a Viterbo restriction map; however, since the Lagrangian L is non-exact, Weinstein neighborhoods of L are not embedded into T^*T^n in an exact way, and so one cannot appeal directly to Theorem 7. One instead appeals to the so-called *truncated Viterbo restriction map* introduced by [Zho22a].

Theorem 11. *Let $\iota : K \subset \Omega$ be a symplectic embedding of a Liouville domain such that the periods of the closed 1-form $\iota^*\lambda_\Omega - \lambda_K$ lie in the subgroup $\rho\mathbb{Z} \subset \mathbb{R}$ for some $\rho > 0$. Then, for any numbers $0 < c < b$ such that $b + c < \rho$ there is a restriction map $\mathfrak{R} : SH_c(\Omega) \to SH_b(K)$ such that:*

- $\mathfrak{R} \circ \Delta = \Delta \circ \mathfrak{R}$,
- $\mathfrak{R}(\zeta_1 * \zeta_2) = \mathfrak{R}(\zeta_1) * \mathfrak{R}(\zeta_2)$,
- $\mathfrak{R} \circ PSS = PSS \circ \mathfrak{R}$,

as in Theorem 7, whenever these expressions make sense; i.e., in the second item we assume that $\zeta_i \in SH_{c_i}(\Omega)$ *and* $\mathfrak{R}(\zeta_i) \in SH_{b_i}(K)$ *for* $i = 1, 2$ *with* $0 < c_i < b_i$ *and* $c_1 + c_2 + b_1 + b_2 < \rho$.

Finally, if ϖ *denotes some collection of free homotopy classes in* Ω*, then* $\mathfrak{R}$ *satisfies:*

- $\mathfrak{R}(SH_c(\Omega, \varpi)) \subset SH_b(K, \iota^*\varpi \cap \kappa)$,

where κ *is the collection of free homotopy classes in the kernel of* $\iota^*\lambda_\Omega - \lambda_K$.

As in [Zho22a], given a Lagrangian $L \subset \Omega$ we apply Theorem 11 to a small Weinstein neighborhood of L. The next step in the argument is to apply the truncated Viterbo restriction map $\mathfrak{R}$ to classes $A_i^{\pm}$ as in Remark 1.2 to conclude classes in $SH(T^*L, \kappa)$ (assuming the rationality constant of L is large enough). Using the fact that L is aspherical enables the use of two algebraic topology results [Lat15, Lemma 5.1 and 5.2], which we use to show $\pi_1(L, pt)$ admit a subgroup $\mathbb{Z}^n$ whose associated covering space is open. This leads to a contradiction, since the covering space is also an aspherical n-manifold, and therefore must be homotopy equivalent to $K(\mathbb{Z}^n, 1) \simeq T^n$ which has non-zero nth homology group (and so cannot be open). The details of this argument are given in §2.5.

Remark 1.4. *As in Theorem 11, it is sometimes convenient to appeal to the direct sum decomposition of* $SH_c(\Omega)$ *into the summands generated by orbits in a given collection of free homotopy classes* ϖ*. It will be obvious from the construction that the Viterbo restriction map associated to an exact embedding* $\iota : K \to \Omega$ *respects this decomposition:* $\mathfrak{R}(SH_c(\Omega, \varpi)) \subset SH_b(K, \iota^*\varpi)$*. We do not need to use this fact in the proof of Theorem 9, but it is useful in certain cases. For instance, if there is an invertible element in* $SH(W, \varpi)$*, then every Legendrian in* $\partial\Omega$ *such that* $\iota^*\varpi = \emptyset$ *must have a Reeb chord.*

1.5. Survey of related results. The chord conjecture due to Arnol'd appears in [Arn86, pp. 15], and it is formulated for S^3 with the standard contact structure. Theorem 10 solves the Arnol'd chord conjecture for $\Lambda \subset ST^*M$ when it applies.

1.5.1. *Higher dilations.* If one allows constraints on the topology of Λ relative the topology of M, then one can deduce quite a bit by combining the following two propositions:

Proposition 1.5 ([Zha16, Corollary 1.1.6] using [Goo85])**.** *A compact orientable manifold* M *is rationally essential*[3] *if and only if the unit element*

[3]Every connected manifold has a classifying map $M \to BG$ where $G = \pi_1(M, pt)$; if the induced map in rational homology kills $[M]$ (we assume M is orientable so $[M]$ is well-defined), then we say M is rationally inessential; otherwise M is said to be rationally essential. Rationally inessential manifolds contain all simply connected manifolds, and form an ideal with respect to Cartesian product (in the space of orientable manifolds).

is non-zero in the S^1-equivariant symplectic homology $SH^{S^1}(M, \mathbb{Q}, \mathfrak{L})$, defined with rational coefficients and twisted by the local coefficient system $\mathfrak{L}$ obtained by transgressing the second Stiefel Whitney class of TM. □

Remark 1.6. *The vanishing of the unit is often expressed as the existence of a "higher dilation." Dilation classes were introduced in* [SS12, Sei14] *and further developed from the framework of S^1-equivariant homology by* [Zha16, Zho21, Zho22b, Zho22a, Li23, Li24].

We note that [ACF16, FP17] *also appeal to the result of* [Goo85]. *In* [FP17] *it is shown that the π_1-sensitive Hofer–Zehnder capacity is finite in the cotangent bundles of all compact rationally inessential manifolds.*

Proposition 1.7 (Truncated Viterbo Restriction [Zho22a])**.** *If $\Lambda \subset \partial\Omega$ is a Legendrian without Reeb chords, there is a restriction morphism:*

$$SH^{S^1}(\Omega, \mathbf{k}, \mathfrak{L}) \to SH^{S^1}(T^*(\Lambda \times S^1), \mathbf{k}, \mathfrak{L}'),$$

which sends the unit element to the unit element; here $\mathbf{k}$ is a coefficient field. The local system $\mathfrak{L}'$ is obtained by restriction the local system $\mathfrak{L}$ to a Weinstein neighborhood of a Lagrangian embedding $\Lambda \times S^1 \to \Omega$. □

By appealing to [Zha16], [Zho22a] concludes:

Claim 1.8 ([Zho22a])**.** *If Λ is an aspherical Legendrian in the unit cotangent bundle of a compact orientable rationally inessential manifold M, then Λ has a Reeb chord for every choice of contact form.* □

Summary of proof. Using Proposition 1.7 and Claim 1.5, one obtains:

$$\text{unit vanishes in } SH^{S^1}(T^*(\Lambda \times S^1), \mathbb{Q}, \mathfrak{L}'),$$

where $\mathfrak{L}'$ is obtained by transgressing some class in $H^2(\Lambda \times S^1, \mathbb{Z}/2\mathbb{Z})$. This is impossible if $\Lambda \times S^1$ is aspherical, as we now explain.

Abbreviate $N = \Lambda \times S^1$. The first step is to appeal to [Abo15] to conclude there is a commutative diagram:

$$\begin{array}{ccc} H_*(N, \mathfrak{o} \otimes \mathbb{Q}) & \xrightarrow{\iota} & H_*(\Lambda N, \mathbb{Q}, \mathfrak{o} \otimes \mathfrak{L}' \otimes \mathfrak{L}) \\ \downarrow \simeq & & \downarrow \simeq \\ H^*(T^*N, \mathbb{Q}) & \xrightarrow{PSS} & SH(T^*N, \mathbb{Q}, \mathfrak{L}'). \end{array}$$

The local system $\mathfrak{L}$ is also obtained by transgressing a class in $H^2(N, \mathbb{Z}/2\mathbb{Z})$, while $\mathfrak{o}$ is the orientation local system for the base manifold. The top arrow ι is an isomorphism onto the contractible summand, since N is aspherical. It follows that the PSS map is also an isomorphism onto the contractible summand.

This implies that the equivariant PSS map:

$$H^*(T^*N, \mathbb{Q}) \otimes \mathbb{Q}[u^{-1}] \to SH^{S^1}(T^*N, \mathbb{Q}, \mathfrak{L}')$$

cannot kill the unit, by a spectral sequence argument applied to the u-adic filtration inherent to S^1-equivariant homology; see [Zha16, Zho22b, Li24]. We recall the argument for the reader's convenience. The contractible summand of $SH^{S^1}(T^*N, \mathbb{Q}, \mathfrak{L}')$ is the homology of a complex:

$$(C_0 \oplus C_+) \otimes \mathbb{Q}[u^{-1}]$$

with deformed differential:

$$d_{S^1} = d + u\Delta + u^2\delta_2 + u^3\delta_3 + \dots$$

where δ_i can be considered as "higher order" BV operators. One sets up the chain complex so that:

- (C_0, d_0) is a subcomplex whose homology computes $H^*(T^*N, \mathbb{Q})$,
- $H(C_0 \oplus C_+, d_0)$ is the contractible summand of $SH(T^*N, \mathbb{Q}, \mathfrak{L}')$,
- the inclusion of C_0 into $C_0 \oplus C_+$ induces the map PSS on homology,
- all higher order terms $\Delta, \delta_2, \delta_3, \dots$ vanish on $C_0 \otimes \mathbb{Q}[u^{-1}]$.

Suppose we have a solution of:

$$d_{S^1}(u^{-k}\zeta_k + \dots + u^{-1}\zeta_1 + \zeta_0) = 1 \in C_0.$$

If $k > 0$ then $d_0\zeta_k = 0$. Since $C_0 \to C_0 \oplus C_+$ is an isomorphism on d_0 homology, we can assume that $\zeta_k = \eta_k + d_0\mu_k$ for $\eta_k \in C_0$. Then, adding $0 = d_{S^1}(d_{S^1}(u^{-k}\mu_k))$ to both sides yields:

$$d_{S^1}(u^{-k}\eta_k + u^{-k+1}(\zeta_{k-1} + \Delta\mu_k) + \dots + u^{-1}(\zeta_1 + \delta_{k-1}\mu_k) + \zeta_0 + \delta_k\mu_k) = 1$$

Now, since all the higher terms in d_{S^1} vanish on $\eta_k \in C_0$, we can simply remove it from the equation and conclude:

$$d_{S^1}(u^{-k+1}(\zeta_{k-1} + \Delta\mu_k) + \dots + u^{-1}(\zeta_1 + \delta_{k-1}\mu_k) + \zeta_0 + \delta_k\mu_k) = 1.$$

By repeating this argument we conclude:

$$d_0(\zeta_0 + \delta_k\mu_k + \delta_{k-1}\mu'_{k-1} + \dots) = 1,$$

contradicting the fact that 1 is not an exact cycle with respect to d_0. □

1.5.2. *Orientation tricks.* Let us note one curiously general example:

Claim 1.9. *Let M be a compact manifold such that $w_2(M)$ evaluates non-trivially on some element in $\pi_2(M)$. For example, $M = N \times \mathbb{C}P^{2n}$ for any positive integer n and any compact smooth manifold N. Then every spin Legendrian in ST^*M, admits a Reeb chord for all contact forms.*

Proof. By [AFO17] (building on [Sei10] in the case $M = \mathbb{C}P^2$), it holds that $SH(T^*M, \mathbb{Q}) = 0$ and so the same argument used in Claim 1.7 can be applied. □

Remark 1.10. *It seems to be a somewhat interesting question how the "spin" assumption in Claim 1.9 can be lifted (without imposing further conditions on the embedding of Λ into ST^*M). One can obtain more consequences from* [AFO17] *if one specifies how the inclusion $\Lambda \to ST^*M$ acts on H^2.*

1.5.3. *Aspherical ambient spaces.* Let us comment that Theorem 10 applies to ST^*T^n, because T^n is a Cartesian product of n copies of S^1. This is notable because T^n is aspherical and hence the symplectic cohomology does not admit any type of higher dilation classes, or any trickery with orientations as in Claim 1.9. In this respect, i.e., applicability to aspherical manifolds, our result is closer to the work of [Iri14] which uses the product structure to bound the Hofer Zehnder capacity of cotangent bundles of manifolds admitting circle actions with non-contractible orbit class.

1.5.4. *Fillable Legendrians.* One can also restrict the class of Legendrians in another ways: by requiring that they admit Lagrangian fillings; see, e.g., [Rit13]. If the fillings are assumed to be exact, then one can use wrapped Floer cohomology; see, e.g., [AS10]. The Arnol'd chord conjecture is established for conormal Legendrians via a comparison with the Morse homology of path spaces, see [BCS25] which builds on work of [AS06, APS08].

1.5.5. *Strong Arnol'd chord conjecture.* An interesting research direction is the "strong" version of the conjecture, as studied in [Kan26, KZ24, Can26], which asks whether there is always a chord with distinct endpoints. It seems difficult to establish the strong Arnol'd conjecture from our methods, since the upper bound we obtain on the length of the Reeb chord is always at least the systole of $\lambda|_{\partial\Omega}$ (so we cannot a priori guarantee chords shorter than all orbits, which is the surefire way to conclude the strong Arnol'd conjecture).

1.5.6. *Time-dependent Reeb flows.* The method is robust, in that it implies $R_{t_2}(\Lambda) \cap R_{t_1}(\Lambda) \neq \emptyset$ for certain numbers $0 \leq t_1 < t_2$, when $t \mapsto R_t$ is a (uniformly fast) time-dependent Reeb flow; see Definition 1.3 and Theorem 9. Such a statement is reminiscient of ones appearing in [CS15]. We note that other methods of producing Reeb chords (such as bubbling analysis in SFT [Abb14]) do not seem to yield such statements for time-dependent flows.

On this topic, a natural question is whether the reintersections can be shown to happen when $t_1 = 0$:

Question 1. *Let Y be a compact contact manifold, and let $\Lambda_s \subset Y$, $s \in [0, \infty)$ be a uniformly fast Legendrian isotopy (in the sense of Definition 1.3). Is it necessarily the case that $\Lambda_s \cap \Lambda_0 \neq \emptyset$ holds for some value $s > 0$?*

Remark 1.11. *It seems likely that wrapped Floer theory establishes a positive answer to Question 1 in the case when (Y, Λ_0) admits an exact Lagrangian filling (W, L) and the* positive wrapped Floer cohomology *$HW^+(L)$, in the sense of, e.g.,* [BCS25] *and the references therein, is non-zero.*

1.5.7. *Cotangent bundles of manifolds with negative sectional curvature.* The restrictions from this paper are related to interactions between product structures, BV operators, and the PSS morphisms. When $\Omega \subset T^*M$, and

M has negative sectional curvature, there are seemingly not enough relations between the top degree PSS classes and the other classes to apply our methods. It is natural to wonder if there a symplectic cohomology based upper bound on the length of Reeb chords of Legendrian knots in $\partial\Omega$ for domains $\Omega \subset T^*\Sigma_g$, where Σ_g is a surface of genus $g > 1$. Note that, since $ST^*\Sigma_g$ is a contact 3-manifold, the existence of Reeb chords is known [HT11, HT13]. One can also compare with the problem on the finiteness of the Hofer-Zehnder capacity of compact subsets in $T^*\Sigma_g$, [Bim24, ABB$^+$26].

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2. Proofs

The contents of this section are:

- §2.1 on the existence of embeddings of stabilized codisk bundles;
- §2.2 on the Künneth morphism;
- §2.4 on the string topology of products of spheres;
- §2.5 on rationality constant of Lagrangians in T^*T^n.

Due to the more-or-less standard nature of the Viterbo restriction map, see [Vit99, McL09, AS10, Rit13], and the fact the truncated version already appears in [Zho22a], we refer the reader to the appendix of the preprint version of this paper [BC26, §A] for the proofs of Theorems 7 and 11.

2.1. From Legendrians to embeddings of stabilized codisk bundles. The goal of this section is to prove Theorem 6. The first part of the construction involves a coisotropic embedding:

Lemma 2.1. *Let $\Omega, \alpha, \Lambda_t, a$ be as in the statement of Theorem 6. There exists an embedding $\varphi : \Lambda \times D(a) \to \Omega$, taking values in the interior, such that:*

$$\varphi^*\lambda = \lambda_{D(a)} \tag{1}$$

where $\lambda_{D(a)}$ is (the pullback of) a Liouville form on $D(a)$ (defining its standard symplectic structure). In particular, φ is a coisotropic embedding.

Proof. First let $\psi : \Lambda \times [0,1] \to \partial\Omega$ be some parametrization of the Legendrian isotopy Λ_t, so that $\psi(\Lambda \times \{t\}) = \Lambda_t$. Then, by the hypotheses of Theorem 6, it holds that ψ is an embedding and:

$$\psi^*\lambda = h\,dt$$

where $h : \Lambda \times [0,1] \to (a, \infty)$ is a smooth function. By compactness of $\Lambda \times [0,1]$, we may suppose that $h > a' > a$ for some a'.

Now consider the modification by a domain dependent Liouville flow:

$$\tilde{\psi}(x,t) = Z_{\log(a'/h(x,t))}(\varphi(x,t)).$$

This is still an embedding (since φ remains an embedding even after quotienting by the Liouville flow), and it takes values in the interior of Ω. An easy computation yields:

$$\tilde{\psi}^*\lambda = a'\,dt.$$

The next step is to extend the embedding as:

$$\phi(x,s,t) = Z_s\tilde{\psi}(x,t) \text{ where } \phi : \Lambda \times (-\infty, 0] \times [0,1] \to \Omega.$$

Another standard computation yields:

$$\phi^*\lambda = a'e^s\,dt.$$

Now we observe that the area of $(-\infty, 0] \times [0,1]$ with respect to the area form $a'e^s\,dt$ is a', and so, since $a' > a$, we can find a smooth area preserving embedding $w : D(a) \to (-\infty, 0] \times [0,1]$ such that $w^*(a'e^s\,dt) = \lambda_{D(a)}$, where $\lambda_{D(a)}$ is some Liouville form on $D(a)$ for its standard symplectic structure.

Finally we define $\varphi : \Lambda \times D(a) \to \Omega$ by the formula:

$$\varphi(x,z) = \phi(x, w(z)).$$

This map φ satisfies the first part of the lemma. It remains to explain why φ is a coisotropic embedding. It is clear that $d\varphi(T\Lambda)$ lies in the ω-orthogonal complement of $\varphi(T\Lambda \times D(a))$, because of the pullback formula (1). By dimensional reasons, it follows that $d\varphi(T\Lambda)$ must equal the entire ω-orthogonal complement. Thus the embedding is coisotropic. □

The next step is to appeal to the following general fact:

Lemma 2.2. *Let N be a compact smooth manifold with boundary, and suppose there are two coisotropic embeddings:*

$$j_1 : N \to (M_1, \omega_1) \text{ and } j_2 : N \to (M_2, \omega_2)$$

such that $j_1^*\omega = j_2^*\omega$. Then a neighborhood of $j_1(N) \subset M$ is symplectomorphic to a neighborhood of $j_2(N) \subset M_2$, and the symplectomorphism can be chosen to send $j_1(x)$ to $j_2(x)$ for $x \in N$.

Proof. See [Got82] and [MS17, Exercise 3.4.18]. □

We apply this fact with:

$$N = \Lambda \times D(a) \quad M_1 = T^*\Lambda \times \mathbb{C} \quad M_2 = \Omega.$$

In particular, we conclude from Lemma 2.2 a symplectic embedding:

$$\Phi : D_\epsilon T^*\Lambda \times D(a) \to \Omega$$

which agrees with the map φ from Lemma 2.1 on $\Lambda \times D(a)$. As it is a symplectic embedding, it follows that:

$$\Phi^*\lambda = pdq + \lambda_{D(a)} + \beta$$

where β is a closed 1-form (here pdq is the standard Liouville form on $T^*\Lambda$). Since Φ agrees with φ on the zero section, and $D_\epsilon T^*\Lambda \times D(a)$ deformation retracts onto $\Lambda \times D(a)$ we conclude that all periods of β vanish, and hence β is an exact 1-form. This completes the proof of Theorem 6. □

2.2. Künneth morphism. A Liouville manifold (W, ω, Z) involves the data of a Liouville vector field Z such that $L_Z\omega = \omega$. Let us denote by Z_s the time-s flow. Hamiltonian Floer theory of (W, ω, Z) is well-established provided one uses Hamiltonian systems and ω-tame almost complex structures that are invariant under Z_s, $s \geq 0$, outside of a compact set; see, e.g., [BC24]. It is in this context that one defines the persistence module $SH_c(\Omega)$ used throughout this paper.

The product $W \times \mathbb{C}$ is another Liouville manifold with diagonal Liouville vector field:

$$Z_{W\times\mathbb{C}} = Z_W + Z_{\mathbb{C}}$$

where $Z_{\mathbb{C}}$ is the radial Liouville vector field on $\mathbb{C}$, with its standard symplectic form.

The naive idea of the Künneth morphism is that, if H_t generates a non-degenerate Hamiltonian system on W (with positive slope b as in the statement of Theorem 8), then:

$$H_t + \tau\pi\left|z\right|^2 \text{ where } \tau = b/a$$

generates a non-degenerate Hamiltonian system on $W \times \mathbb{C}$, and the vector space underlying the Floer complex $CF(H_t + \tau\pi\left|z\right|^2)$ is identified with the Floer complexes $CF(H_t)$ computed in W. If one picks a split almost complex structure $J_{W\times\mathbb{C}} = J_W \oplus J_{\mathbb{C}}$, then the differential respects this identification, and one obtains an isomorphism of complexes:

$$CF(H_t + \tau\pi\left|z\right|^2, J_{W\times\mathbb{C}}) \to CF(H_t, J_W). \tag{2}$$

Since $\tau \in (0, 1)$, this chain level identification also respects the unit element (this is no longer true when $\tau \notin (0, 1)$).

The problem in using this observation in symplectic cohomology theory is that $H_t + \pi\left|z\right|^2$ and $J_W + J_{\mathbb{C}}$ are not Z-invariant outside of a compact set, and so our preferred framework (as used in [BC24, BC25] and throughout

the rest of this paper) cannot be applied. Existing literature [Oan08, McL09] successfully establishes a "Künneth isomorphism" between the colimits:

$$SH(W \times \mathbb{C}) \simeq SH(W) \otimes SH(\mathbb{C}).$$

Such a statement is not ideal for our purposes since $SH(\mathbb{C}) = 0$; we instead require a "filtered" Künneth map $\mathfrak{K}$ as in Theorem 8.

2.2.1. *Choice of almost complex structure.* As in Theorem 8, fix:

- a Liouville domain Q, with radial coordinate r_1,
- an ω-tame almost complex structure J_1 on the completion W, which is invariant under Z_s, $s \geq 0$, outside of $Q = \{r_1 \leq 1\}$,
- the standard complex structure J_2 on $\mathbb{C}$,
- the radial coordinate $r_2 = a^{-1}\pi |z|^2$ for the Liouville domain $D(a)$.

The complex structure:

$$Z_s^* J_1 = dZ_s^{-1} J_1 dZ_s$$

on W can be considered as "neck stretched," loosely speaking.

Let $\ell(r)$ be a smooth approximation of $\max\{\log(r), 0\}$, such that:

- $\ell(r) = \log(r)$ when $r \geq 1$,
- $\ell(r) \leq 0$ for $r \leq 1$,
- $\ell(r)$ is constant on the interval $r \in [0, \frac{1}{2}]$.

See §2.2.2 for an explicit choice of such ℓ.

Define an almost complex structure on $T(W \times \mathbb{C})$:

$$J = Z^*_{-\ell(r_2)}(J_1) \oplus J_2. \tag{3}$$

Then we have:

Lemma 2.3. *The almost complex structure J is Liouville equivariant outside of the rectangle $\{r_1 \leq 1\} \cap \{r_2 \leq 1\}$. That is to say, if (w_1, w_2) lies outside of this rectangle, and $s > 0$, it holds that $(Z_s^* J)_{(w_1,w_2)} = J_{(w_1,w_2)}$. Moreover, J is genuinely split:*

$$J_{(w_1,w_2)} = (Z^*_{-\ell(0)} J_1)_{w_1} \oplus (J_2)_{w_2}$$

on the strip $\{r_2 \leq 1/2\}$.

Proof. Let $r_i = r_i(w_i)$. If r_2 is larger than 1, we compute:

$$\begin{aligned}(Z_s^* J)_{(w_1,w_2)} &= (Z^*_{s-\log(e^s r_2)} J_1)_{w_1} \oplus J_{2,w_2} \\ &= (Z^*_{-\log(r_2)} J_1)_{w_1} \oplus J_{2,w_2}, \\ &= (Z^*_{-\ell(r_2)} J_1)_{w_1} \oplus J_{2,w_2} = (J)_{(w_1,w_2)}.\end{aligned}$$

On the other hand if r_1 is larger than 1 while r_2 is smaller than 1, we compute:

$$\begin{aligned}(Z_s^* J)_{(w_1,w_2)} &= (Z^*_{s-\ell(e^s r_2)} J_1)_{w_1} \oplus J_{2,w_2} \\ &= J_{1,w_1} \oplus J_{2,w_2} \\ &= (Z^*_{-\ell(r_2)} J_1)_{w_1} \oplus J_{2,w_2},\end{aligned}$$

where we use the fact that J_1 is equivariant outside of $\{r_1 \geq 1\}$, and:

- $s - \ell(e^s r_2) \geq 0$, in the second equality,
- $-\ell(r_2) \geq 0$, in the third inequality.

Finally, if r_2 is less than $1/2$, the splitting of J holds from the requirement that $\ell(r_2) = \ell(0)$ is constant in this region. □

2.2.2. *Hamiltonian systems for the Künneth morphism.* We continue with the set-up of §2.2.1. The construction of the Künneth map is defined in a similar way to the "naive" Künneth map described in §2.2; see equation (2). As in the statement of Theorem 8, we will find a domain $K \subset Q \times D(a)$ and directly identify a chain complex computing $SH_b(K)$ with a chain complex computing $SH_b(Q)$; this identification produces the desired map $\mathfrak{K}$.

The primary goals of §2.2 are to construct this map and prove it respects $PSS(1)$; a secondary goal is to make the map $\mathfrak{K}$ canonical. For the sake of canonicity, we introduce some functions via explicit formulas:

Definition 2.4. *Define a function $\beta : \mathbb{R} \to [0,1]$ by the formula:*

$$\beta(s) := \begin{cases} 0 & \text{for } s \leq 0 \\ m^{-1} \int_0^s e^{-1/\sigma^2} e^{-1/(\sigma-1)^2} d\sigma & \text{for } s \in [0,1] \\ 1 & \text{for } s \geq 1 \end{cases}$$

where $m = \int_0^1 e^{-1/\sigma^2} e^{-1/(\sigma-1)^2} d\sigma$. This function is a standard smooth cut-off function which increases monotonically from 0 to 1 on the interval $[0,1]$. Also define:

- $f(r) := \int_0^r \beta(2s-1)ds + 1 - \int_0^1 \beta(2s-1)ds$,

so that f is convex, non-decreasing, constant on $[0, \frac{1}{2}]$, and satisfies $f(r) = r$ for $r \geq 1$. Finally define:

- $\ell(r) := \log(f(r))$,

which makes the choice of $\ell(r)$ in §2.2.1 canonical.

The map $\mathfrak{K}$ is defined via the following family of Hamiltonian systems.

Definition 2.5. *Let $C^\infty(Q \times \mathbb{R}/\mathbb{Z})$ be the topological space of compactly supported, time-dependent, smooth functions P_t on Q, and let $\mathcal{U}$ be an open neighborhood of 0 in $C^\infty(Q \times \mathbb{R}/\mathbb{Z})$.*

Let us say that a time-dependent Hamiltonian function H_t on $W \times \mathbb{C}$ is admissible for the Künneth map for $Q \times D(a)$ and slope $b < a$, relative $\mathcal{U}$ and a number $\delta > 0$, provided that:

$$H_t = be^{\ell(r_2)}\delta f(\delta^{-1}e^{-\ell(r_2)}r_1) + br_2 + \beta(2 - 2r_2)P_t \tag{4}$$

where $P_t \in \mathcal{U}$ and:

- *(K1) all orbits of H_t lie in the submanifold $\{r_2 = 0\} = W \times \{0\}$, and,*
- *(K2) the moduli space of Floer cylinders for (H_t, J) is cut transversally, where J is given by (3).*

We denote by $\mathcal{O}_b(\delta, \mathcal{U})$ the set of such systems H_t satisfying (K1) and (K2) with $P_t \in \mathcal{U}$.

Definition 2.6. *In the context of Definition 2.5, define:*

$$K_\delta := \left\{\delta e^{\ell(r_2)} f(\delta^{-1}e^{-\ell(r_2)}r_1) + r_2 \leq 1\right\} \subset Q \times D(a).$$

Then K_δ is a smooth Liouville domain[4] *defining the Liouville structure on $W \times \mathbb{C}$ such that $H_t = r_{K_\delta}$ holds outside of a compact set (here r_{K_δ} is the radial coordinate for the domain K_δ) when H_t is defined by (4).*

Remark 2.7. *To simplify the exposition, we will not discuss changes of almost complex structure and will always use J given by (3) for a fixed J_1. This slight lack of canonicity is easily remedied by including J_1 as an auxiliary choice, and this strategy works without any problems (except for making the notation more cumbersome). We adopt a middle ground and try to make the construction canonical with regard to the "significant" choice of P_t.*

Lemma 2.8. *For any slope $b < a$ and any open set $\mathcal{U}$ and sufficiently small number δ as in Definition 2.5 the set $\mathcal{O}_b(\delta, \mathcal{U})$ is non-empty.*

Proof. For (K1), we compute the Hamiltonian vector field X_{H_t} with respect to the splitting $TW \oplus T\mathbb{C}$:

$$X_{H_t} = \begin{bmatrix} bf'(\delta^{-1}e^{-\ell(r_2)}r_1)X_{r_1} + \beta(2 - 2r_2)X_{P_t} \\ (bS - 2\beta'(2 - 2r_2)P_t)X_{r_2} \end{bmatrix} \tag{5}$$

where:

- $S = \ell'(r_2)e^{\ell(r_2)}\delta[f(\rho) - \rho f'(\rho)] + 1$ abbreviating $\rho = \delta^{-1}e^{-\ell(r_2)}r_1$.

In particular, by picking δ and P_t small enough, we may ensure that:

$$0 < bS - 2\beta'(2 - 2r_2)P_t < a,$$

since S can be made as close to 1 as desired by making δ small (and P_t can also be made vanishingly small). Since the minimal period of X_{r_2} is a

[4] To check $K_\delta \subset Q \times D(a)$, observe that:

$$\max\{r_1, r_2\} \leq cf(c^{-1}r_1) + r_2 \text{ for all } c > 0$$

since f is convex and $f(x) = x$ for $x \geq 1$.

(recall that $r_2 = a^{-1}\pi |z|^2$ is the radial coordinate for $D(a)$) it follows that any initial point lying above the region $r_2 \neq 0$ cannot lie on a one-periodic orbit of X_{H_t}. Thus (K1) holds.

Part (K2) follows easily from (K1) and the formula (5) — one can simply use the generic perturbation term P_t to render all orbits non-degenerate and all Floer cylinders regular (as all Floer cylinders must pass through the region where P_t is active). Here it is important that $b \notin Spec(Q)$ so that all orbits actually lie in Q (where P is active). □

2.2.3. *Floer cylinders for the Künneth morphism.* In this section we prove structural results for the Floer complex $CF(H_t, J)$ provided $H_t \in \mathcal{O}_b(\delta, \mathcal{U})$ and δ and the open set $\mathcal{U}$ are sufficiently small.

Lemma 2.9. *For sufficiently small δ and $\mathcal{U}$ as in Definition 2.5 all Floer cylinders for (H_t, J) where $H_t \in \mathcal{O}_b(\delta, \mathcal{U})$ lie in the submanifold $W \times \{0\}$.*

Moreover, every Floer differential cylinder for $(H_t|_{W\times\{0\}}, J|_{W\times\{0\}})$ remains a valid Floer differential cylinder for H_t, J.

Proof. By (K1), we already know that all orbits of $H_t \in \mathcal{O}_b(\delta, \mathcal{U})$ lie in the submanifold $W \times \{0\}$. To prove that the connecting trajectories for (H_t, J) also lie in this submanifold, we appeal to a compactness argument.

The key is the formula (5) and the fact that $\ell(r_2)$ is constant on the region $r_2 \leq 1/2$. Indeed, if $r_2 \leq 1/2$, then (5) simplifies to the split system:

$$X_{H_t} = \begin{bmatrix} bf'(\delta^{-1}e^{-\ell(0)}r_1)X_{r_1} + X_{P_t} \\ bX_{r_2} \end{bmatrix}.$$

Since J is also split in the region $r_2 \leq 1/2$ (Lemma 2.3), we conclude that any Floer cylinder u contained in the region $\{r_2 \leq 1/2\}$ equals $u = (u_1, u_2)$ where u_2 is a Floer cylinder for (bX_{r_2}, J_2). Since $b \in (0, a)$, the only Floer cylinders for (bX_{r_2}, J_2) are the constant ones located at the origin. Thus, any Floer cylinder for (H_t, J) not contained in $W \times \{0\}$ must intersect the region $\{r_2 > 1/2\}$. This conclusion is stable under taking limits: if we consider a sequence $\delta_n \to 0$ and $\mathcal{U}_n$, so that $P_{n,t} \in \mathcal{U}_n$ implies $P_{n,t} \to 0$, then any sequence of Floer cylinders $u_n = (u_{1,n}, u_{2,n})$ intersecting $\{r_2 > 1/2\}$ will have a subsequence so that $u_{2,n}$ converges (after reparametrization by a translation) to a Floer cylinder for bX_{r_2}. This is because since the coefficient:

$$(bS - 2\beta'(2 - 2r_2)P_t) \quad \text{from (5)}$$

converges uniformly to b as $n \to \infty$. This completes the first part of the proof.

The second part, that Floer differential cylinders for the restricted data remain valid when considered in the larger space follows from standard arguments for split Hamiltonian systems and split almost complex structures (noting that H_t, J are both split in a neighborhood of $W \times \{0\}$). Briefly,

one shows that the linearized operator at a solution $u = (u_1, 0)$ also splits into a diagonal matrix:

$$D_u = \begin{bmatrix} D_{u_1} & 0 \\ 0 & \bar{\partial} + 2\pi b/a \end{bmatrix}$$

since $\bar{\partial}+2\pi b/a$ is an isomorphism, since $b \in (0, a)$, acting on the appropriate Sobolev spaces:

$$W^{1,p}(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, \mathbb{C}) \to L^p(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, \mathbb{C}),$$

the regularity and rigidity of the solution u_1 is equivalent to the regularity and rigidity of the solution u. □

Due to Lemma 2.9, we can pick $\delta > 0$ and $\mathcal{U}$ small enough so that, for each $H_t \in \mathcal{O}_b(\delta, \mathcal{U})$, there is a tautological isomorphism of complexes:

$$\mathfrak{K}_{H_t} : CF(H_t, J) \simeq CF(H_t|_{W\times\{0\}}, J|_{W\times\{0\}}).$$

This map should be thought of as one representation of the Künneth map satisfying Theorem 8; in the next section we explain how to make $\mathfrak{K}$ "canonical," i.e., independent of the auxiliary choices used in H_t.

2.2.4. *Canonicity of the Künneth map.* Recall from §1 that $SH_b(Q)$ is defined to be the (formal) inverse limit of the Floer cohomologies of systems which agree with br_Q outside of a compact set. Thus, for $\delta > 0$ and $\mathcal{U}$ small enough such that $H_t \in \mathcal{O}_b(\delta, \mathcal{U})$ satisfies Lemma 2.9 we have the following diagram:

$$\begin{array}{ccc} SH_b(K_\delta) & \xrightarrow{\mathfrak{K}_\delta} & SH_b(Q) \\ \downarrow & & \downarrow \\ HF(H_t, J) & \xrightarrow{\mathfrak{K}_{H_t}} & HF(H_t|_{W\times\{0\}}, J|_{W\times\{0\}}) \end{array} \tag{6}$$

where $K_\delta \subset Q \times D(a)$ is as in Definition 2.6. The upper horizontal map is well-defined, for fixed H_t, by inverting the right vertical map. However, it a priori depends on the choice of P_t used in the definition of H_t. In this section we show the result is independent of P_t.

Lemma 2.10. *The upper horizontal map in* (6) *is independent of the choice of $H_t \in \mathcal{O}_b(\delta, \mathcal{U})$, provided that δ and $\mathcal{U}$ are sufficiently small.*

Proof. It suffices to prove that the maps $\mathfrak{K}_{H_t}$ commute with the continuation maps arising from changing P_t. This argument follows the same lines as Lemma 2.9, except Floer cylinders are replaced by continuation cylinders. The details are left to the reader. □

2.2.5. *Compatibility of the Künneth map with the unit element.* Fix δ and $\mathcal{U}$ small enough that Lemma 2.10 applies; this fixes a domain K_δ and a map $\mathfrak{K}_\delta : SH_b(K_\delta) \to SH_b(Q)$. The goal of this section is proving:

Lemma 2.11. *For δ small enough, the isomorphism $\mathfrak{K}_\delta : SH_b(K_\delta) \to SH_b(Q)$ preserves the unit elements, i.e., $\mathfrak{K}_\delta(PSS(1)) = PSS(1)$.*

Proof. The proof is similar to the proof of Lemma 2.9 and 2.10; one shows (by a compactness argument) that all of the solutions contributing to $PSS(1)$ remain entirely in the submanifold $W \times \{0\}$.

The argument that solutions contributing to $PSS(1)$ for the restricted data are rigid/regular if and only if they are rigid/regular for the unrestricted data follows a similar argument as the one used in the proof of Lemma 2.9. However, here it is truly necessary to use that $b/a \in (0,1)$, while the proof of Lemma 2.9 technically only requires that $b/a \notin \mathbb{Z}$. The reason is that the linearized operator for a PSS solution $u : \mathbb{C} \to W$ appears in the form:

$$D_u = \begin{bmatrix} D_{u_1} & 0 \\ 0 & \bar{\partial} + \beta(-s)2\pi b/a \end{bmatrix}$$

in cylindrical coordinates $z = e^{-2\pi(s+it)}$; here we agree to use the cut-off function β from Definition 2.4 in the definition of the PSS equation, which involves an interpolation between the zero Hamiltonian and H_t. If $b/a \in (0,1)$, then the operator:

$$\bar{\partial} + \beta(-s)2\pi b/a : W^{1,p}(\mathbb{C},\mathbb{C}) \to L^p(\mathbb{C},\mathbb{C}) \tag{7}$$

is an isomorphism, as can be established using a Fourier series argument. The argument then proceeds as in Lemma 2.9. □

Proof of Theorem 8. Fix δ and $\mathcal{U}$ small enough that Lemmas 2.9, 2.10, and 2.11 apply; this fixes a domain $K = K_\delta$ and a map $\mathfrak{K} : SH_b(K) \to SH_b(Q)$. By what we have shown above, this map $\mathfrak{K}$ satisfies Theorem 8. □

2.3. Submersions and Legendrian lifts. In this section we prove Theorem 10. The proof is an application of simple facts about symplectic reductions. The first step is to recall that a smooth submersion $\pi : E \to M$, where E is a compact manifold, is automatically a smooth proper fiber bundle (this result is often attributed to Ehresmann).

Let $\pi : E \to M$ be a smooth proper fiber bundle. Define:

$$N = \{(p,q) \in T^*E : p \text{ vanishes on } \ker d\pi\};$$

then N is a coisotropic submanifold of T^*E (see [MS17, §5.4]).

Lemma 2.12. *The leaves of the characteristic foliation of N (that is to say, the integrable submanifolds tangent to the $d\lambda$-complement to TN) consist of the submanifolds:*

$$\mathcal{F}_{x,y} = \{(p,q) \in T^*E : p = x \circ d\pi \text{ and } \pi(q) = y\}, \tag{8}$$

where $(x, y) \in T^*M$ is in the complement of the zero section.

Proof. It is easy to see that the tangent space of each leaf $\mathcal{F}_{x,y}$ is contained in the $d\lambda$-complement to TN. Since:

- $\dim \mathcal{F}_{x,y} = \dim E - \dim M$, and
- $\dim N = \dim E + \dim M$,

we conclude from dimension reasons that $\dim \mathcal{F}_{x,y}$ equals the dimension of the characteristic foliation. □

Let SN be the spherization[5] of the vector bundle $N \to E$ and consider the following diagram:

$$\begin{array}{ccc} SN & \xrightarrow{\subset} & ST^*E \\ \downarrow{\scriptstyle \pi} & & \\ ST^*M. & & \end{array} \tag{9}$$

It is a standard fact of symplectic reductions that $\pi^{-1}(\Lambda) \subset SN$ is a Legendrian in ST^*E for every Legendrian $\Lambda \subset ST^*M$.

Lemma 2.13. *The map π in (9) is proper. In particular, if Λ is a compact Legendrian in ST^*M, then $\pi^{-1}(\Lambda) \subset N$ is a compact Legendrian in ST^*E.*

Proof. The leaves of the characteristic foliation on N are the sets $\mathcal{F}_{x,y}$ appearing in (8); in particular, if $K \subset T^*M$ is a compact set, then $(p_n, q_n) \in \pi^{-1}(K)$ is a compact set, because if $(x_n, y_n) \in K$ converge then q_n converge, after passing to a subsequence, (since $\pi : E \to M$ is proper and y_n converges), and it follows automatically that $p_n = x_n \circ d\pi$ converges.

If $\Lambda \subset ST^*M$ is a compact set, then there is a compact set $K \subset T^*M - M$ whose projection to the ideal boundary ST^*M is Λ. Then $\pi^{-1}(K) \subset N$ is a compact set, by the first paragraph of the proof. It is clear that $\pi^{-1}(K)$ is in $T^*E - E$, and it is easy to see that the ideal projection of $\pi^{-1}(K)$ equals $\pi^{-1}(\Lambda) \subset ST^*E$; thus $\pi^{-1}(\Lambda)$ is compact. □

The final step in the proof of Theorem 10 is to prove:

Lemma 2.14. *If Λ_s is a uniformly fast Legendrian isotopy in ST^*M (see Definition 1.3), then $\pi^{-1}(\Lambda_s)$ is a uniformly fast Legendrian isotopy in ST^*E.*

Proof. Consider the reduction map $\pi : N \to T^*M$. We compute:

$$\pi^*\lambda = pdq; \tag{10}$$

[5]The reader is warned that the symbol "S" is used both for *spherization* (dimension lowering operation applied to vector bundles) and *symplectization* (dimension raising operation applied to cooriented contact manifolds). The meaning of the symbol should be inferred from the context.

writing $\lambda = xdy$, this follows from the relationship between $(p,q) \in N$ and its projection $\pi(p,q) = (x,y)$ where $p = x \circ d\pi$ and $y = \pi(q)$. The result then follows easily: let K_s be some lift of Λ_s to the symplectization $T^*M - M$. Because Λ_s is uniformly fast, and because of (10), the velocity vectors of curves contained in $\pi^{-1}(K_s)$ are uniformly positive when inserted into $r^{-1}pdq = r^{-1}\pi^*\lambda$, for any radial function $r : T^*E - E \to (0,\infty)$. This implies $\pi^{-1}(\Lambda_s)$ is uniformly fast, as desired. □

Proof of Theorem 10. Under the hypotheses of Theorem 10, we conclude from Lemma 2.14 that $\pi^{-1}(\Lambda_{t_2})$ must reintersect $\pi^{-1}(\Lambda_{t_1})$, for some $0 \leq t_1 < t_2$, by Theorem 9, and thus Λ_{t_2} must reintersect Λ_{t_1}, as desired. □

Remark 2.15. *The same argument proves the following: suppose E is a compact manifold and ST^*E solves the Arnol'd chord conjecture, in the sense that all compact Legendrians admit a non-constant Reeb chord for every choice of Reeb flow, and suppose there is a submersion $E \to M$. Then ST^*M also solves the Arnol'd chord conjecture. This is because any Reeb flow on ST^*M is generated by a positive* 1*-homogeneous radial function r_M on T^*M, which lifts to a positive* 1*-homogeneous radial function $r_M \circ \pi$ on N, which can be extended (non-canonically) to a positive* 1*-homogeneous radial function r_E on T^*E. The ideal restriction of the Hamiltonian flow of X_{r_E} is a Reeb flow which preserves SN and is related to the Reeb flow corresponding to r_M via the map $SN \to ST^*M$.*

Remark 2.16. *Remark 2.15 can be placed in a more general context. Suppose Y_1 and Y_2 are two compact cooriented contact manifolds, and suppose there is a coisotropic submanifold $N \subset SY_1$ which is invariant under the Liouville flow and which admits a* proper *Liouville equivariant reduction map:*

$$\pi : N \to SY_2,$$

where the word "reduction" means that the fibers of this map are the leaves of the characteristic foliation of N, and that $\pi^* d\lambda_2 = d\lambda_1|_N$, where λ_i is the canonical Liouville form on SY_i. It follows from Liouville equivariance that $\pi^*\lambda_2 - \lambda_1 = 0$. The above argument then goes through and one concludes that Y_1 solving the Arnol'd chord conjecture implies Y_2 solves the Arnol'd chord conjecture.

2.4. String topology and products of spheres. In this section we prove Theorem 4 concerning classes in $H_*(\Lambda M)$ for $M = S^{n_1} \times \cdots \times S^{n_k}$. First, we construct classes A and B for the case of $M = S^n$. The key part of this step is the notion of *completing manifolds* as in [Oan15, §6] which relates classes in the spherical tangent bundle STS^n with classes in ΛS^n. This step is done in §2.4.1. In §2.4.2 we construct a special class in $H_n(STS^n)$ using the notion of a generalized section. Afterwards in §2.4.4 we use the case of one sphere to construct the classes in $M = S^{n_1} \times \cdots \times S^{n_k}$.

Throughout this section we use the bordism class approach to the homology of the loop space, as in [BC25].

2.4.1. *Completing manifolds.* Let STS^n denote the spherical tangent bundle. Each point v determines its *orthogonal equator*, which is the unique equator passing through the basepoint of v and which is orthogonal to v at that basepoint. Consider the sphere bundle of pairs:

$$E = \{(x, v) \in S^n \times STS^n : x \text{ lies in the orthogonal equator of } v\}.$$

Define the map $F : E \to \Lambda S^n$ which sends $(x, v) \in E$ to the circle obtained by intersecting S^n with the affine plane passing through x and v (this means the affine plane intersects the basepoint of v and is tangent to v). This circle is parametrized by $\mathbb{R}/\mathbb{Z}$ with constant speed in the unique way that the tangent vector at 0 points in the direction of v.

This procedure yields the *completing manifolds operation*:

$$\gamma : H_*(STS^n) \to H_{*+n-1}(\Lambda S^n)$$

as follows: given a class $f : X \to STS^n$, the class γf is the composition of the two upper horizontal morphisms in the following diagram:

$$\begin{array}{ccccc} f^*E & \longrightarrow & E & \longrightarrow & \Lambda S^n \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \\ X & \xrightarrow{f} & STS^n. & & \end{array}$$

2.4.2. *Generalized section.* In this section we construct a special class:

$$s : \mathbb{R}P^n \to STS^n$$

to which we will apply the completing manifolds construction of §2.4.1 to obtain a class in $H_{2n-1}(\Lambda S^n)$. The construction of s is as a *generalized section*, i.e., that $\pi \circ s : \mathbb{R}P^n \to S^n$ has degree 1. Even though $STS^n \to S^n$ does not admit a section for even values of n, it does admit a generalized section: after a blow up of the base space, transforming S^n to $\mathbb{R}P^n$, it admits a section. We describe this with greater detail.

The first step is to recall the *unoriented blow-down map* $b : \mathbb{R}P^n \to S^n$.

Lemma 2.17. *Let $N \in S^n$ denote the north pole, and let stereo $: S^n \setminus N \to \mathbb{R}^n$ be the stereographic projection. There is a unique smooth map $b : \mathbb{R}P^n \to S^n$ such that:*

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}P^n \\ \downarrow{\scriptstyle stereo^{-1}} & & \downarrow{\scriptstyle b} \\ S^n \setminus N & \longrightarrow & S^n \end{array}$$

where the upper horizontal map is the parametrization:

$$(x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n]$$

associated to the zeroth affine coordinate chart on $\mathbb{R}P^n$.

Proof. Uniqueness is obvious since the domain of the zeroth affine chart is dense. Moreover, the map obviously must extend by collapsing the hyperplane divisor at infinity corresponding to $x_0 = 0$ to the north pole N.

It remains only to check this map is actually smooth. First recall that the inverse of the stereographic projection map is given by:

$$stereo^{-1}(x_1, \dots, x_n) := \frac{(x_1^2 + \cdots + x_n^2 - 1, 2x_1, \dots, 2x_n)}{x_1^2 + \cdots + x_n^2 + 1} \tag{11}$$

To prove this extends smoothly from the zeroth affine coordinate chart to all of $\mathbb{R}P^n$, we check how the formula transforms when we move to the kth affine chart parametrized by $[y_0 : \cdots : y_{k-1} : 1 : y_{k+1} : \cdots : y_n]$; the transition function is $x_j = y_j/y_0$, with the convention that $y_k = 1$. In these new coordinates, (11) is given by:

$$\frac{(y_1^2 + \cdots + y_n^2 - y_0^2, 2y_1y_0, \dots, 2y_ny_0)}{y_0^2 + y_1^2 + \cdots + y_n^2}, \tag{12}$$

which evidently extends smoothly to the entire affine chart (it was initially only defined when $y_0 \neq 0$, as that is where the kth affine chart intersects the zeroth affine chart). Since k was arbitrary, we conclude the result. □

Lemma 2.18. *There exists a smooth map $s : \mathbb{R}P^n \to STS^n$ such that:*

$$\begin{array}{ccc} & \overset{s}{\curvearrowright} & STS^n \\ & & \downarrow \pi \\ \mathbb{R}P^n & \xrightarrow{b} & S^n. \end{array}$$

In the zeroth affine chart, s can be chosen to agree with the push-forward of a constant vector field on $\mathbb{R}^n$ via the inverse of the stereographic projection.

Proof. The derivative of $stereo^{-1}(x)$ with respect to the x_1 variable extends to the following vector field on S^n:

$$V(q) = (q_1(1 - q_0), 1 - q_0 - q_1^2, -q_1q_2, \dots, -q_1q_n).$$

We will prove that the ray spanned by $V \circ b$ extends smoothly to a map s.

In the affine chart (12) with $k > 0$, one computes:

$$V \circ b = \frac{(2y_1y_0, y_0^2 - y_1^2 + y_2^2 + \dots, 2y_1y_2, \dots, 2y_1y_n)}{(2y_0^2)^{-1}(y_0^2 + y_1^2 + \cdots + y_n^2)^2}$$

Thus we can set:

$$s([y_0 : \cdots : y_n]) = \mathbb{R}_+(2y_1y_0, y_0^2 - y_1^2 + y_2^2 + \dots, 2y_1y_2, \dots, 2y_1y_n).$$

This ray is well-defined; indeed, if the generator vanishes, then:

$$y_1^2 = y_0^2 + y_2^2 + \cdots + y_n^2 \implies y_1 \neq 0, \tag{13}$$

and the other entries $2y_1y_i = 0$ yield $y_0 = y_2 = \cdots = y_n = 0$ which contradicts (13). This completes the proof. □

2.4.3. *Manipulation of classes in the free loop space of a sphere.* Apply the completing manifolds operation from §2.4.1 to the map $s : \mathbb{R}P^n \to STS^n$ constructed in §2.4.2 to produce the class $B = \gamma s \in H_{2n-1}(\Lambda S^n)$. Similarly, let $P \in H_{n-1}(\Lambda S^n)$ be the class obtained from the point class in STS^n, and let pt denote the point class in $H_0(\Lambda S^n)$ consisting of a single constant loop. Our first lemma in this section is:

Lemma 2.19. *It holds that $B * pt = P$, where $*$ denotes the Chas-Sullivan product.*

Proof. If we pick the representative of pt which is a constant loop not based at the north pole N, this follows easily from the definition of the Chas-Sullivan product in terms of transverse intersections; see [BC25, §2.1].

Briefly, $B * pt$ considers those loops in B whose base points are pt; since:

- the base points of B cover the blow-down map b,
- pt is a regular value of b, and,
- there is a unique point in $b^{-1}(pt)$,

we conclude that $B * pt$ has the same domain as P, namely $E|_{pt}$; see §2.4.1. The only difference is that loops in P are parametrized with unit speed while loops in $B * pt$ are parametrized differently; nonetheless, the two classes are equal in $H_{n-1}(\Lambda S^n)$. □

Remark 2.20. *In the case $n = 1$, there are two connected components of STS^n, and so P is not a well-defined class, but rather there are two classes depending on the orientation (and $B * pt$ will equal one of these, depending on how the section s is chosen).*

Following the terminology in [BC25, §2.2] introduce the *action class*:

$$A(x;t) = (\cos(2\pi t)x_0 - \sin(2\pi t)x_1, \sin(2\pi t)x_0 + \cos(2\pi t)x_1, x_2, \ldots, x_n),$$

associated to the rotation circle action so that $A \in H_n(\Lambda S^n)$, and similarly let $A^{-1}(x;t) = A(x;-t)$. We have:

Lemma 2.21. *We have $A * A^{-1} = [S^n]$. In the case $n > 1$, we have $A^{-1} = A$.*

Proof. The first part is a direct consequence of [BC25, §2.2.1]. The second part follows from the fact that $\pi_1(SO(n+1)) = \mathbb{Z}/2\mathbb{Z}$ for $n > 1$. □

Lemma 2.22. *It holds that $\Delta P = A^{-1}$ in $H_n(\Lambda S^n)$; in the case $n = 1$ we require that P is obtained by picking a point in the positively oriented component of STS^1.*

Proof. In [BC25, §2.2.2] we construct an explicit class $D \in H_{n-1}(\Lambda S^n)$ using an open book decomposition of S^n and show that $\Delta D = A^{-1}$. In this proof we will show that the class P is homotopic to the class D, and so the desired result will follow.

Fix $v \in STS^n$ and let E_v be the equator orthogonal to v, as in §2.4.1. Let us think of $v \in STS^n_q$ as a pair $(q, \mathbb{R}_+ w)$ where $\mathbb{R}_+ w$ is the ray through the origin spanned by a unit vector w and q is a basepoint on S^n.

For any two distinct points in $x, y \in E_v$ there is a unique two-dimensional subspace Π which contains w and the vector $x - y$. Let us denote by:

$$\Gamma(x, y) = (x + \Pi) \cap S^n = (y + \Pi) \cap S^n$$

parametrized as a circle $\mathbb{R}/\mathbb{Z} \to S^n$ via the formula:

$$t \mapsto \frac{(x+y) + \cos(t)(x-y)}{2} + \frac{\sin(t)\,|x-y|}{2} w.$$

with constant speed $2^{-1}\,|x-y|$. By the completing manifolds construction, the class P is represented by the map:

$$P : x \in E_v \mapsto \Gamma(q, x). \tag{14}$$

For concreteness, let us fix $w = \partial_0$, $q = (0, 1, 0, \ldots, 0)$, so $E_v = \{0\} \times S^{n-1}$. Define:

$$\ell(0, x_1, \ldots, x_n) = x_1 \qquad \rho(x_0, \ldots, x_n) = (0, -x_1, x_2, \ldots, x_n)$$

The class D considered in [BC25, §2.2.2] is (a smoothing of) the map:

$$D : x \in E_v \mapsto \begin{cases} \Gamma(\rho(x), x) & \text{if } \ell(x) \leq 0, \\ x & \text{if } \ell(x) \geq 0. \end{cases} \tag{15}$$

This map is not smooth at the interface $\ell(x) = 0$ but it is continuous, and therefore represents a well-defined bordism class. The class parametrizes all loops passing through one page of an open book decomposition; we require the piecewise smooth formula to switch between the circle action on one page and the constant loops on the opposite page.

Claim 2.23. *The classes P and D given in* (14) *and* (15) *are homotopic as maps $E_v \to \Lambda S^n$.*

Proof of claim. We construct a continuous homotopy $G : E_v \times [0,1] \to \Lambda S^n$ interpolating between D and P at the two extremities (one obtains a smooth homotopy between P and a smoothing of D by standard smooth approximation results). We define:

$$G : (x, r) \in E_v \in [0,1] \mapsto \begin{cases} \Gamma(\rho_r(x), x) & \text{if } \ell(x) \leq r, \\ x & \text{if } \ell(x) \geq r, \end{cases} \tag{16}$$

where $\rho_r : E_v \cap \{\ell \leq r\} \to E_v \cap \{\ell \geq r\}$ is any continuous map which satisfies $\rho_r(x) = x$ if $\ell(x) = r$ and $\rho_0(x) = \rho(x)$ and $\rho_1(x) = q$ for all x. We fix:

$$\rho_r(0, x_1, \ldots, x_n) = (0, \beta_r(x_1), \alpha_r(x_1) x_2, \ldots, \alpha_r(x_1) x_n)$$

where:

$$\beta_r(x) = \frac{x+1}{r+1}(r-1) + 1$$

and α_r is determined by the condition that $|\rho_r(0, x_1, \ldots, x_n)| = 1$. Then G is the desired homotopy with $G(x, 0) = D(x)$ and $G(x, 1) = P(x)$; see Figure 1. □

Since we have shown that $\Delta D = A^{-1}$ in [BC25, Lemma 17], it follows from the claim that $\Delta P = A^{-1}$, as desired. □

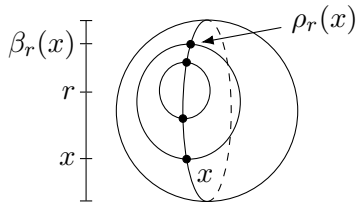


FIGURE 1. An illustration of the loop $G(x, r)$ with $x_1 < r$.

This concludes the first part of Theorem 4, since $A * \Delta(B * pt) = [S^n]$ by combining Lemmas 2.19, 2.21, and 2.22.

2.4.4. *The product of spheres.* For the second part on the product of spheres, we require the following notion:

Definition 2.24. *Let $F_1 : X_1^{n_1} \times \mathbb{R}/\mathbb{Z} \to M_1$ and $F_2 : X_2^{n_2} \times \mathbb{R}/\mathbb{Z} \to M_2$ represent classes in $H_{n_1}(\Lambda M_1)$ and $H_{n_2}(\Lambda M_2)$ respectively. Then we define:*

$$F_1 \times F_2 : (X_1 \times X_2) \times \mathbb{R}/\mathbb{Z} \to M_1 \times M_2$$

by the formula:

$$(F_1 \times F_2)(x_1, x_2; t) = (F_1(x_1; t), F_2(x_2; t)),$$

so that $F_1 \times F_2 \in H_{n_1+n_2}(\Lambda(M_1 \times M_2))$.

Lemma 2.25. *Given classes F_1, F_2 as in Definition 2.24, then:*

- *$\Delta(F_1 \times F_2) = F_1 \times \Delta F_2$ provided that $F_1(x, t) = F_1(x, 0)$, i.e., F_1 is a class of constant loops.*

Given also F_1', F_2' as in Definition 2.24, then:

- *$(F_1 \times F_2) * (F_1' \times F_2') = (F_1 * F_1') \times (F_2 * F_2')$.*

Proof. This follows easily from the definition of the BV operator and the Chas-Sullivan product; see [BC25, §2.1]. □

Completion of proof of Theorem 4. Set:

- $A_i := S^{n_1} \times \cdots \times A_{n_i} \times \cdots \times S^{n_k}$, and
- $B_i := S^{n_1} \times \cdots \times B_{n_i} \times \cdots \times S^{n_k}$.

The classes $A_{n_i} \in H_{n_i}(\Lambda S^{n_i})$, and $B_{n_i} \in H_{2n_i-1}(S^{n_i})$ are from the construction in §2.4.3 and S^{n_i} is, by abuse of notation, the class of constant loops corresponding to the sphere S^{n_i}.

It follows from Lemma 2.25 and the results of §2.4.3 that:

- $C_i = [S^{n_1} \times S^{n_2} \times \cdots \times S^{n_i} \times pt \times pt \times \cdots]$,

where C_i is defined in terms of the $A_1, \ldots, A_i$ and $B_1, \ldots, B_i$ as in the statement of Theorem 4. □

2.5. Rationality constants of Lagrangians in cotangent bundles of tori. In this section we prove Theorem 5. Fix a fiberwise starshaped subdomain $\Omega \subset T^*L$, consider a compact Lagrangian submanifold L in the interior of Ω. The goal is to prove that either L is exact, or $\rho(L)$ is bounded above by a constant which depends only on Ω.

2.5.1. *Non-exact Weinstein neighbourhood.* The first step is to use the Weinstein neighborhood theorem to conclude a symplectic embedding $\iota : K \to \Omega$ of a disk cotangent bundle $K = DT^*L$ which extends the embedding $L \to \Omega$. It follows that the period group of the closed one-form:

$$\iota^*\lambda_\Omega - \lambda_K$$

is valued in $\rho(L)\mathbb{Z}$. This holds because K deformation retracts onto L and so the periods of $\iota^*\lambda_\Omega - \lambda_K$ are the same as the periods of $\lambda_\Omega|_L$.

2.5.2. *Classes in the free-loop space of cotangent bundle of a torus.* The next step introduces certain classes in $SH(T^*T^n)$; these classes were alluded to in Remark 1.2.

Lemma 2.26. *There exist classes $A_i^\pm \in SH(T^*T^n)$, $i = 1, \ldots, n$ such that*

$$A_i^+ * A_i^- = PSS(1)$$

and such that $A_i^\pm$ lies in the free homotopy class of $t \mapsto \pm te_i$.

Proof. This is well-known, and follows from the arguments of §2.4.3 regarding action classes, the arguments in §2.4.4 regarding cartesian products, and the Viterbo isomorphism (using the framework of [BC25]). □

Therefore, having fixed $\Omega \subset T^*T^n$, we can pick the slope $c > 0$ large enough that $A_i^+ * A_i^- = PSS(1)$ holds in $SH_c(\Omega)$ for each $i = 1, \ldots, n$.

2.5.3. *Truncated Viterbo restriction.* Combining the set-up of §2.5.1 and §2.5.2, we can apply Theorem 11 to conclude:

Corollary 12. *If the Lagrangian $L \subset \Omega$, as in §2.5.1, has $\rho(L) > 2c$, where c is as in §2.5.2, then there are classes $B_i^\pm = \mathfrak{R}(A_i^\pm)$ so that:*

- $B_i^+ * B_i^- = PSS(1)$

- $B_i^{\pm} \in SH(T^*L, \kappa_i^{\pm})$, *where* $\kappa_i^{\pm}$ *is the collection of free homotopy classes of loops* γ *lying in the kernel of* $\iota^*\lambda_\Omega - \lambda_K$ *and satisfying that* $\iota(\gamma)$ *lies in the free homotopy class of* $t \mapsto \pm te_i$.

2.5.4. *From symplectic cohomology back to string topology.* In this subsection, we import one of the main results from [AS06, Abo15] that there is a BV algebra map from $SH(T^*L)$ to $HM_*(\Lambda L)$. We emphasize here that $HM_*(\Lambda L)$ uses a Morse model rather than the bordism classes model used in §2.4.

Theorem 13. *There is a graded BV-algebra* $HM_*(\Lambda L)$ *over* $H^*(T^*L)$ *with:*

- *the BV operator has degree* $+1$,
- *the product has degree* $-n$,
- *the algebra map sends* $H^*(T^*L)$ *to* $HM_{n-*}(\Lambda L)$.
- *there is a direct sum decomposition into summands* $HM_*(\Lambda M, \kappa')$ *corresponding to free homotopy classes* κ' *of loops in* M.

The group $HM_d(\Lambda L, \kappa')$ *is isomorphic to the degree* d *summand* $H_d(\Lambda_{\kappa'} L)$ *of the singular homology of the component* $\Lambda_{\kappa'} L$ *of free loop space* ΛL.

There is a map of BV algebras:

$$\Psi : SH(T^*L) \to HM_*(\Lambda L),$$

defined by counting half-infinite Floer cylinders with boundary conditions on the zero section; the map Ψ *respects the BV operator, the product, and the algebra structure over* $H^*(T^*L)$*; the operation also respects free homotopy classes, in that* $SH(T^*L, \kappa)$ *is mapped into* $HM_*(\Lambda L, \pi(\kappa))$ *where* π *is the obvious projection.*

Proof. The construction of this map is the main topic of [Abo15]. We comment that the algebra map $H^*(T^*L) \to HM_{n-*}(\Lambda L)$ is given by a Thom isomorphism style map (take a class in $H^*(T^*L)$, intersect it with the zero section, and take the corresponding cycle of constant loops). □

Referring to the notation and conclusions of Corollary 12, we conclude:

Corollary 14. *Assume the context of Corollary 12; there are non-zero classes* $C_i \in H_*(\Lambda_{\kappa_i} L)$ *so that:*

- $\deg(C_i) \geq n$,
- κ_i *is the collection of free homotopy classes of loops* γ *lying in the kernel of* $\iota^*\lambda_\Omega - \lambda_K$ *and satisfying that* $\iota(\gamma)$ *lies in the free homotopy class of* $t \mapsto \pm te_i$*, for some choice of* $\pm$ *sign.*

Proof. Let $C_i^{\pm} = \Psi(B_i^{\pm})$ where Ψ is as in Theorem 13. Since $C_i^+ * C_i^-$ equals the unit element of degree n, we conclude that one of C_i^+ or C_i^- must have degree at least n. Let us suppose the sign is $\epsilon \in \{+, -\}$. Let κ_i be the image of the corresponding κ_i^{ϵ} under $\pi : T^*L \to L$, and set C_i equal to the

image of C_i^ϵ under the identification between $HM_*(\Lambda L, \kappa_i)$ and $H_*(\Lambda_{\kappa_i} L)$. It follows from the construction that C_i, κ_i satisfy the statement. □

2.5.5. *Two facts about aspherical manifolds.* We recall two facts about the string topology aspherical manifolds taken directly from [Lat15, §5].

Lemma 2.27 ([Lat15, Lemma 5.1])**.** *Let L be a connected manifold, and let Z be the centralizer of an element $[\gamma] \in \pi_1(L, pt)$, and let $\pi : L' \to L$ be the covering space associated to the subgroup Z, and let γ' be a lift of γ. Then the induced projection $\pi : \Lambda_{\gamma'} L' \to \Lambda_\gamma L$ is a homeomorphism.* □

Lemma 2.28 ([Lat15, Lemma 5.2])**.** *Let L be a connected aspherical manifold, and let L', γ' be as in Lemma 2.28. Then the evaluation at the basepoint map $\Lambda_{\gamma'} L' \to L'$ is a homotopy equivalence.* □

We apply these lemmas in our sitation. Let C_i, κ_i be as in Corollary 14. Pick elements $\gamma_i \in \kappa_i$ based at a chosen basepoint $pt \in L$ so that we may consider $[\gamma_i] \in \pi_1(L, pt)$.

Lemma 2.29. *In the above context, and assuming that the Lagrangian L is aspherical, then there are powers $p_i \geq 1$ such that $[\gamma_i]^{p_i}$ pairwise commute for $i = 1, \ldots, n$.*

Proof. The proof has two steps; first we show the centralizer Z_i of $[\gamma_i]$ is a finite index subgroup. From Lemma 2.27, the projection $\Lambda_{\gamma_i'} L' \to \Lambda_{\gamma_i} L$ is a homeomorphism, where $\Pi : L' \to L$ is a covering associated to Z. Because the non-zero class C_i has degree at least n, and C_i lifts to a class in $\Lambda_{\gamma_i'} L'$, Lemma 2.28 implies L' has a non-vanishing homology in some degree at least n. Since L' is a closed n manifold, this implies that L' is compact, and so the covering $L' \to L$ is finite. Consequently, Z_i is a finite-index subgroup.

The second step is to show there is p_i such that $[\gamma_i]^{p_i} \in Z_1 \cap \cdots \cap Z_n$. Set $q_j := [\pi_1(L) : Z_j]$; then for each element $g \in \pi_1(L, pt)$ we have that $g^{q_j} \in Z_j$. Setting $p_i = \prod_{j \neq i} q_j$ completes the proof. □

Lemma 2.30. *Assume the context of Lemma 2.29 (which assumes there is an aspherical Lagrangian $L \subset \Omega$, as in §2.5.1, with $\rho(L) > 2c$, where c is as in §2.5.2). The subgroup:*

$$G = \left\{ [\gamma_1]^{p_1 k_1} \cdots [\gamma_n]^{p_n k_n} : k_i \in \mathbb{Z} \right\}$$

is a finite index subgroup of $\pi_1(L)$ which is isomorphic to $\mathbb{Z}^n$, and which lies in the kernel of the 1-form $\iota^* \lambda_\Omega$ where $\iota : L \to \Omega$ is the inclusion.

Proof. To prove that G is isomorphic to $\mathbb{Z}^n$, it suffices to show that:

$$[\gamma_1]^{p_1 k_1} \cdots [\gamma_n]^{p_n k_n} = 1 \implies k_1 = \cdots = k_n = 0.$$

This holds since $\gamma_i \in \kappa_i$ and $\iota(\kappa_i)$ is in the free homotopy class of $\pm e_i$, so:

$$\iota_*([\gamma_1]^{p_1 k_1} \cdots [\gamma_n]^{p_n k_n})$$

lies in a non-trivial free homotopy class.

To prove that G is finite index, we argue as follows: since L is aspherical, so is any covering space $L' \to L$. Let L' be the covering space associated with the subgroup G. Since $G \cong \mathbb{Z}^n$ we have that L' is an Eilenberg-MacLane space $K(\mathbb{Z}^n, 1)$, which further implies that L' is homotopy equivalent to T^n. From this, we conclude that L' is a closed manifold and hence the covering $L' \to L$ is finite.

The last part of the statement follows from the fact that each κ_i lies in the kernel of $\iota^*\lambda_\Omega$, as established in Corollary 14. □

2.5.6. *Proof of Theorem 5.* By Lemma 2.30, if $\iota : L \to \Omega$ is a Lagrangian embedding with $\rho(L) > 2c$ where c is as in §2.5.2, then the kernel of $\iota^*\lambda_\Omega$ contains a finite index subgroup G. The kernel of a non-zero homomorphism to $\mathbb{R}$ has an infinite index. This implies $\iota^*\lambda_\Omega = 0$, as desired. □

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