ON THE RIGIDITY OF TRANSLATED POINTS

DYLAN CANT AND JAKOB HEDICKE

ABSTRACT. We show that there exist contact isotopies of the standard contact sphere whose time-1 maps do not have any translated points which are optimally close to the identity in the Shelukhin-Hofer distance. This proves the sharpness of a theorem of Shelukhin on the existence of translated points for contact isotopies of Liouville fillable contact manifolds with small enough Shelukhin-Hofer norm.

1. Introduction

A translated point is a contact analogue of a fixed point due to Sandon; see [San11a, San11b, San12, San13]. Briefly, a *translated point* of a contactomorphism ψ, relative a contact form α, is a point $x \in \Sigma$ such that x and $\psi(x)$ lie on the same trajectory of the Reeb vector field R^α, where $\Sigma = \{x : (\psi^*\alpha)_x = \alpha_x\}$ is the set where the *scaling factor* is 1, and R^α is the vector field satisfying $\alpha(R^\alpha) = 1$ and $d\alpha(R^\alpha, -) = 0$.

Translated points are amenable to study using symplectic techniques such as generating functions, Floer theory, and other variational principles; see, e.g., [Giv90, Giv91, AF10, AM13, She17, MU19, GKPS21, Oh21, Oh22, All22b, All22a, AA23, DUZ23, Can23].

One should think of translated points as having similar rigidity to Lagrangian intersections or Reeb chords between Legendrians: provided the contactomorphism ψ is close enough to the identity map, translated points of ψ must exist, but they can disappear for sufficiently large deformations. Indeed, the [She17, Theorem B] makes this precise:

If ψ is the time-1 map of a contact isotopy and α is a choice of contact form on the ideal boundary of a Liouville manifold, and:

$$(1) \qquad 2 \inf_{s \in \mathbb{R}} \mathrm{dist}_\alpha(\psi, R_s^\alpha) < \textit{minimal length of a closed } \alpha\textit{-Reeb orbit,}$$

then ψ has a translated point relative the contact form α.

Here the distance is the *Shelukhin-Hofer distance for contactomorphisms* introduced in [She17], whose definition is reviewed in §1.1. The left hand side in (1) is known as the *α-oscillation norm* of ψ.

See also [Oh22] for a proof of this result when Y is not assumed to be Liouville fillable.

Our main result is a non-existence result for translated points, and it proves that Shelukhin's result is optimal in that it cannot be improved without further hypotheses; see §1.2 for further discussion.

Theorem 1. *For any $\epsilon > 0$, there exist contact isotopies ψ_t of S^{2n+1}, $n \geq 1$, with its standard contact structure, whose time-1 maps have no translated points relative the standard contact form α, and which satisfy:*

$$2 \inf_{s \in \mathbb{R}} \mathrm{dist}_\alpha(\psi_1, R_s^\alpha) < 1 + \epsilon;$$

note that 1 is the minimal period of a closed Reeb orbit for the standard contact form.

The construction of ψ_t is as a composition $\psi_t = \kappa_t \gamma_t$ where:

(1) γ_t is a specific focusing[1] contact isotopy whose scaling-factor-1 set Σ concentrates near a single point ζ on S^{2n+1} and whose image $\gamma_1(\Sigma)$ is arbitrarily close to the antipodal point $-\zeta$, and
(2) κ_t is a strict contact isotopy which displaces $-\zeta$ from the Reeb orbit through ζ.

The goal of this paper is to show that this can be done in such a way that ψ_t has length at most $(1 + \epsilon)/2$, and ψ_1 has no translated points. It then follows that ψ_t has oscillation energy at most $1 + \epsilon$, as desired.

1.1. Shelukhin-Hofer distance. The following pseudo-distance between contactomorphisms (in the universal cover) was introduced in [She17]; given contact isotopies[2] $\psi_{0,t}, \psi_{1,t}$, define:

$$\mathrm{dist}_\alpha(\psi_{0,t}, \psi_{1,t}) := \inf \mathrm{length}_\alpha(\psi_{s,1}),$$

where the infimum is over all squares[3] $\psi_{s,t}$ in the contactomorphism group extending $\psi_{i,t}$, $i = 0, 1$ and satisfying $\psi_{s,0} = \mathrm{id}$. The length of the path $\psi_{s,1}$ is defined by the formula:

$$\mathrm{length}_\alpha(\psi_{s,1}) := \int_0^1 \max_{y \in Y} \left| \alpha_{\psi_{s,1}(y)}(\partial_s \psi_{s,1}(y)) \right| \mathrm{d}s;$$

---

[1]The construction is related to [Can24] which uses a different a focusing isotopy.
[2]Note that isotopies ψ_t are supposed to satisfy $\psi_0 = \mathrm{id}$.
[3]The image of such a square appears as a triangle since $\psi_{s,0} = \mathrm{id}$ holds for all s.

we note that if $\partial_s \psi_{s,1} = V_s(\psi_{s,1})$, then the maximum in the integrand is also the maximum of the s-dependent contact Hamiltonian[4] $\alpha_y(V_s(y))$. Because we infimize over squares, but only measure the distance along the top edge of the square, the pseudo-distance should be considered as being defined on the universal cover (considered here as a quotient space of the contact isotopy group). As usual, the Shelukhin-Hofer norm of an isotopy ψ_t is its Shelukhin-Hofer distance to the identity.

1.2. On the factor of two. There is a difference of conventions between the Hofer norms used in [AF10] and [She17] which has the result of changing certain expressions by a factor of two. With the conventions in §1.1, [She17, Conjecture 31] is equivalent to: *if ψ_t is a contact isotopy whose α-Shelukhin-Hofer oscillation norm is less than $\rho(\alpha)$ then ψ_1 has translated points.* Here we follow [She17] and denote the minimal length of a closed α-Reeb orbit by the symbol $\rho(\alpha)$.

Our construction in Theorem 1, and the fact that the above conjecture is known to hold on the standard contact sphere (see [She17, Oh22, Can23]), proves that this statement of the conjecture is sharp, i.e., cannot be improved without further hypotheses.

1.2.1. Translated chains. In the final section §2.5, we explain why our construction also has no *translated chains*, as defined in [FSZ23].

Recall that a translated chain of a contactomorphism ψ of order $k \in \mathbb{N}$, relative a contact form α, is a point x so that:

$$(R_s^\alpha \psi)^k(x) = x \text{ and } (((R_s^\alpha \psi)^k)^* \alpha)_x = \alpha_x,$$

for some $s \in \mathbb{R}$. In other words, x is a fixed point of the kth iterate of the contactomorphism $R_s^\alpha \psi$, and the scaling factor of the kth iterate at x is 1. We prove that:

Theorem 2. *There exist contact isotopies ψ_t of the standard contact sphere with* $\text{length}_\alpha(\psi_t) \leq 1/2 + \epsilon$, *and so ψ_1 has no translated chains with respect to the standard contact form α.*

---

[4]An essential feature of contact geometry is the isomorphism $V \mapsto \alpha(V)$ between contact vector fields X and smooth functions. The function $\alpha(V)$ is called the *contact Hamiltonian*, see e.g. [Gei08, Section 2.3] for further details.

also supported in his research at the Institut de mathématique d'Orsay by funding from the ANR project CoSy.

2. Construction of the contact isotopy

As explained above, the construction of the contact isotopy ψ_t is as a composition $\psi_t = \kappa_t \gamma_t$. The contact isotopy γ_t is a cut-off version of the contact isotopy generated by a height function on the sphere. As we will see below, height functions on spheres generate contact isotopies which are *focusing*, in the sense that they have two fixed points, one of which is attracting and one of which is repelling. The results of §2.1 and §2.2 are concerned with the analysis of these contact isotopies. It is important for our argument that we have an explicit description of the scaling-factor-1 set for such contact isotopies.

The contact isotopy generated by a height function has an unbounded Shelukhin-Hofer length, and so it will be necessary to cut-off the isotopy. The bulk of the argument is concerned with the cut-off operation, especially how it affects the scaling-factor-1 set. This part of the paper occupies §2.3.

The proof is completed in §2.4 where we construct the displacing isotopy κ_t and show that $\psi_t = \kappa_t \gamma_t$ has a time-1 map without translated points and a small enough Shelukhin-Hofer length.

2.1. *A special contact vector field on the sphere.* Consider $\mathbb{R}^{2n+2}$ with coordinates $(p, q, x_1, y_1, \ldots, x_n, y_n)$ and the Liouville form:

$$\lambda = \frac{1}{2}(p\mathrm{d}q - q\mathrm{d}p) + \frac{1}{2}\sum_{i=1}^{n}(x_i\mathrm{d}y_i - y_i\mathrm{d}x_i).$$

Let $B(r) = \left\{\pi\left|z\right|^2 = r\right\}$ be the ball of symplectic capacity r. A special role is played by the sphere $\partial B(1)$ since λ restricts to the standard contact form whose Reeb flow is 1-periodic. We will consider the following vector fields tangent to $\partial B(1)$:

$$\frac{1}{2\pi}R = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p} + \sum_{i=1}^{n}x_i\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial x_i},$$

$$\frac{1}{2\pi}F_i = p\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial p} + y_i\frac{\partial}{\partial q} - q\frac{\partial}{\partial y_i},$$

$$\frac{1}{2\pi}JF_i = p\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial p} + q\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial q},$$

where J is the standard complex structure on $\mathbb{R}^{2n+2}$. Then R is the Reeb vector field and F_i, JF_i, $i = 1, \ldots, n$, span the standard contact distribution (the plane of complex tangencies).

Introduce the vector field:

$$V = \frac{1}{2} \sum_{i=1}^{n} (y_i F_i - x_i J F_i).$$

This is not a contact vector field, and it is tangent to the contact distribution of $\partial B(1)$.

Lemma 3. *The vector field:*

$$X = pR + V$$

is a contact vector field on $\partial B(r)$; indeed:

$$\mathrm{d}(\lambda(X)) + \mathrm{d}\lambda(X, -) = -2\pi q\lambda.$$

The restriction of X to $\partial B(1)$ has contact Hamiltonian equal to p.

We recall that the contact Hamiltonian of a contact vector field X on (Y, α) is simply the function $\alpha(X)$; see, e.g., [Gei08, Section 2.3]. In our case, $Y = \partial B(1)$ and the contact form is the pullback $\alpha = \lambda|_{\partial B(1)}$.

Proof. By equivariance under rescaling, it suffices to prove the lemma on the sphere $\partial B(1)$ bounding the ball of capacity 1. In this case, $\lambda(X) = p$, and hence one needs to show that $\mathrm{d}p + \mathrm{d}\lambda(X, -) = -2\pi q\lambda$ holds on $\partial B(1)$. We do the computation in $\mathbb{R}^4$, although the general case follows the same exact computation. Alternatively, one can think of x, y as being vector valued coordinates.

One computes:

$$\mathrm{d}\lambda(X, -) = \pi(py\mathrm{d}y - y^2\mathrm{d}p + qy\mathrm{d}x + px\mathrm{d}x - qx\mathrm{d}y - x^2\mathrm{d}p).$$

Using $\pi(p^2 + q^2 + x^2 + y^2) = 1$ we have:

$$\mathrm{d}p + \mathrm{d}\lambda(X, -) = \pi(py\mathrm{d}y + qy\mathrm{d}x + px\mathrm{d}x - qx\mathrm{d}y + p^2\mathrm{d}p + q^2\mathrm{d}p).$$

Finally, using $p\mathrm{d}p + q\mathrm{d}q + y\mathrm{d}y + x\mathrm{d}x = 0$ we have:

$$\mathrm{d}p + \mathrm{d}\lambda(X, -) = -\pi q(p\mathrm{d}q - q\mathrm{d}p + x\mathrm{d}y - y\mathrm{d}x) = -2\pi q\lambda.$$

Thus the desired formula holds. $\square$

2.2. Projection to the p, q plane. We will abuse notation and refer to vector fields on $D(1) = \{\pi(p^2 + q^2) \leq 1\}$ by the symbols:

$$\begin{cases} R = 2\pi(p\partial_q - q\partial_p), \\ V = (1 - \pi p^2 - \pi q^2)\partial_q, \\ X = pR + V; \end{cases}$$

this is justified because R, V, X are related[5] to the vector fields consid-
ered in §2.1 under the projection $\partial B(1) \to D(1)$.

2.2.1. An integral estimate. One computes:

$$(2) \qquad \begin{cases} \mathrm{d}p(X) = -2\pi pq, \\ \mathrm{d}q(X) = 1 + \pi(p^2 - q^2), \end{cases}$$

so that the vertical line $p = 0$ is an invariant set. We will give an exact
solution of this flow in §2.2.2. It follows from (2) that:

$$(3) \qquad \frac{p\mathrm{d}q(X) - q\mathrm{d}p(X)}{2\pi(p^2 + q^2)} = \frac{p(1 + \pi(p^2 + q^2))}{2\pi(p^2 + q^2)}.$$

The left hand side is $\frac{1}{2\pi}\mathrm{d}\theta(X)$ where $\mathrm{d}\theta$ is the closed differential form
on $\mathbb{C}^\times$ which measures winding angles around zero, and hence, for any
flow line $z : \mathbb{R} \to \{p > 0\}$ for the vector field X, we have:

$$\int_{-\infty}^{\infty} p(z(t)) \le \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta_{z(t)}(X(z(t)))dt \le \frac{1}{2}.$$

We have used $(1 + \tau)/(2\tau) \ge 1$ for $\tau = \pi(p^2 + q^2)$, (3), and that the
integral of $z^*\mathrm{d}\theta$ is the total angle traced out by the flow line which is
bounded from above by π.

Remark. Since p is the contact Hamiltonian for the contact vector field
X, the above estimate is related to the Shelukhin-Hofer length of the
isotopy defined in §1.1. We will use this result in §2.3.4.

2.2.2. Exact solution of the flow. If one sets $z = p + iq$, then the flow
lines of X satisfy:

$$(4) \qquad z' = i\pi z^2 + i,$$

whose time t flow is a biholomorphism of the disk. Moreover, if one
introduces a new coordinate $w \in \mathbb{R} \times [0, \pi]$, and uses the following
biholomorphism from the strip to the disk:

$$(5) \qquad z = \frac{ie^w + 1}{\sqrt{\pi}(e^w + i)},$$

then a computation yields:

$$z' = \frac{-2e^w}{\sqrt{\pi}(e^w + i)^2}w' \text{ and } i\pi z^2 + i = \frac{-4e^w}{(e^w + i)^2}.$$

In particular, the ODE (4) reduces to the ODE $w' = 2\sqrt{\pi}$.

---

[5]Here we recall that V_1 is related to V_2 under a map f if $V_2 \circ f = \mathrm{d}f \circ V_1$. If this
holds, then f maps integral curves of V_1 to integral curves of V_2.

2.2.3. Determining where the scaling factor is 1. This description of the flow gives a nice characterization of the set where the scaling exponent g for the time T flow φ_T is zero (here $\varphi_T^* \alpha = e^g \alpha$). This set is the inverse image of its projection to the p, q plane since:

$$g(p, q, x, y) = -2\pi \int_0^T q \circ \varphi_t(p, q, x, y)\mathrm{d}t,$$

and $q \circ \varphi_t(p, q, x, y)$ can be determined using the solution to (4). To deduce this formula, we differentiate $\varphi_t^* \alpha = e^{g_t} \alpha$ with respect to t, using Cartan's formula, and conclude that $g = g_T$ is the integral of $\mathrm{d}(\alpha(X))(R) \circ \varphi_t$ over the interval $[0, T]$; see also [Gei08, Section 2.3.2] for a general formula for the scaling factor of a contact flow.

Let us denote this projection by C. Writing $z = p + iq$, and $w = a + ib$ we have:

$$\sqrt{\pi}z = \frac{ie^{w+\bar{w}} + e^w + e^{\bar{w}} - i}{e^{w+\bar{w}} + 1 + e^{\bar{w}}i - e^w i} \implies \sqrt{\pi}q = \frac{e^{2a} - 1}{e^{2a} + 1 + 2e^a \sin(b)}.$$

In particular, we conclude that:

$$z(a + ib) \in C \iff \int_a^{a + 2\sqrt{\pi}T} \frac{e^{2\tau} - 1}{e^{2\tau} + 1 + 2e^\tau \sin(b)}\mathrm{d}\tau = 0.$$

This integral can be solved explicitly since:

$$\frac{d}{d\tau}(\ln(2\sin(b)e^\tau + e^{2\tau} + 1) - \tau) = \frac{e^{2\tau} - 1}{e^{2\tau} + 1 + 2e^\tau \sin(b)};$$

with a small computation one obtains:

$$C = \left\{z = z(a + ib) : a = -\sqrt{\pi}T \text{ and } b \in [0, \pi]\right\}.$$

Since $w' = 2\sqrt{\pi}$, one sees that $\varphi_t(C)$ is always a circular arc orthogonal to the boundary, $\varphi_{T/2}(C)$ is the horizontal line $q = 0$, and $\varphi_T(C)$ is the reflection of C under the line $q = 0$.
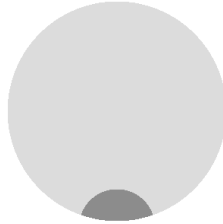

FIGURE 1. The regions separated by C. One sees C is a circular arc orthogonal to the boundary of the disk.

2.3. *The cut-off flow.* For any function $f : \mathbb{R} \to \mathbb{R}$, introduce the cut-off version:

$$X_f := f(p)R + f'(p)V,$$

where V denotes the vector field introduced in section 2.1. A direct computation shows that $\mathrm{d}(f(p)) + \mathrm{d}\alpha(X_f, -) = -2\pi q f'(p)\alpha$; this computation shows that X_f is a contact vector field on $\partial B(1)$. By abuse of notation, we also denote the projection to $D(1)$ by X_f.

2.3.1. *Equivariance under reflection.* The flow of $f(p)R + f'(p)V$ is equivariant with respect to the reflection through the line $p = 0$ provided that $f(-p) = -f(p)$.

For this reason we will always assume that f is an odd function such that $f(p) = p$ holds for p near 0; when we define $f : [0, \pi^{-1/2}] \to \mathbb{R}$ we implicitly extend f to be an odd function on $[-\pi^{-1/2}, \pi^{1/2}]$.

Thus it is sufficient to analyze the dynamics in the right half $\{p \geq 0\}$. We will therefore assume that all trajectories lie in $\{p \geq 0\}$ unless otherwise stated.

2.3.2. *The Shelukhin-Hofer length.* Our construction outlined in §2 is a time-dependent contact isotopy generated by:

$$X_t = f_t(p)R + f'_t(p)V,$$

where f_t is a family of odd functions, for $t \in [0, T]$. It is clear that the Shelukhin-Hofer length of this isotopy can be computed exactly as:

$$\mathrm{length}_\alpha = \int_0^T \max f_t(p) dt.$$

2.3.3. *Discontinuous degeneration of the cut-off vector field.* For a given number $\eta \in [0, \pi^{-1/2}]$, consider:

$$f_{\delta,\eta}(p) = \eta + \delta - \delta\mu(\delta^{-1}(\eta - p + \delta)),$$

where μ is a convex function so that $\mu(x) = x$ for $x \geq 1$ and $\mu(x) = 1/2$ for $x \leq 0$.
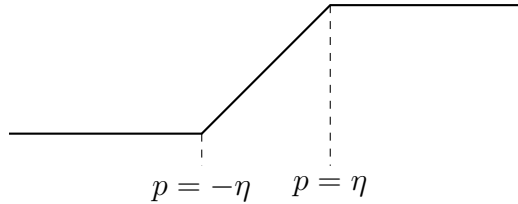

$$p = -\eta \qquad p = \eta$$

FIGURE 2. The function $f_{\delta,\eta}$ is a smoothed out version of the above piecewise linear function.

By construction we have (always assuming $p \geq 0$):

$$
(6) \qquad f_{\delta,\eta}(p) = \begin{cases} p & \text{for } p \leq \eta, \\ \eta + \delta/2 & \text{for } p \geq \eta + \delta, \end{cases}
$$

Notice that $\lim_{\delta \to 0} X_{f_{\delta,\eta}}$ converges pointwise to:

$$
X_\eta = \begin{cases} pR + V & \text{for } p \leq \eta, \\ \eta R & \text{for } p > \eta, \end{cases}
$$

This limit vector field is not continuous; however, it is continuous along the boundary, since V vanishes on the boundary.

2.3.4. The time-dependent cut-off flow. Fix a sequence $\delta_n \to 0$, and let $z : [0, T] \to \partial D(1)$ be an integral curve for the flow of $X = pR + V$, in the region where $p > 0$. Let $\eta(t) = p(z(t))$. The time dependent contact Hamiltonian $f_{n,t}(p) = f_{\delta_n, \eta(t)}(p)$ defines a sequence of contact vector fields $X_{n,t}$ as in §2.3.3. Let $\varphi_{n,t}$ be the resulting time-dependent isotopy (not to be confused with the original flow φ_t).

The length of $\varphi_{n,t}$ is bounded by:

$$
\text{length}_\alpha(\varphi_{n,t}) \leq \frac{1 + \delta_n T}{2},
$$

where we have used the bound on the length in §2.3.2, and the fact:

$$
\max f_{\delta_n, \eta(t)}(p) = \eta(t) + \delta_n/2,
$$

together with the integral estimate from §2.2.1:

$$
\int_0^T \eta(t)dt = \int_0^T p(z(t))dt \leq \frac{1}{2}.
$$

It is very important that the length of $\varphi_{n,t}$ converges to $1/2$ as $n \to \infty$. As in §2.3.3, we will also refer to the pointwise limit:

$$
X_t = \lim_{n \to \infty} X_{n,t}.
$$

The limit is uniform on compact subsets of $\{(t, u) : p(u) \neq \eta(t)\}$.

2.3.5. Set of piecewise trajectories. Introduce the set $\mathcal{M}(z)$ of continuous maps $u : [0, T] \to D(1) \cap \{p \geq 0\}$ so that:

(M1) $p(u(t)) = \eta(t)$ holds for finitely many values of t, or $u(t) = z(t)$ holds identically,

(M2) on the open set where $p(u(t)) \neq \eta(t)$, u is a flow line for the limit flow X_t,

where $z : [0, T] \to \partial D(1)$ is a flow line as in §2.3.4. It is clear that u is smooth on the open set in (M2). Let us call the finite set of times where $p(u(t)) = \eta(t)$ *crossing times.*

Lemma 4. *Each $u \in \mathcal{M}(z)$, $u \neq z$, has either zero or one crossing time.*

Proof. If $u(t_*) = z(t_*)$ holds for some time t_* then since $u(t)$ and $z(t)$ solve the same ODE on $[t_*, t_* + \delta)$ and on $(t_* - \delta, t_*]$, by (M2), we can conclude by uniqueness that $u(t) = z(t)$ holds identically. Thus we may suppose that $u(t) \neq z(t)$ for all times t.

Suppose there is a time t_* so that $p(u(t_*)) = \eta(t_*)$. One computes a two-sided derivative:

$$\left. \frac{\partial}{\partial t} \right|_{t=t_*} p(u(t)) - \eta(t) = 2\pi \eta(t_*)(q(z(t_*)) - q(u(t_*))).$$

In particular, if $z(t_*)$ is on the lower half of the circle $\{q < 0\}$, then the difference $q(z(t)) - q(u(t))$ is negative and so the relative p-coordinate is decreasing; this means that $u(t)$ lies where $p(u(t)) < \eta(t)$ for t slightly larger than t_*. Such crossings are shown on the left of Figure 3; let us call such crossings *entrances*. The other type of crossing, called an *exit*, is when $z(t_*)$ is on the upper half of the circle $\{q > 0\}$ and the same argument shows that $p(u(t)) > \eta(t)$ for t slightly larger than t_*; see the right side of Figure 3.

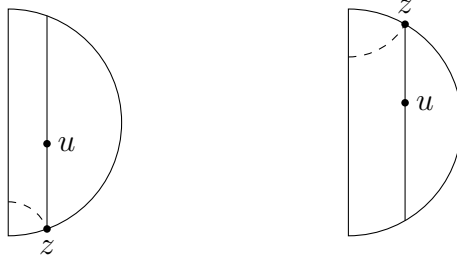It therefore follows that one cannot have an entrance followed by an entrance, or an exit followed by an exit.

FIGURE 3. In the scenario on the right, u enters the region where the flow in $pR + V$; on the left, u exits this region. The dashed arc is denoted by $C(t)$.

A topological argument shows that one cannot have an exit followed by an entrance or vice-versa. One considers the circular arc $C(t)$ passing through $z(t)$ orthogonal to $\partial D(1)$; see the dashed arc in Figure 3. By the dynamics of $pR + V$, each flow line of X_η which starts on $C(0)$ remains on $C(t)$ for all times. A flow line which does not begin on $C(0)$ can never cross $C(t)$; thus if u starts on top of $C(0)$, then u can enter but never exit, while if u starts below $C(0)$ then u can exit but never enter. Thus we have proved there is at most one crossing time; we note that it is possible for a trajectory to never enter or exit.   □

2.3.6. Scaling factor for the piecewise flow. Let $z : [0, T] \to \partial D(1)$ be a boundary trajectory of $X = pR + V$, as above. For each $u \in \mathcal{M}(z)$, introduce the *scaling exponent*:

$$g(u) = -2\pi \int_{t_1}^{t_2} q(u(t))dt,$$

where $[t_1, t_2] \subset [0, T]$ is the maximal interval so that $p(u(t)) \leq \eta(t)$ holds for all $t \in [t_1, t_2]$.

Let us now suppose that $q(z(T/2)) = 0$. By the analysis in §2.2.3, it follows that $g(z) = 0$; moreover, $g(u) = 0$ holds for every trajectory u so that $u(t) \in C(t)$ for all times t. The converse also holds:

Lemma 5. *In the above setting, $g(u) = 0$ if and only if $u(t)$ lies on the circular arc orthogonal to $\partial D(1)$ through $z(t)$.*

Bear in mind that $u(t)$, $z(t)$ remain in $\{p > 0\}$, as explained in §2.3.1.

Proof. Because $q(z(T/2)) = 0$ it follows that $p(z(T/2)) = \pi^{-1/2}$ and hence every $u \in \mathcal{M}(z)$ must satisfy $p(u(t)) \leq p(z(t))$ for some t. In particular, a trajectory can either:

(1) start with $p(u(0)) \geq p(z(0))$ and enter later on,
(2) start with $p(u(0)) \leq p(z(0))$ and exit later on, or
(3) satisfy $p(u(t)) \leq p(z(t))$ for all $t \in [0, T]$.

Let us suppose that $u(t)$ is not on the circular arc through $z(t)$. In cases (1) and (2), let t_* be the time for which $p(u(t_*)) = p(z(t_*))$. In case (1), we have that $t_* < T/2$ and:

$$g(u) = -2\pi \int_{t_*}^{T} q(u(t))dt.$$

On $[t_*, T]$, u is a flow line for $pR + V$. Since $u(t_*)$ lies above $C(t_*)$ (because $u(0)$ lies above $C(0)$) it follows that:

$$(7) \qquad g(u) < -2\pi \int_{t_*}^{T} q(v(t))dt,$$

where $v : [0, T] \to D(1)$ is a trajectory for $pR + V$ which starts on the arc $C(0)$; indeed, one picks the trajectory v so that $v(t_* + \tau) = u(t_*)$ holds for some $\tau > 0$, and the analysis in §2.2.3 implies (7).
Since $v(0) \in C(0)$, it holds that:

$$-2\pi \int_{t_*}^{T} q(v(t))dt \leq 0,$$

indeed, the integral over $[0, T]$ is zero, and the contribution due to $[0, t_*]$ is non-negative since $t_* \leq T/2$. Therefore $g(u) < 0$, as desired.

The argument in case (2) is similar. In this case the exit time satisfies $t_* \geq T/2$, and:

$$g(u) = -2\pi \int_0^{t_*} q(u(t))dt > -2\pi \int_0^{t_*} q(v(t))dt \geq 0,$$

for some trajectory $v : [0, T] \to D(1)$ which starts on $C(0)$.

Finally, for case (3) it follows immediately from §2.2.3 that $g(u) \neq 0$, since u is a flow-line for $pR+V$ for all of $[0, T]$ and does not start on the set where the scaling-exponent is zero, namely $C(0)$. This completes the proof.   □

2.3.7. *Limiting behaviour of the cut-off flow.*

Lemma 6. *A sequence of flow-lines $u_n : [0, T] \to D(1)$ for $X_{n,t}$, as defined in §2.3.4, has a subsequence which converges in the C^0 topology to a solution $u \in \mathcal{M}(z)$ as $n \to \infty$.*

Proof. To begin, observe that the vector fields $X_{n,t}(p, q)$ are uniformly bounded for $(p, q) \in D(1)$ and $t \in [0, T]$. Therefore, by the Arzelà-Ascoli theorem, u_n has a subsequence which converges in C^0 to a limiting continuous map u. It remains to show that $u \in \mathcal{M}(z)$.

Let X_t be the discontinuous limit of $X_{n,t}$ as in §2.3.4.

First we claim the following: *if $u(t_*) \in \partial D(1)$ holds for some time t_*, then $u(t) \in \partial D(1)$ for all times $t \in [0, T]$.* The claim is proved by proving a differential inequality for the function

$$\ell(p, q) = 1 - \pi(p^2 + q^2).$$

Indeed, along any flow line of $X_{n,t}$, we have:

$$d\ell(X_{n,t}) = f'_{n,t}(p)d\ell(\partial_q)\ell;$$

see §2.2. Since $|f'_{n,t}(p)|$ is bounded by 1 (by construction), and $|d\ell(\partial_q)|$ is bounded by some constant C, we have:

$$\left| \tfrac{d}{dt}\ell \right| \leq C\ell,$$

along any flow line. This differential inequality integrates to:

$$(8) \qquad\qquad \ell(u_n(t)) \leq e^{C|t-t_*|}\ell(u_n(t_*)).$$

Our claim then follows from (8) and the convergence of u_n to u.

Suppose that $u(t_*) \in \partial D(1)$ holds for some time t_*. Given ϵ, there is a small neighbourhood of $(t_*, u(t_*))$ in $[0, T] \times D(1)$ so that:

$$|X_{n,t}(v) - X_{t_*}(u(t_*))| \leq \epsilon$$

for n sufficiently large and (t, v) in this neighbourhood. This is because the discontinuous term in X_t vanishes along the boundary; see §2.3.3. Thus, for h small enough, we have:

$$\frac{u(t_* + h) - u(t_*)}{h} = \lim_{n \to \infty} \int_0^1 X_n(u_n(t_* + \tau h))d\tau = X_{t_*}(u(t_*)) + O(\epsilon),$$

and hence u is differentiable at t_* and $u'(t_*) = X_{t_*}(u(t_*))$. In particular, if $u(t_*) \in \partial D(1)$ for any time t_*, then either $u(t) = z(t)$ for all times t or $u(t) \neq z(t)$ for all times t. Thus we may suppose that $u(t) \neq z(t)$ for every $t \in [0, T]$ (noting that if $u(t) = z(t)$ holds for all t then (M1) and (M2) are trivially satisfied).

To show (M1) we need to prove that $p(u(t)) = \eta(t)$ holds for only finitely many times t. To do so, we introduce the function:

$$E_n(t) = p(u_n(t)) - \eta(t).$$

For any n, we have from (2) and (6) that:

$$\tfrac{d}{dt} E_n(t) = F_n(t) := 2\pi q(z(t))\eta(t) - 2\pi q(u_n(t)) f_{\delta_n, \eta(t)}(p(u_n(t))).$$

In the limit $n \to \infty$ the right hand side converges uniformly to the continuous function:

$$F(t) = 2\pi q(z(t))\eta(t) - 2\pi q(u(t)) \max \{p(u(t)), \eta(t)\}.$$

Since $E_n(t)$ converges uniformly to the limit $E(t) := p(u(t)) - \eta(t)$ and $F_n(t)$ converges uniformly to $F(t)$, it follows that $E(t)$ is continuously differentiable and $E'(t) = F(t)$. If $E(t_*) = 0$, it then holds that:

$$E'(t_*) = 2\pi\eta(t)(q(z(t)) - q(u(t))).$$

Since $u(t_*) \neq z(t_*)$ it holds that $E'(t_*) \neq 0$; in other words, the level set $\{t : E(t) = 0\}$ is transverse and is a finite set. This proves (M1).

The second property (M2) is established as follows. Pick a time t_* so that $E(t_*) \neq 0$. By continuity, $p(u_n(t))$ remains a uniform distance away from from $\eta(t)$ for n large enough and for t in some interval around t_*. Thus, for t near t_*, we may assume that $(t, u_n(t))$ lies in a region of $[0, T] \times D(1)$ where $X_{n,t}$ converges uniformly to X_t. It follows that the continuous limit $u(t)$ is a flow-line for X_t away from the zero set of E. Therefore property (M2) is satisfied, as desired. $\square$

2.3.8. Scaling factor for the cut-off flow. In this section we analyze the set where the cut-off flow has scaling factor equal to 1. The idea is fairly simple: we will use the convergence result of §2.3.7 with §2.3.6 to prove that the cut-off flow has a scaling-factor-1-set which is well-approximated by the circle $C(0)$.

Let $X_{n,t}$ be the cut-off contact vector field and let $\varphi_{n,t}$ be its contact isotopy as in §2.3.4. We suppose that $z : [0, T] \to \partial D(1)$ is chosen so that $q(z(T/2)) = 0$. Then the circle $C(t)$ passing through $z(t)$ which is orthogonal to $\partial D(1)$ satisfies:

(1) $C(0)$ is the scaling-factor-1 set for the original flow φ_t,
(2) $C(t) = \varphi_t(C(0)) = \varphi_{n,t}(C(0))$.

The fact that $\varphi_t(C(0)) = \varphi_{n,t}(C(0))$ holds because $\varphi_t(C(0))$ always remains in the region where the cut-off vector field agrees with the original vector field $pR + V$. The fact that $C(t) = \varphi_t(C(0))$ holds because φ_t is a Möbius transformation of the disk, as shown in §2.2.2. The goal in this section is to prove:

Lemma 7. *For given $T > 0$ and $\epsilon > 0$, the number n can be chosen large enough that the scaling-factor-1 set $\Sigma_{n,T}$ for $\varphi_{n,T}$ is such that its images $\varphi_{n,T}(\Sigma_{n,T})$ are ϵ-close to $C(T)$.*

Proof. Recall that $\varphi_{n,T}^* \alpha = e^{g_{n,T}} \alpha$ where:

$$g_{n,T}(x) = \int d(f_{n,t}(p))(R) dt = \int f'_{n,t}(p) dp(R) dt = -2\pi \int f'_{n,t}(p) q dt,$$

where the integral is along the flow line for $X_{n,t}$ starting at x.

Let us argue by contradiction; if the images of the scaling-factor-1 set do not concentrate near $C(t)$ as $n \to \infty$, then we can find a sequence of flow lines $u_n(t)$ so that $u_n(t)$ becomes at least ϵ far from $C(t)$ (for some t) and so that $g_{n,T}(u_n(0)) = 0$.

Using the result of §2.3.7, we can pass to a subsequence so that u_n converges uniformly to a limit $u \in \mathcal{M}(z)$. It is clear that $u \neq z$, and hence u has an isolated crossing with the set $p = \eta(t)$. For any $\rho_2 > 0$, there is $0 < \rho_1 < \rho_2$ and an interval I of length ρ_2 so that $u(t)$ is ρ_1 far from $\{p = \eta(t)\}$ outside I.

By the C^0 convergence of u_n to u, it follows that u_n remains $\rho_1/2$ far from $\{p = \eta(t)\}$ outside of I, for n large enough.

Observe that $F_n(t, p) = -2\pi f'_{n,t}(p) q$ converges uniformly to:

$$F(t, p) = \begin{cases} -2\pi q & \text{if } p \leq \eta(t) \\ 0 & \text{if } p > \eta(t) \end{cases}$$

on compact subsets of the complement of the locus $\{p = \eta(t)\}$. Since:

$$(t, u_n(t))$$

remains in such a compact set for $t \notin I$ (the compact set is the complement of the $\rho_1/2$ neighbourhood around $\{p = \eta(t)\}$) we conclude

that:

$$\int_0^T |F_n(t, u_n(t)) - F(t, u(t))|\, dt - \int_I |F_n(t, u_n(t)) - F(t, u(t))|\, dt,$$

is $o(1)$ as $n \to \infty$. However, since F_n, F are bounded from above by a constant (independently of n), we conclude that:

$$\int_0^T |F_n(t, u_n(t)) - F(t, u(t))|\, dt = O(\rho_2) + o(1) \text{ as } n \to \infty,$$

where we use that $|I| = \rho_2$. Since ρ_2 was arbitrary, we conclude that

$$\lim_{n \to \infty} \int_0^T |F_n(t, u_n(t)) - F(t, u(t))|\, dt = 0.$$

In particular, $\int_0^T F(t, u(t)) dt = 0$, since $\int_0^T F_n(t, u_n(t)) dt = 0$.
This implies by §2.3.6 that $u(t) \in C(t)$, contradicting the C^0 convergence of u_n to u. This contradiction completes the proof. $\qquad\square$

2.4. Displacing the set where the scaling factor is 1. Let $\varphi_{n,t}$ be the contact isotopy constructed in §2.3.4; suppose also that the flow line $z : [0, T] \to \partial D(1)$ used in its definition is such that $q(z(T/2)) = 0$. By taking T sufficiently large, we may suppose that:

(1) $C(0)$ lies in a small neighbourhood U_- of $-i\pi^{-1/2}$ and $C(T)$ lies in a small neighbourhood U_+ of $i\pi^{-1/2}$.

Then by §2.3.4 and §2.3.8 we can take n large enough that:

(2) the length of $\varphi_{n,t}$, for $t \in [0, T]$, is as close to $1/2$ as desired,
(3) the set Σ_n where $\varphi_{n,t}^* \alpha = \alpha$ satisfies that the projection of $\varphi_{n,t}(\Sigma_n)$ to $D(1)$ is as close to the travelling circular arc $C(t)$ as desired.

These two properties are achieved by taking n sufficiently large.

Fix now a small number ϵ. We claim:

Lemma 8. *There exists a strict contact isotopy κ_t of $\partial B(1)$ whose Shelukhin-Hofer length is at most $\epsilon/4$ and whose time 1 map sends the point lying above $i\pi^{-1/2}$ into the fiber over $p = q = 0$.*

Proof. Apply [Ush14, Theorem 1.6] to the point:

$$A = [i\pi^{-1/2} : 0 : \cdots : 0] \in \mathbb{C}P^n = \partial B(1)/(\mathbb{R}/\mathbb{Z})$$

to conclude a Hamiltonian isotopy with Hofer norm less than $\epsilon/4$ which sends A to some point in the hyperplane divisor lying above $p = q = 0$. Then use the fact that Hamiltonian isotopies of $\mathbb{C}P^n$ with small Hofer

norm lift to strict contact isotopies of S^{2n+1} with small Shelukhin-Hofer norm, as shown in, e.g., [She17, pp. 1192]. $\qquad\qquad\square$

Fix such an isotopy κ_t. By picking T large enough, we may suppose that:

$$(9) \qquad\qquad \kappa_1(\mathrm{pr}^{-1}(U_+)) \subset \mathrm{pr}^{-1}(D(\epsilon))$$

where the disk $D(\epsilon)$ is disjoint from U_-; here $\mathrm{pr} : \partial B(1) \to D(1)$ is the projection to the p-q-plane. We also pick n large enough that:

$$(10) \qquad\qquad \Sigma_n \subset \mathrm{pr}^{-1}(U_-) \text{ and } \varphi_{n,T}(\Sigma_n) \subset \mathrm{pr}^{-1}(U_+).$$

Introduce $\gamma_t = \varphi_{n,tT}$. We claim that the isotopy $\psi_t = \kappa_t\gamma_t$ has a time-1 map without translated points, and has length at most $(1 + \epsilon)/2$.

Indeed, the scaling factor 1 set for ψ_1 is the same as the scaling factor 1 set for $\varphi_{n,T}$, namely Σ_n, since κ_t is strict. By (9), (10) and the fact that there are no Reeb flow lines joining $\mathrm{pr}^{-1}(U_-)$ to $\mathrm{pr}^{-1}(D(\epsilon))$, since the Reeb vector field is rotational, it follows that there are no Reeb flow lines joining Σ_n to $\psi_1(\Sigma_n)$. Thus ψ_1 has no translated points.

The contact Hamiltonian generating $\kappa_t\gamma_t$ is $k_t + Th_{n,tT} \circ \kappa_t^{-1}$, where $k_t, h_{n,t}$ are the contact Hamiltonians for κ_t and $\varphi_{n,t}$; this easily yields the bound on the length of ψ_t provided the length of $\varphi_{n,t}$ is less than $1/2 + \epsilon/4$. This completes the proof of Theorem 1.

2.5. Non-existence of translated chains. We prove Theorem 2

The original time T flow φ_t, with T large, has the property that: the circle $C(0)$ divides $D(1)$ into regions where scaling factor is bigger than or smaller than 1, and if $\mathrm{pr}(x)$ lies in $D(\epsilon)$, then the scaling exponent at x is strictly negative. Here $D(\epsilon)$ is as in §2.4.

In particular, if we pick n sufficiently large, we can ensure that, if x has scaling factor less than 1 for $\varphi_{n,T}$, then $\varphi_{n,T}(x) \in \mathrm{pr}^{-1}(U_+)$. On the other hand, we can also ensure that, if x has scaling factor more than 1 for $\varphi_{n,T}$, then $x \in \mathrm{pr}^{-1}(U_-)$. Finally, if $x \in \mathrm{pr}^{-1}(D(\epsilon))$, then the scaling exponent of $\varphi_{n,T}$ at x is strictly negative.

Let $\gamma_t = \varphi_{n,Tt}$ and $\psi_t = \kappa_t\gamma_t$, as above. Since R_s^α and κ_1 are strict, the scaling factor of $R_s^\alpha\psi_1$ at x equals the scaling factor of $\varphi_{n,T}$ at x. Let us abbreviate $\eta = R_s^\alpha\psi_1$.

Since R_s^α preserves $\mathrm{pr}^{-1}(D(\epsilon))$ and $\kappa_1(\mathrm{pr}^{-1}(U_+)) \subset \mathrm{pr}^{-1}(D(\epsilon))$, we conclude that:

$$(11) \qquad \text{if } x \text{ has scaling factor} \leq 1 \text{ for } \eta, \text{ then } \eta(x) \in \mathrm{pr}^{-1}(D(\epsilon)).$$

Now suppose that η^k has a fixed point x with scaling factor 1.

It is easy to show that the scaling factor of η^k at x is the product of the scaling factors at $x, \eta(x), \eta^2(x), \ldots, \eta^{k-1}(x)$; see, e.g., [Can24]. Therefore, at least one point $\eta^j(x)$, $j = 0, \ldots, k-1$, must have scaling factor at most 1. It then follows from (11) that $\eta^{j+1}(x) \in \mathrm{pr}^{-1}(D(\epsilon))$. Therefore the scaling exponent of η at $\eta^{j+1}(x)$ is strictly negative, so $\eta^{j+2}(x)$, $\eta^{j+3}(x)$, etc, are all contained in $\mathrm{pr}^{-1}(D(\epsilon))$.

It then follows that x must lie in $\mathrm{pr}^{-1}(U_-)$, otherwise the product of scaling factors would be strictly less than 1. However, $\mathrm{pr}^{-1}(U_-)$ and $\mathrm{pr}^{-1}(D(\epsilon))$ are disjoint, since U_- and $D(\epsilon)$ are disjoint. This completes the proof. $\qquad\square$

References

[AA23]   S. Allais and P.-A. Arlove. Spectral selectors and contact orderability. arXiv:2309.10578, 2023.

[AF10]   P. Albers and U. Frauenfelder. Leaf-wise intersections and Rabinowitz Floer homology. *Journal of Topology and Analysis*, 2(1):77–98, 2010.

[All22a] S. Allais. Morse estimates for translated points on unit tangent bundles. arXiv:2205.13946, 2022. À paraître dans Ann. Inst. Fourier.

[All22b] S. Allais. On the minimal number of translated points in contact lens spaces. *Proceedings of the American Mathematical Society*, 150:2685–2693, 2022.

[AM13]   P. Albers and W. J. Merry. Translated points and Rabinowitz Floer homology. *J. Fixed Point Theory Appl.*, 13:201–214, 2013.

[Can23]   D. Cant. Shelukhin's Hofer distance and a symplectic cohomology barcode for contactomorphisms. arXiv:2309.00529, September 2023.

[Can24]   D. Cant. Contactomorphisms of the sphere without translated points. *J. Symplectic Geom.*, 22(1):1–10, 2024.

[DUZ23]   D. Djordjević, I. Uljarević, and J. Zhang. Quantitative characterization in contact Hamiltonian dynamics - I. arXiv:2309.00527, 2023.

[FSZ23]   M. Fraser, S. Sandon, and B. Zhang. Contact non-squeezing at large scale via generating functions. arXiv:2310.11993, 2023.

[Gei08]   H. Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

[Giv90]   A.B. Givental. Nonlinear generalization of the Maslov index. In *Theory of singularities and its applications*, volume 1 of *Adv. Sov. Math.*, pages 71–103. AMS, 1990.

[Giv91]   A.B. Givental. The nonlinear Maslov index. In *Geometry of low-dimensional manifolds*, Lecture Note Series, pages 35–43. London Math. Soc., 1991.

[GKPS21] Gustavo Granja, Yael Karshon, Milena Pabiniak, and Sheila Sandon. Givental's non-linear Maslov index on Lens spaces. *Int. Math. Res. Not.*, 2021(23):18225–18299, 2021.

[MU19]   W. J. Merry and I. Uljarević. Maximum principles in symplectic homology. *Israel J. Math.*, 229:39–65, 2019.

[Oh21]   Y-G. Oh. Geometry and analysis of contact instantons and entanglement of Legendrian links I. arXiv:2111.02597, 2021.

[Oh22]   Y-G. Oh. Contact instantons, anti-contact involution and proof of Shelukhin's conjecture. arXiv:2209.03548, 2022.

[San11a]   S. Sandon. Contact homology, capacity and non-squeezing in $\mathbb{R}^{2n} \times S^1$ via generating functions. *Ann. Inst. Fourier*, 61(1):145–185, 2011.

[San11b]   S. Sandon. Equivariant homology for generating functions and orderability of lens spaces. *J. Symp. Geom.*, 9(2):123–146, 2011.

[San12]   S. Sandon. On iterated translated points for contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times \mathrm{S}^1$. *Int. J. Math.*, 23(2), 2012.

[San13]   S. Sandon. A Morse estimate for translated points of contactomorphisms of spheres and projective spaces. *Geom. Dedicata*, 165:95–110, 2013.

[She17]   E. Shelukhin. The Hofer norm of a contactomorphism. *J. Symp. Geom.*, 15, 2017.

[Ush14]   M. Usher. Submanifolds and the Hofer norm. *J. Eur. Math. Soc.*, 16:1571–1616, 2014.

Email address: dylan@dylancant.ca

INSTITUT DE MATHÉMATIQUE D'ORSAY, UNIVERSITÉ PARIS-SACLAY, BÂTIMENT 307, RUE MICHEL MAGAT, F-91405 ORSAY CEDEX, FRANCE

Email address: jakob.hedicke@gmail.com

CENTRE DE RECHERCHES MATHÉMATIQUES, UNIVERSITÉ DE MONTRÉAL, PAVILLON ANDRÉ-AISENSTADT, 2920 CHEMIN DE LA TOUR, MONTRÉAL, H3T 1J4, CANADA