# THE SPECTRAL DIAMETER OF A SYMPLECTIC ELLIPSOID

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ABSTRACT. The spectral diameter of a symplectic ball is shown to be equal to its capacity; this result upgrades the known bound by a factor of two and yields a simple formula for the spectral diameter of a symplectic ellipsoid. We also study the relationship between the spectral diameter and packings by two balls.

## 1. *Introduction*

**1.1.** *Spectral diameter as a capacity.* A well-known construction in Floer theory associates a *spectral invariant* to a compactly supported Hamiltonian system $\varphi_t$ on a convex-at-infinity symplectic manifold $W$. The sum of the spectral invariants of $\varphi_t$ and its inverse is called the *spectral norm* $\gamma(\varphi_t)$. For an open set $U \subset W$ one can therefore consider the *spectral diameter*:

$$\gamma(U) = \sup\{\gamma(\varphi_t) : \varphi_t \text{ is supported in } U\}.$$

Such a quantity is a symplectic capacity for $U$ in the sense of, e.g., [CHLS07], and has been considered in [Sch00, FS07, Mai22]; for further discussion see §2.1.6. Our main result is the exact formula for the spectral diameter of a symplectic ellipsoid in $W = \mathbb{C}^n$:

**Theorem 1.1.** *The spectral diameter of the ellipsoid:*

$$E(a_1, \dots, a_n) = \{z : \sum \pi a_i^{-1}|z_i|^2 < 1\},$$

*with* $a_1 \leq \cdots \leq a_n$*, is equal to:*

$$\gamma(E(a_1, \dots, a_n)) = \begin{cases} a_n & \text{if } a_n \in [a_1, 2a_1], \\ 2a_1 & \text{if } a_n \in [2a_1, \infty); \end{cases} \tag{1}$$

*in particular,* $\gamma(B(1)) = 1$ *and* $\gamma(Z(1)) = 2$.

To the authors' knowledge, the equality $\gamma(B(1)) = 1$ has so far not appeared in the literature. The inequality $\gamma(Z(1)) \leq 2$ follows from a displacement energy bound and has been observed before; see §1.2 and §2.2.

**1.2.** *Outline of argument.* The argument proving Theorem 1.1 is divided into three main steps:

(1) *The case of a cylinder*: $\gamma(E(1,\infty,\dots))\le 2$. This step is proved using a well-known upper bound on the spectral diameter in terms of the displacement energy; see §2.2 and §2.3.
(2) *The case of a long ellipsoid*: $\gamma(E(1,\dots,2))\ge 2$. This lower bound uses the standard moment map $\mathbb{R}^{2n}\to[0,\infty)^n$ and toric geometry. Briefly, one shows that $E(1,\dots,2)$ contains two balls of capacity 1, and then explicitly constructs systems supported in these two balls to obtain the stated lower bound; see §2.5 and §2.6.
(3) *The case of a ball*: $\gamma(B(1))=1$. This step is the most delicate; the argument relies on the Hamiltonian circle action on $\mathbb{C}^n$ which rotates all the coordinates, and analyzing the effect on the action filtration of Floer homology groups. The proof is given in §2.9.

Assuming (1), (2), and (3), we prove the theorem. It is well-known that:

$$\gamma(\sqrt{a}U)=a\gamma(U)$$

for open sets $U\subset\mathbb{R}^{2n}$; see, e.g., [CHLS07, §2]. Therefore, using (1) and (2), if $a_n\in[2a_1,\infty)$, we have:

$$\gamma(E(a_1,\dots,a_n))=a_1\gamma(E(1,\dots,a_n/a_1))=2a_1,$$

because $E(1,\dots,2)\subset E(1,\dots,a_n/a_1)\subset E(1,\infty,\dots)$. On the other hand, if $a_n\in[a_1,2a_1]$, then a similar argument yields:

$$\gamma(E(a_1,\dots,a_n))=\frac{a_n}{2}\gamma(E(2a_1/a_n,\dots,2))\ge\frac{a_n}{2}\gamma(E(1,\dots,2))=a_n,$$

while (3) implies:

$$\gamma(E(a_1,\dots,a_n))\le\gamma(E(a_n,\dots,a_n))=a_n,$$

so $\gamma(E(a_1,\dots,a_n))=a_n$ for $a_n\in[a_1,2a_1]$. This is what we wanted to show.

**1.3.** *The spectral diameter and packings by two balls.* Part of the argument used in the proof of Theorem 1.1 involves the additivity of the spectral diameter with respect to packings by two balls. We state this result as it is of independent interest:

**Theorem 1.2.** *Let $U$ be an open set in an aspherical and convex-at-infinity symplectic manifold $W$. Suppose there exists a symplectic embedding:*

$$B(a_1)\sqcup B(a_2)\to U;$$

*then $a_1+a_2\le\gamma(U)$.*

Such additivity does not hold for packings by three or more balls; for example, the cylinder in $\mathbb{C}^n$ contains infinitely many disjoint balls with a given capacity but has a bounded spectral diameter.

Theorem 1.2 also enables one to conclude:

**Corollary 1.3.** *The spectral diameter of a polydisk:*

$$P(a_1,\dots,a_n) = D(a_1)\times\cdots\times D(a_n),$$

*with* $a_1 \le \cdots \le a_n$*, is equal to* $2a_1$ *if* $n \ge 2$*; if* $n=1$ *it is equal to* $a_1$*.*

The spectral diameter obstructs symplectic embeddings of $P(a,a)$ into $B(a')$ unless $2a \le a'$; when $n=2$, this embedding obstruction recovers the result proved in [EH90]. In higher dimensions the spectral diameter obstruction is weaker than the one obtained in [EH90].

**1.4.** *A two-ball capacity.* Consider the capacity:

$$c_{2B}(U) := \sup\{a+b : \text{ there exists a symplectic } B(a)\sqcup B(b)\subset U\}.$$

Our method proves that $c_{2B}(U)=\gamma(U)$ whenever $U$ is a symplectic ellipsoid or polydisk. This begs the question: *what is the largest class* $\mathcal{C}$ *of domains* $U$ *for which* $c_{2B}(U)=\gamma(U)$*?* The construction of [Her98] shows that the class $\mathcal{C}$ does not contain certain starshaped domains; it is based on the fact that arbitrarily small neighborhoods of the torus $\partial D(r)\times\cdots\times\partial D(r)$ have a large spectral diameter (proportional to $r$) but with a small $c_{2B}$ capacity.

The equality $c_{2B}(B(1)) = \gamma(B(1)) = 1$ implies Gromov's two-ball theorem [Gro85, 0.3.B]. Gromov's method yields $c_{2B}(\mathbb{C}P^n)=1$; this fact can also be proved using spectral diameter. Indeed, it follows from §2.6.1 that:

$$c_{2B}(\mathbb{C}P^n) \le \gamma(\mathbb{C}P^n).$$

When combined with the result of [EP03] that $\gamma(\mathbb{C}P^n)\le 1$ this upper bound establishes the two-ball theorem for $\mathbb{C}P^n$.

**1.5.** *On the spectral displacement energy.* The *spectral displacement energy* of a precompact open set $U\subset W$ is the value:

$$e_\gamma(U) = \inf\{\gamma(\psi_t) : \psi_1(U)\cap U = \emptyset\};$$

such a quantity was considered in [Vit92, §4]. Combining the displacement energy bound of §2.2 and the existence of a certain compactly supported Hamiltonian system displacing $B(1)$ from itself it is possible to prove the following:

**Theorem 1.4.** *The spectral displacement energy* $e_\gamma(B(1))$ *is equal to 1.*

The construction of the system is recalled in §2.3. This equality shows one does not obtain the spectral diameter $\gamma(B(1))=1$ directly from the displacement energy bound from §2.2. Indeed, the displacement energy bound yields:

$$\gamma(B(1)) \le 2e_\gamma(B(1)) = 2,$$

which is suboptimal in view of Theorem 1.1.

**1.6.** *Spectral diameter of special balls in projective space.* Consider $\mathbb{C}P^n$ with the Fubini-Study symplectic form, normalized so that the class of $\mathbb{C}P^1$ has symplectic area 1. In this case, it is well-known that the symplectic structure on $\mathbb{C}P^n$ is determined as the symplectic reduction of $\partial B(1) \subset \mathbb{C}^{n+1}$.

Let $a \in (0, 1)$. The function $S_a = \pi|z_0|^2 - a$ is well-defined on $\mathbb{C}P^n$ via the quotient map $\partial B(1) \to \mathbb{C}P^n$, and generates a 1-periodic Hamiltonian circle action on $\mathbb{C}P^n$.

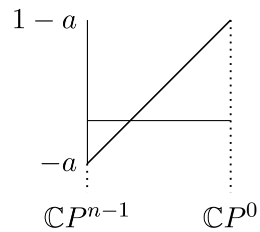


FIGURE 1. Schematic illustration of $S_a$, graphed as a function of $\pi|z_0|^2$. The point $\mathbb{C}P^0$ is the maximum of $S_a$ and corresponds to the line $z_1 = \cdots = z_n = 0$. The divisor $\mathbb{C}P^{n-1}$ is the Morse-Bott minimum of $S_a$ and represents all lines passing through the hyperplane $z_0 = 0$.

The set $\{S_a \geq 0\}$ is symplectomorphic to a ball of capacity $1 - a$, a fact whose verification is left to the reader. Let us call a ball in $\mathbb{C}P^n$ *special* if it is obtained by applying a Hamiltonian diffeomorphism to $\{S_a \geq 0\}$, for some $a \in (0, 1)$. Beyond dimensions $n = 1, 2$, it does not seem to be known whether the image of every embedding of a closed ball into $\mathbb{C}P^n$ is special.

Using methods similar to our proof of Theorem 1.1, we prove:

**Theorem 1.5.** *The spectral diameter of a special ball is equal to its capacity.*

Here the spectral diameter is computed using the spectral invariants within $\mathbb{C}P^n$ and uses the coefficient field $\mathbb{Z}/2$.

A special ball can be parametrized by an embedding $i : B(a) \to \mathbb{C}P^n$. Thus any compactly supported Hamiltonian system $\psi_t$ on $B(a)$ can be "pushed forward" to $\mathbb{C}P^n$ by the formula $i\psi_t i^{-1}$, extended to the complement of the ball as the identity system. It is natural to wonder whether:

$$\gamma_{\mathbb{C}^n}(\psi_t) = \gamma_{\mathbb{C}P^n}(i\psi_t i^{-1}) \tag{2}$$

holds for every system $\psi_t$. As we show in §2.10, the equality (2) fails in general (our example requires $a$ to be close to 1). This is noteworthy as it shows we cannot simply appeal to the known bound on the spectral diameter of $\mathbb{C}P^n$ from [EP03] to deduce $\gamma_{\mathbb{C}^n}(\psi_t) \leq 1$ for all $\psi_t$ supported in $B(1)$.

Interestingly enough, if (2) holds for even an arbitrarily small special ball, then Theorem 1.5 can be used to recover Theorem 1.1 for balls. Establishing sufficient conditions to ensure (2) in the presence of symplectic spheres seems to be a non-trivial task, even for small balls, and we save further study of (2) for future research.

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## 2. *Proofs*

**2.1.** *Floer homology in convex-at-infinity manifolds.* This section is concerned with a recollection of various Floer theoretic objects used in this paper.

**2.1.1.** *Cappings.* For each fixed point $x$ of the time-one map $\psi_1$, a *representative capping* is a smooth map $u : [-1, 1] \times \mathbb{R}/\mathbb{Z} \to W$ so that $u(-1, t) = x$ and $u(1, t) = \psi_t(x)$. Representative cappings are considered up to equivalence: the difference of two representative cappings forms a sphere and if this sphere has zero symplectic area then the representatives are declared to be equivalent. An equivalence class of representatives will be referred to as a *capping.* Capped orbits are denoted as pairs $(x, u)$.

Requiring that $u(-1, t) = x$ has the following advantage:

**Lemma 2.1.** *If $u$ is a capping of $x$ then $\bar{u}(s, t) = \psi_t^{-1}(u(-s, t))$ is a capping of $x$ for the system $\psi_t^{-1}$, and the action of $\bar{u}$ is minus the action of $u$.* □

**2.1.2.** *Action.* To each capping one can associate an action:

$$\mathcal{A}(\psi_t; x, u) = \int H_t(\psi_t(x))\mathrm{d}t - \int u^*\omega,$$

where $H_t$ is the normalized generator for a contact-at-infinity Hamiltonian system $\psi_t$. For simplicity we suppose that $W$ is connected, and we consider two classes of normalization in this paper:

(1) If $W$ is open, and $Y_0$ is a chosen connected component of the ideal boundary of $W$, then a Hamiltonian function $H_t$ is normalized if it is one-homogeneous in the non-compact end corresponding to $Y_0$; see [AAC23] for the definitions of *ideal boundary* and *one-homogeneous.*
(2) If $W$ is closed, then a Hamiltonian function $H_t$ is normalized if the integral of $H_t\omega^n$ over $W$ vanishes for each $t$.

In (1) different choices of $Y_0$ give different normalizations. The crucial properties are that the set of normalized Hamiltonians is a vector subspace and a constant normalized function is zero.

**2.1.3.** *The Floer homology vector space.* Let $\psi_t$ be a contact-at-infinity system and suppose that $\psi_1$ has non-degenerate fixed points.

Define $\mathrm{CF}(\psi_t)$ to be the $\mathbb{Z}/2$-vector space of semi-infinite sums generated by capped orbits $(x,u)$ of $\psi_t$, requiring that $\mathcal{A}(\psi_t;x,u)\ge L$ holds for only finitely many terms in the sum, for each $L$.

**2.1.4.** *The Floer homology differential.* The Floer differential depends on a choice of almost complex structure $J_t$, although different choices give isomorphic chain complexes. It is defined as usual in fixed point Floer homology; see, e.g., [DS93, Sei15]. The relevant moduli space $\mathcal{M}(\psi_t,J_t)$ is the space of twisted holomorphic curves:

$$\begin{cases} w:\mathbb{C}\to W,\\ \partial_s w+J_t(w)\partial_t w=0,\\ \psi_1(w(s,t+1))=w(s,t).\end{cases}$$

In order for the cylinder $u(s,t)=\psi_t(w(s,t))$ to solve a smooth PDE, we require that $J_t$ is $\psi_1$-*twisted-periodic*, i.e., $J_{t+1}(w)=\mathrm{d}\psi_1^{-1}J_t(\psi_1(w))\mathrm{d}\psi_1$.

By counting rigid-up-to-translation elements in $\mathcal{M}(\psi_t,J_t)$, with $w(-\infty)$ considered as input and $w(+\infty)$ considered as output one obtains a map:

$$d_{\psi_t,J_t}:\mathrm{CF}(\psi_t)\to\mathrm{CF}(\psi_t).$$

One uses the cylinder $u$ to determine the capping of the output in terms of the capping of the input. With these homological conventions, the Floer differential decreases action. The homology of $(\mathrm{CF}(\psi_t),d_{\psi_t,J_t})$ is denoted[1] by $\mathrm{HF}(\psi_t)$.

**2.1.4.a.** It follows that $J_{-t}$ is $\psi_1^{-1}$ twisted periodic which yields an inversion identification $\iota:\mathcal{M}(\psi_t,J_t)\to\mathcal{M}(\psi_t^{-1},J_{-t})$ given by $\iota(w)(s,t)=w(-s,-t)$.

**2.1.5.** *Reeb flows.* This section is only relevant when $W$ is open. We describe the set-up on $\mathbb{C}^n$, although everything holds verbatim on a general convex-at-infinity manifold $W$ if one replaces $\pi|z|^2$ by a suitable function $r$.

Let $R_{\delta,s}$ be the Hamiltonian flow generated by:

$$\mu_\delta(\pi|z|^2-1)+1$$

where $\mu_\delta$ is a convex cut-off function so that:

(1) $\mu_\delta(x)$ is the constant $\delta/2$ for $x\le 0$,
(2) $\mu_\delta(x)=x$ for $x\ge\delta$,
(3) $\mu_\delta'(x)>0$ for $x>0$.

It is important that $R_{\delta,s}$ agrees with the flow of $\pi|z|^2$ for $\pi|z|^2>1+\delta$, i.e., the ideal restriction of $R_{\delta,s}$ is the standard one-periodic Reeb flow.

[1] To be pedantic, $\mathrm{HF}(\psi_t)$ should be defined as a limit of the homologies of $(\mathrm{CF}(\psi_t),d_{\psi_t,J_t})$ as $J_t$ varies over all admissible complex structures.

**2.1.6.** *Spectral invariants.* For any class $a \in \mathrm{HF}(\psi_t)$, let:

$$c(\psi_t; a) := \inf\{\sup_i \mathcal{A}(\psi_t; x_i, u_i) : \text{the cycle } \textstyle\sum_i (x_i, u_i) \text{ represents } a\};$$

loosely speaking, $c(\psi_t; a)$ is a homological min-max over all representative cycles; see, e.g., [BP94, Sch00, Oh05, FGS05, FS07, Ush08].

When $\psi_t$ is convex-at-infinity and its ideal restriction is a negative Reeb flow there is a distinguished class $1 \in \mathrm{HF}(\psi_t)$ which plays the role of the unit element for the pair-of-pants product; see §2.1.9. This is initially defined for non-degenerate systems. One extends the definition of $\mathrm{HF}(\psi_t)$ and the unit element to all systems whose ideal restriction is a non-positive Reeb flow as an inverse limit over continuation maps. The spectral invariant $c(\psi_t; 1)$ extends to the limit; we refer the reader to [AAC23] for details on continuation maps and this extension.

**2.1.7.** *The spectral norm.* The *spectral norm* of a compactly supported system $\psi_t$ is defined by:

$$\gamma(\psi_t) := c(\psi_t; 1) + c(\psi_t^{-1}; 1) = \lim_{s \to 0} c(R_{-st}\psi_t; 1) + c(R_{-st}\psi_t^{-1}; 1), \tag{3}$$

where $R_s$ is one of the Reeb flows constructed in §2.1.5 (the parameters going into the definition of $R$ do not matter in the limit $s \to 0$). It is crucial that $\psi_t$ is compactly supported, as the ideal restrictions of $\psi_t$ and $\psi_t^{-1}$ are then both non-positive.

A similar definition of a spectral norm in the case of $\mathbb{C}^n$ appears in [Vit92] using the framework of generating functions. The paper which introduced the spectral norm using the framework of Floer theory in aspherical manifolds is [Sch00], and the extension to convex-at-infinity $W$ is due to [FS07].

**2.1.8.** *Spectral diameter.* As explained in the introduction, one obtains a *spectral diameter*[2] $\gamma(U)$, for any open set $U \subset W$, as the supremum of $\gamma(\psi_t)$ over systems $\psi_t$ with compact support in $U$. The spectral diameter as a capacity is considered in [Sch00, FS07]; an earlier capacity appears in [Vit92], namely the *spectral capacity*:

$$c(U) := \sup\{c(H_t; 1) : H_t \text{ has compact support in } U\},$$

which is known to be a normalized capacity on $\mathbb{C}^n$. See §2.1.10 for our conventions used to define $c(H_t; 1)$ on closed manifolds.

**2.1.9.** *Sub-additivity for spectral invariants and pair-of-pants product.* Suppose that $\varphi_t, \psi_t$ are two contact-at-infinity systems. The pair-of-pants product operation is a map:

$$*_{\mathrm{PP}} : \mathrm{HF}(\varphi_t) \otimes \mathrm{HF}(\psi_t) \to \mathrm{HF}(\varphi_t \psi_t),$$

[2]If one defines the distance $d(\varphi_t, \psi_t) = \gamma(\varphi_t^{-1}\psi_t)$ then $\gamma(U)$ is the diameter of the set of all systems supported in $U$.

defined by counting solutions to Floer's equation over a pair-of-pants surface, as in, e.g., [Sch95, Sch00, Sei15, KS21, AAC23]. The precise structure of Floer's equation involves the choice of a Hamiltonian connection over the pair-of-pants, and using a connection with zero curvature produces sharp energy estimates in terms of the actions of asymptotics. These energy estimates imply the sub-additivity of the spectral invariants:

$$c(\varphi_t\psi_t; a *_{\mathrm{PP}} b) \le c(\varphi_t; a) + c(\psi_t; b).$$

For further discussion we refer the reader to [AAC23, §2.4] which describes in detail the pair-of-pants product in convex-at-infinity symplectic manifolds.

It is important to note that, if $\varphi_t = R_{-\epsilon t}$ is a Reeb flow with a small negative speed, and $a = 1$ is the unit element, then $1 *_{\mathrm{PP}} b$ is the image of $b$ under the continuation map $\mathrm{HF}(\psi_t) \to \mathrm{HF}(\varphi_t\psi_t)$; see [KS21].

**2.1.10.** *Non-normalized Hamiltonians.* It is sometimes convenient to generalize the definition to allow non-normalized Hamiltonian functions via the rule:

$$c(H_t + f(t); a) = c(\psi_t; a) + \int_0^1 f(t)dt,$$

where $H_t$ is the normalized generator for $\psi_t$, and $f(t)$ is a time-dependent shift. If we use the symbol $c(\psi_t; a)$, then we require using the normalized generator. On the other hand, if we use the symbol $c(H_t; a)$, then we allow $H_t$ to be non-normalized.

**2.2.** *The displacement energy bound.* In this section we recall the displacement energy bound on the spectral norm:

**Proposition 2.2.** *Let $W$ be a rational, semipositive, and convex-at-infinity symplectic manifold, and let $\psi_t, \varphi_t$ be two compactly supported Hamiltonian systems such that $\psi_1$ displaces the support of $\varphi_t$. Then $\gamma(\varphi_t) \le 2\gamma(\psi_t)$. Moreover, if $W$ is open, then $c(\varphi_t; 1) \le \gamma(\psi_t)$.*

*Proof.* See [Gin05] which proves the result in the both the open and the closed case, assuming $W$ is aspherical; see also [Vit92, HZ94, Sch00, Oh05, Ush10b]. The argument extends easily from aspherical to rational and semipositive, all that is required is that spectral invariants are well-defined and are sub-additive (we use semipositivity to ensure their well-definedness) and valued in the action spectrum (this is why we assume rationality). □

**2.3.** *An upper bound to the capacity of a cylinder.* In this section we bound the spectral displacement energy of the cylinder following closely the ideas in [Pol01, §2.4].

A disc $B(1) \subset \mathbb{C}$ is symplectomorphic to a square with sides of length 1; in particular, $Z(1)$ is symplectomorphic to $R = (0,1)^2 \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$. Therefore, it is enough to show that $\gamma(R) \leq 2$. Consider the Hamiltonian system:

$$\psi_t(x_1, y_1, \dots, x_n, y_n) = (x_1, y_1 + t, \dots, x_n, y_n),$$

generated by $H(x, y) = x_1$. Here, we identify $z_j = x_j + iy_j$ for all $j$. Then:

$$\psi_1(R) \cap R = \emptyset,$$

and the displacement occurs in $(0,1) \times (0,2) \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$. Of course $\psi_t$ is neither compactly supported nor normalized.

Let $\varphi_t$ be a Hamiltonian system supported inside $(0,1)^2 \times K \subset R$, for a compact set $K \subset \mathbb{C}^{n-1}$. Consider a Hamiltonian system $\psi'_t$ generated by a function $H'$ obtained by cutting-off $H$ outside an arbitrarily small neighbourhood of $(0,1) \times (0,2) \times K$; we can ensure that

$$\max H' - \min H' \leq 1 + \delta$$

for some small $\delta$. This construction yields a system whose time-one map $\psi'_1$ displaces the support of $\varphi_t$. Moreover, by §2.2:

$$\gamma(\varphi_t) \leq 2\gamma(\psi'_t) \leq 2(\max H' - \min H') = 2 + 2\delta.$$

Here we use the well-known estimate that the spectral norm is less than the Hofer norm; see, e.g., [KS21]. Since $\varphi_t$ is arbitrary and $\delta$ can be chosen arbitrarily small, we conclude the desired result $\gamma(Z(1)) = \gamma(R) \leq 2$.

**2.4.** *Displacement energy of a ball.* In this section we show that the spectral displacement energy $e_\gamma(B(1))$ of $B(1)$ is equal to 1; this is the statement of Theorem 1.4.

We begin by showing $e_\gamma(B(1)) \geq 1$. Let $\psi_t$ be a Hamiltonian system such that $\psi_1$ displaces $B(1)$ and, hence, the support of any Hamiltonian system $\varphi_t$ supported therein. Proposition 2.2 implies $c(\varphi_t; 1) \leq \gamma(\psi_t)$.

It is known that one can find a bump function supported in $B(1)$ generating a Hamiltonian system $\varphi_t$ whose spectral invariant $c(\varphi_t; 1)$ is arbitrarily close to 1; indeed, this follows from the construction in §2.6. In particular, one concludes that $\gamma(\psi_t) \geq 1$; this proves the desired inequality.

We now show that $e_\gamma(B(1)) \leq 1$; to that effect, it is enough to find a Hamiltonian system $\psi'_t$ whose time-1 map displaces $B(1)$ and $\gamma(\psi'_t) \leq 1$. The system constructed in §2.3 satisfies the desired properties. This concludes the proof of Theorem 1.4.

**2.5.** *Ball packings of toric domains.* This section explains the toric approach to ball packings following [Sch05]. This is used to show that:

(1) the ellipsoid $E(a_1, \dots, a_n)$ with $a_n \geq 2a_1$, and
(2) the polydisk $P(a_1, \dots, a_n)$,

each contain two disjoint balls whose capacities are arbitrarily close to $a_1$. Then Theorem 1.2 bounds their spectral diameter from below by $2a_1$. On the other hand, the displacement energy bound of §2.2 and the construction in §2.3 bounds their spectral diameter from above by $2a_1$. Such arguments played a role in the proof of Theorem 1.1 and the results in §1.4.

Define the moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}$ by $\mu(z) = (\pi|z_1|^2, \dots, \pi|z_n|^2)$. Given a domain $D \subset \mathbb{C}^n$, its image $\mu(D)$ will be referred to as its *toric image*. It is easy to check that the toric image of an ellipsoid is a simplex while the toric image of a polydisk is a rectangular parallelepiped. A domain $D$ is called *toric* provided $D = \mu^{-1}(\mu(D))$; a moment's thought will reveal the both ellipsoids and polydisks are toric domains.

The standard technique used to construct a ball packing of a toric domain is to decompose its toric image into simplices, each of which are supposed to represent embedded symplectic balls. The key is the following symplectomorphism $\Phi$ between $\mathbb{R}^n_{>0} \times T^n$ and $\mathbb{C}^n \setminus \{z_1 \cdots z_n = 0\}$:

$$(a_1, \dots, a_n, \theta_1, \dots, \theta_n) \mapsto \frac{1}{\sqrt{\pi}}(\sqrt{a_1}e^{2\pi i\theta_1}, \dots, \sqrt{a_n}e^{2\pi i\theta_n}),$$

where $(a_1, \dots, a_n)$ are the coordinates on $\mathbb{R}^n_{>0}$, $(\theta_1, \dots, \theta_n)$ are the coordinates on $T^n = \mathbb{R}^n/\mathbb{Z}^n$; the symplectic form on the domain is $\sum \mathrm{d}a_i \wedge \mathrm{d}\theta_i$.

It is a rather deep fact that:

**Lemma 2.3.** *If $\Delta(a) \subset \mathbb{R}^n_{>0}$ is the open simplex consisting of open convex combinations of $0, ae_1, \dots, ae_n$, then the Gromov width of $\mu^{-1}(\Delta(a))$ is $a$.*

*Proof.* For the proof see, e.g., [Sch05, §3.1]. This is not immediate; indeed:

$$\mu^{-1}(\Delta(a)) = B(a) \setminus \{z_1 \dots z_n = 0\},$$

so any embedding of a ball must miss the removed parts. However, one can still embed balls with capacity arbitrarily close to $a$. □

Introduce the group $G = \mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ of special affine transformations. Then $G$ acts on $\mathbb{R}^n$ in a natural way, and the action extends to an action on $\mathbb{R}^n \times T^n$ by canonical transformations (in particular, the action is via symplectomorphisms). As a consequence, we conclude the following corollary of Lemma 2.3: *if the toric image of a toric domain $\Omega$ contains a disjoint union $\Delta(a) \sqcup g(\Delta(a))$ where $g \in G$, then $\Omega$ contains two disjoint symplectic balls whose capacities are each arbitrarily close to $a$.*

Thus to prove that the ellipsoid $E(a_1, \dots, a_n)$ with $a_n \geq 2a_1$, and the polydisk $P(a_1, \dots, a_n)$ each contain two disjoint balls of capacity arbitrarily close to $a_1$, it suffices to cut its toric image into subsimplices; this can be done and is shown in Figure 2.

**2.6.** *Spectral diameter and packings of two balls.* We present the proof of Theorem 1.2 which provides a lower bound on the spectral diameter of domains in aspherical manifolds based on their packings by two balls. As in the statement of the theorem, let $U \subset W$ be an open domain in $W$, and let:

$$B(a) \sqcup B(b) \to U$$

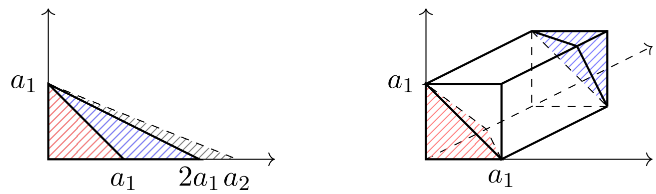


FIGURE 2. Decomposing the toric image of the ellipsoid (shown in dimension 2) and the polydisk (shown in dimension $n = 3$) into standard simplices. In the picture on the right we have set $a_1 = a_2$.

be an embedding of a disjoint union of Darboux balls. We will show that $\gamma(U) \geq a+b$ by explicitly constructing systems supported in these two balls. Other works have estimate spectral norms of systems supported in a disjoint union of balls, e.g., [Sey15, HLRS16, Ish16, Tan22, GT23]; these results are much more general and apply to any systems. Our result only requires a lower bound and can be deduced by elementary means.

The reason for assuming that $W$ is aspherical is so that the action is associated directly to the orbits (there is a single capping) and the spectral invariants lie in a compact nowhere dense action spectrum.

Consider the following Hamiltonian function:

$$H_{a,\eta,\delta}(z) = \mu_\delta(\eta(a - \pi|z|^2)) - \delta/2$$

where $0 < \eta < 1$ is any number, $\mu_\delta$ is the cut-off function in §2.1.5, and $\delta$ is much smaller than $\eta a$. It is important to note that this system has:

(1) a 1-periodic orbit at $z = 0$ whose action is $\eta a - \delta/2$,
(2) a family of 1-periodic orbits when $\pi|z|^2 \geq a$ whose actions are 0.

Let $K_{a,b,\eta,\mu}$ be the system on $U$ obtained by implanting $H_{a,\eta,\delta}$ in the image of the ball $B(a)$ and the system $-H_{b,\mu,\delta}$ on the image of the ball $B(b)$.

Let $\psi_t$ be the generated system. One concludes:

(a) the action spectrum of $\psi_t$ is the set $\{0, \eta a - \delta/2, \delta/2 - \mu b\} + C$, and,
(b) $\{0, \delta/2 - \eta a, \mu b - \delta/2\} - C$ is the spectrum of $\psi_t^{-1}$.

Here $C$ is a constant shift needed to normalize $K$, and is only necessary when $W$ is closed. Hence, the possible values for the spectral norm of $\psi_t$ are from the following set:

$$\{\eta a - \delta/2, \mu b - \delta/2, \eta a + \mu b - \delta\}.$$

We now argue by cases. The key idea is to exploit continuity and spectrality of the spectral invariants, and non-degeneracy of the spectral norm.

In the first case we have $c(\psi_t) = \eta a - \delta/2 + C$ and $c(\psi_t^{-1}) = -C$; by taking $\eta \to 0$ and using continuity of the spectral invariants we obtain a system that is not the identity and has zero spectral norm which is a contradiction as the spectral norm is non-degenerate.

The second case, when $c(\psi_t) = C$ and $c(\psi_t^{-1}) = \mu b - \delta/2 - C$, is similarly ruled out. Hence the only possibility is that $\gamma(\psi_t) = \eta a + \mu b - \delta$. By taking $\delta$ to 0 and $\eta, \mu$ to 1 we conclude $\gamma(U) \geq a + b$, as desired.

**2.6.1.** *Beyond the aspherical case.* The analogue of Theorem 1.2 in the non-aspherical case is more subtle, and the argument in §2.6 requires some modification because the action functional becomes multivalued. In this section we refine the argument by taking into account the indices of the orbits.

**Proposition 2.4.** *Suppose that $(M^{2n}, \omega)$ is a closed symplectic manifold and:*

$$\omega(u) \neq 0 \implies |c_1(u)| \geq n + 1$$

*for all $u \in \pi_2(M)$. If there is a symplectic embedding $B(a) \sqcup B(b) \to U$, where $U \subset M$ is an open set, then $\gamma(U)$ is at least $a + b$.*

*Proof.* Consider the Hamiltonian $K_{a,b,\eta,\mu}$ as above, and take a small Morse perturbation $K'$ of $K_{a,b,\eta,\mu}$. For $\eta, \mu$ less than 1, the only orbits of $K'$ are its critical points. Using the constant cappings, the maximum has action approximately $a\eta$, the minimum has action approximately $-b\mu$, and all others critical points have actions close to zero. The Conley-Zehnder indices of these orbits are related to their Morse indices in such a way that any non-constant recapping will not contribute to the spectral invariant of the unit, and in this fashion one concludes that $\gamma(K') \approx a\eta + b\mu$; taking the limit $\eta, \mu \to 1$ yields the desired result. □

**2.7.** *Duality and inversion in rational symplectic manifolds.* We suppose throughout this section that $(W, \omega)$ is *rational*, i.e., $\omega(\pi_2(M))$ is a discrete subgroup of $\mathbb{R}$; the minimal positive generator of this subgroup is denoted by $\rho$ and is called the *rationality constant* of $(W, \omega)$.

The goal is to relate the spectral invariants of $\psi_t^{-1}$ with the spectral invariants of $\psi_t$. For related discussion we refer the reader to [EP03, Ush10a, LZ18]. We will show:

**Lemma 2.5.** *Let $(W, \omega)$ be rational, and let $\psi_t$ be a contact-at-infinity Hamiltonian system with non-degenerate fixed points. For any class $b \in \mathrm{HF}(\psi_t^{-1})$, we have:*

$$c(\psi_t^{-1}; b) = -\inf\{c(\psi_t; a) : \langle a, b\rangle = 1\};$$

*where $\langle -, -\rangle$ denotes the duality pairing $\mathrm{HF}(\psi_t) \otimes \mathrm{HF}(\psi_t^{-1}) \to \mathbb{Z}/2$ defined on generators by $\langle (y, v), (x, u)\rangle = 1$ if and only if $x = y$ and $u = \bar{v}$; see §2.7.1 for more details.*

*Proof.* The proof uses the duality isomorphism in §2.7.1 for rational symplectic manifolds. For the rest of the argument we refer the reader to the proof of [EP03, Lemma 2.2]. □

**2.7.1.** *The duality isomorphism.* Let $(y, v)$ be a capped orbit of $\psi_t$, and consider the capped orbit of $\psi_t^{-1}$ given by $(y, \bar{v})$ given by $\bar{v}(s, t) = \psi_t^{-1}(v(-s, t))$ as in §2.1.1. Let us denote by:

$$\langle -, - \rangle : \mathrm{CF}(\psi_t) \otimes \mathrm{CF}(\psi_t^{-1}) \to \mathbb{Z}/2,$$

the pairing defined by:

$$\langle (y, v), (x, u) \rangle = \begin{cases} 1 & x = y \text{ and } u = \bar{v}, \\ 0 & \text{otherwise,} \end{cases}$$

where the equality between $u$ and $\bar{v}$ holds in the space of symplectic cappings, as described in §2.1.1.

We first observe that, if $\sum(y_j, v_j)$ and $\sum(x_i, u_i)$ are semi-infinite sums of the correct type to define Floer chains, then only finitely many terms in each can contribute to their pairing, because:

$$\langle (y_j, v_j), (x_i, u_i) \rangle \neq 0 \text{ if and only if } \mathcal{A}(\psi_t; y_j, v_j) + \mathcal{A}(\psi_t^{-1}; x_i, u_i) = 0,$$

and only finitely many terms in each sum are equal to the negative of a term in the other sum.

It follows from the identification of §2.1.4.a that this pairing satisfies:

$$\langle d(y, v), (x, u) \rangle + \langle (y, v), d(x, u) \rangle = 0,$$

and therefore the map:

$$\alpha \in \mathrm{CF}_{\leq -A}(\psi_t) \mapsto \langle \alpha, - \rangle \in \mathrm{Hom}(\mathrm{CF}_{\geq A}(\psi_t^{-1}), \mathbb{Z}/2) \tag{4}$$

is a chain map, where $\mathrm{CF}_{\geq A} = \mathrm{CF}/\mathrm{CF}_{<A}$. When $(W, \omega)$ is rational it is not hard to see that (4) is an isomorphism on chain level; briefly, the reason is that any element of $\mathrm{Hom}(\mathrm{CF}_{\geq A}(\psi_t^{-1}), \mathbb{Z}/2)$ can be regarded as an infinite sum of the form $\sum_j \langle (y_j, v_j), - \rangle$ with $\mathcal{A}(\psi_t; y_j, v_j) \leq -A$ (any such infinite sum is well-defined as an element of the dual space). The rationality assumption ensures that there are only finitely many capped orbits of $\psi_t$ with action in a given compact interval, and hence $\sum(y_j, v_j)$ is well-defined as an element of $\mathrm{CF}_{\leq -A}(\psi_t)$. This proves that (4) is surjective; the proof of injectivity is easier and is left to the reader.

**2.8.** *Naturality transformations.* Given a contact-at-infinity system $\phi_t$ with $\phi_1 = \mathrm{id}$ whose orbits $\phi_t(x)$ are contractible, one can associate a chain level *naturality transformation*

$$\mathfrak{n} : \mathrm{CF}(\psi_t) \to \mathrm{CF}(\phi_t \psi_t)$$

for any other contact-at-infinity system $\psi_t$. There is a close relationship between naturality transformations and the *Seidel representation* of [Sei97].

Bearing in mind that $W$ is assumed to be connected, the operation depends on an auxiliary choice of capping of one of the orbits $\phi_t(x)$, namely, a cylinder $u : [-1, 1] \times \mathbb{R}/\mathbb{Z} \to W$ so that $u(-1, t) = x$ and $u(1, t) = \phi_t(x)$. For any other choice of point $y \in W$, one can take any path $\eta(s)$ from $x$ to $y$ and define $u_y$ to be the capping of $y$ relative $\phi_t$ obtained by concatenating:

(1) the inverse of $\eta$ (from $y$ to $x$),
(2) the capping $u$ of $x$,
(3) the cylinder $\phi_t(\eta)$ from $\phi_t(x)$ to $\phi_t(y)$.

Since $\phi_t$ has zero flux, $u_y$ is independent of $\eta$ up to equivalence.

The transformation $\mathfrak{n}$ sends the capped orbit $(y, v)$ in $\mathrm{CF}(\psi_t)$ to the concatenation $(y, u_y \# \phi_t(v))$. It is immediate from our definition of the Floer homology differential in terms of the time-1 map that $\mathfrak{n}$ is a chain map; see, e.g., [KS21, pp. 3308] for a similar construction.

One computes that $\mathfrak{n}$ shifts action values according to the action of the capped orbit $(y, u_y)$ of the system $\phi_t$. Since $(y, u_y)$ is a critical point for the action functional (on the covering space of cappings), this action shift is independent of $y$. As in [KS21, Proposition 31], it follows that:

**Lemma 2.6.** *If $a \in \mathrm{HF}(\psi_t)$, then $c(\phi_t\psi_t; \mathfrak{n}(a)) = c(\psi_t; a) + \mathcal{A}(\phi_t; x, u)$.* □

**2.8.1.** *A particular naturality transformation.* In this section we consider the particular loop $\phi_t$ on $W = \mathbb{C}^n$ generated by $H = \pi|z|^2$. This Hamiltonian function is normalized and it generates the $\mathbb{R}/\mathbb{Z}$-action $\phi_t(z) = e^{2\pi i t} z$. Therefore $\phi_t$ induces a naturality transformation on Floer homology:

$$\mathfrak{n} : \mathrm{HF}(\psi_t) \to \mathrm{HF}(\phi_t\psi_t).$$

Since $\mathbb{C}^n$ is symplectically aspherical, there is a unique choice of auxiliary capping $(x, u)$; let us therefore take $x = 0$ and $u$ to be the constant capping. It follows that $\mathfrak{n}$ is action preserving. To be precise:

**Lemma 2.7.** *Let $a \in \mathrm{HF}(\psi_t)$ where $\psi_t$ is as above. Then the spectral invariant of $a$ equals the spectral invariant of $\mathfrak{n}(a) \in \mathrm{HF}(\phi_t\psi_t)$.* □

Consider $\mathrm{HF}(\psi_t R_{-\epsilon t})$ for $\epsilon \in (0, 1)$ and where $\psi_t$ is compactly supported. It is well-known that this group is 1-dimensional over $\mathbb{Z}/2$. Indeed, the dimension is independent of the choice of $\psi_t$ and one can find a representative so that $\psi_t R_{-\epsilon t}$ has a single non-degenerate orbit located at $z = 0$. It must therefore hold that the unit element is the sole non-zero element.

Let us then define $\mathrm{pt} \in \mathrm{HF}(\phi_t\psi_t R_{-\epsilon t})$ to be the image of 1 under $\mathfrak{n}$. We call this element the "point class" since $\phi_t\psi_t R_{-\epsilon t}$ has a positive slope at infinity (namely $1 - \epsilon$), and so the generator should be thought of as the minimum of a Morse function (rather than the unit element which should be thought of as the maximum of a Morse function). Since $\mathfrak{n}$ is an isomorphism, the point class generates $\mathrm{HF}(\phi_t\psi_t R_{-\epsilon t})$. This point class plays a role in §2.9.

Using continuation or naturality maps, we therefore define $\mathrm{pt} \in \mathrm{HF}(\psi_t R_{st})$ for any positive slope $s \in (0, 1)$ if $\psi_t$ is compactly supported. This element is natural, i.e., is preserved under continuation maps, because continuation maps commute with naturality maps; see, e.g., [CHK23, §2.2.5].

**2.9.** *The spectral diameter of a ball.* This section proves that the spectral diameter of $B(a)$ equals $a$. The lower bound is well-known and follows from a simplified version of the construction in §2.6. It is then enough to prove that $\gamma(\psi_t) \le a$ for all systems $\psi_t$ compactly supported in $B(a)$. Without loss of generality, let us set $a = 1$, and fix a system $\psi_t$ supported in $B(1)$.

The proof splits into five claims. The argument involves the naturality transformation $\mathfrak{n}$ from §2.8.1 which sends the unit class 1 to the class pt. It is important that 1 generates $\mathrm{HF}(R_{-st}\psi_t)$ and pt generates $\mathrm{HF}(R_{st}\psi_t)$ for any $s \in (0,1)$ and any compactly supported $\psi_t$; i.e., there are two relevant Floer homologies and each is one-dimensional.

**Claim 2.8.** *Let $s \in (0,1)$ and for $\delta > 0$ let $R_s = R_{\delta,s}$ be as in §2.1.5; then:*

*(i)* $\gamma(\psi_t) = 2s(1+\delta/2) + c(R_{-st}\psi_t; 1) + c(R_{-st}\psi_t^{-1}; 1)$,
*(ii)* $c(R_{-st}\psi_t^{-1}; 1) = -c(\psi_t R_{st}; \mathrm{pt})$.

*Let $\phi_t$ be the loop generated by $H = \pi|z|^2$, and suppose $s > 1/2$; then:*

*(iii)* $c(R_{st}\psi_t; \mathrm{pt}) + c(\phi_t R_{-2st}; 1) \ge c(\phi_t R_{-st}\psi_t; \mathrm{pt})$,
*(iv)* $c(\phi_t R_{-st}\psi_t; \mathrm{pt}) = c(R_{-st}\psi_t, 1)$, *and*
*(v)* $c(\phi_t R_{-2st}; 1) \le (1-2s)(1+\delta/2)$.

Before we prove the claim, we use it to determine the spectral diameter of the ball $B(1)$. We estimate:

$$\begin{aligned}
\gamma(\psi_t) &= 2s(1+\delta/2) + c(R_{-st}\psi_t; 1) + c(R_{-st}\psi_t^{-1}; 1)\\
&= 2s(1+\delta/2) + c(R_{-st}\psi_t; 1) - c(\psi_t R_{st}; \mathrm{pt})\\
&\le 2s(1+\delta/2) - c(\phi_t R_{-st}\psi_t; \mathrm{pt}) + c(\phi_t R_{-2st}; 1) + c(R_{-st}\psi_t; 1)\\
&= 2s(1+\delta/2) + c(\phi_t R_{-2st}; 1)\\
&\le 2s(1+\delta/2) + (1-2s)(1+\delta/2) = 1 + \delta/2.
\end{aligned}$$

The $j$th (in)equality uses the $j$th item in Claim 2.8. Since $\delta$ can be chosen arbitrarily small, we conclude $\gamma(\psi_t) \le 1$. We now prove the five claims.

*Proof of Claim 2.8.* We begin by recalling the definition of the spectral norm:

$$\gamma(\psi_t) := \lim_{\epsilon \to 0} c(R_{-\epsilon t}\psi_t; 1) + c(R_{-\epsilon t}\psi_t^{-1}; 1).$$

Therefore to prove (i) it is enough to prove the following identity for all systems $\psi_t$:

$$c(R_{-st}\psi_t; 1) + (s-\epsilon)(1+\delta/2) = c(R_{-\epsilon t}\psi_t; 1). \tag{5}$$

Let $K_t$ generate $\psi_t$ so that $R_{-\sigma t}\psi_t$ is generated by:

$$K_t^\sigma = K_t - \sigma(\mu_\delta(\pi|z|^2 - 1) + 1), \tag{6}$$

where we use the fact that $\psi_t$ and $R_s$ commute with each other. The cut-off function $\mu_\delta$ is defined in §2.1.5.

Observe that the 1-periodic orbits of the system $R_{-\sigma t}\psi_t$ are independent of $\sigma \in (0,1)$, and all remain in the ball $B(1)$; indeed, outside of the ball we have $R_{-\sigma t}\psi_t = R_{-\sigma t}$ which rotates with a non-zero speed because of our construction in §2.1.5. The action of the orbits depends on $\sigma$, as is clear from equation(6). Therefore a continuity argument (and the nowhere density of the action spectrum) implies that the spectral invariant must change according to:

$$\frac{\partial}{\partial\sigma}c(R_{-\sigma t}\psi_t;1) = -(1+\mu_\delta(\pi|z|^2-1)) = -(1+\delta/2),$$

This implies equation (5), and completes the proof of (i).

The second item (ii) follows from the duality formula proved in Lemma 2.5 and the fact that the only non-zero class in $\mathrm{HF}(\psi_t R_{st})$ is the point class pt.

The third item (iii) is an immediate consequence of the subadditivity of the spectral invariants proved in §2.1.9.

Item (iv) follows from the fact that the naturality transformation defined by $\phi_t$ is action preserving and it sends the unit $1 \in \mathrm{HF}(R_{-st}\psi_t)$ to the point class $\mathrm{pt} \in \mathrm{HF}(\phi_t R_{-st}\psi_t)$ which is the only non-zero class in this group; see §2.8 for further discussion.

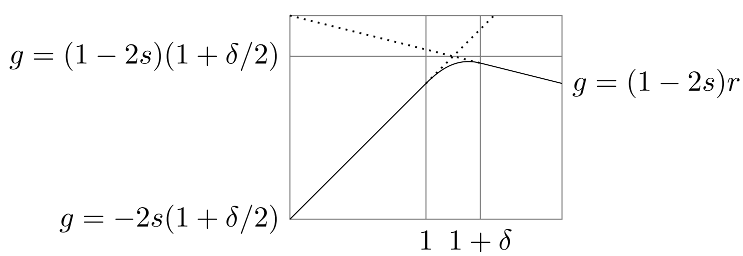


Figure 3. The graph of $g(r)$; its graph remains below the dashed lines $g = (1-2s)r$ and $g = r - 2s(1+\delta/2)$ and coincides with one of these lines when $r \notin (1, 1+\delta)$.

To establish (v) we argue as follows; first the Hamiltonian function $G$ generating $\phi_t R_{-2st}$ is equal to:

$$G = \pi|z|^2 - 2s(\mu_\delta(\pi|z|^2-1)+1) = g(\pi|z|^2);$$

see Figure 3 for the graph of $g(r)$.

The $g$-coordinate of the intersection of the two dashed lines in Figure 3 equals $(1-2s)(1+\delta/2)$. It follows that every orbit of $\phi_t R_{-2st}$ has action bounded from above by this amount, assuming of course that $1/2 < s < 1$, and hence:

$$c(\phi_t R_{-2st};1) \leq (1-2s)(1+\delta/2),$$

which is what we wanted to show. □

**2.10.** *Comparison with the bound of Entov and Polterovich.* It is shown in [EP03] that the spectral diameter of $\mathbb{C}P^n$ is equal to the symplectic area of the class of a line, which we normalize to be 1. It also holds that $\mathbb{C}P^n$ contains a symplectically embedded copy of $B(1)$; let us fix the standard embedding $e : B(1) \to \mathbb{C}P^n$ whose image is the complement of the hyperplane divisor. Then, for any Hamiltonian system $\psi_t$ with compact support in $B(1)$, we can implant $\psi_t$ as a compactly supported system $e\psi_t e^{-1}$ in $\mathbb{C}P^n$, and know that $\gamma_{\mathbb{C}P^n}(e\psi_t e^{-1}) \leq 1$.

It is therefore natural to wonder whether $\gamma_{B(1)}(\psi_t) \leq \gamma_{\mathbb{C}P^n}(e\psi_t e^{-1})$ holds, as such a relation would imply our result from §2.9. However, this strategy does not work because:

**Proposition 2.9.** *There exist systems $\psi_t$ supported in $B(1)$ so that:*

$$\gamma_{\mathbb{C}P^1}(e\psi_t e^{-1}) < \gamma_{B(1)}(\psi_t);$$

*the difference between the two sides can be made arbitrarily close to 1.*

We state the result only for the case $n = 1$ as the argument is very simple in that case, although the authors expect a similar phenomenon holds in higher dimensions.[3]

*Proof.* Let $\psi_t$ be the Hamiltonian system generated by a radial bump function $H = f(\pi|z|^2)$ where $f$ is non-increasing, equals 1 on $[0, 1-\epsilon]$ and rapidly cuts off to zero so as to have compact support in $[0, 1)$. A straightforward consideration of the spectrum shows that $\gamma_{B(1)}(\psi_t) \geq 1 - \epsilon$.

On the other hand, by taking $\epsilon \to 0$, we arrange that $e\psi_t e^{-1}$ is supported in any chosen neighbourhood $U$ of the divisor $\mathbb{C}P^{n-1}$, which is a single point when $n = 1$. The argument is finished by showing that:

$$\inf\{\gamma(U) : \mathbb{C}P^{n-1} \subset U\} = 0; \tag{7}$$

in words, the spectral diameter of small neighborhoods of the divisor is small. In the low-dimensional case $n = 1$, (7) follows from the displacement energy bound of §2.2, completing the proof. □

**2.11.** *Spectral diameter of special balls in $\mathbb{C}P^n$.* The proof of Theorem 1.5 is based on the following two lemmas, in which we denote by the homology class in $M = \mathbb{C}P^n$ represented by the hyperplane by $\Gamma \in H_{2n-2}(M)$, the fundamental class by $[M]$ and the point class by [pt].

**Lemma 2.10.** *For any Hamiltonian system $\psi_t$ on $M = \mathbb{C}P^n$, we have:*

$$\gamma(\psi_t) + c(\varphi_t\psi_t; [M]) - c(\varphi_t\psi_t; \Gamma) = 1;$$

*where $\varphi_t$ is the loop generated by $S_a$.*

[3] One approach to Proposition 2.9 in higher dimensions is to exploit the fact that $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ is *stably displaceable* (see [Gür08, §4.3]), and that the spectral invariants satisfy a Künneth formula ([EP09, Theorem 5.1]); see [Bor12, §3.6] for a related discussion.

Introduce $U = \{S_a > 0\}$, i.e., $U$ is the interior of the standard special ball of capacity $1 - a$. Then:

**Lemma 2.11.** *For any Hamiltonian system $\psi_t$ supported in $U$, we have:*

$$c(\varphi_t\psi_t; [M]) - c(\varphi_t\psi_t; \Gamma) \geq a.$$

*where $\varphi_t$ is the loop generated by $S_a$.*

From the above lemmas, it follows that the spectral diameter of $U$ is at most $1 - a$. Combining this upper bound with the lower bound furnished by Proposition 2.4 yields Theorem 1.5.

In the proofs of both lemmas we allow ourselves to work with non-normalized Hamilltonians on $\mathbb{C}P^n$, using the conventions in §2.1.10.

**2.11.1.** *Proof of Lemma 2.10.* The proof is a simple application of the naturality transformation associated to the loop generated by $S_a$ and the inverse loop generated by $\bar{S}_a$.

The naturality transformation associated to $S_a$ sends the class represented by the maximum of a perfect Morse function to the class of the index $2n-2$ critical point. Consequently by Lemma 2.6:

$$c(H_t; [M]) = c(S_a \# H_t; \Gamma) + \text{const},$$

for every Hamiltonian $H_t$ supported in $U$. The constant term is the action of a capped orbit of $S_a$ and can be determined by sending $H_t \to 0$, in which case it must equal $c(0; [M]) - c(S_a; \Gamma)$. Since $S_a$ can be perturbed to a perfect Morse function whose index $2n - 2$ critical point has action $-a$ (see Figure 1), and the Hessian at this critical point is small enough, it follows that $c(0; [M]) - c(S_a; \Gamma) = a$, and hence the above equation becomes:

$$c(H_t; [M]) = c(S_a \# H_t; \Gamma) + a. \tag{8}$$

The next stage of the argument is similar but is based instead on the naturality transformation generated by $\bar{S}_a = -S_a$. This naturality transformation sends the class of the maximum to the class of the minimum, and hence:

$$c(S_a \# H_t; [M]) = c(\bar{S}_a \# S_a \# H_t; [\text{pt}]) + \text{const} = c(H_t; [\text{pt}]) + \text{const}.$$

The constant term can again be determined by sending $H_t \to 0$, in which case it equals $c(S_a; [M])$ which equals $1 - a$, i.e., the critical value of the maximum as shown in Figure 1. Thus:

$$c(S_a \# H_t; [M]) = c(H_t; [\text{pt}]) + 1 - a. \tag{9}$$

To complete the proof add together (8) and (9), for $H_t$ generating $\psi_t$. □

**2.11.2.** *Proof of Lemma 2.11.* Let $U = \{S_a > 0\}$, as in the statement, and suppose that $H_t$ is a Hamiltonian function supported in $U$ generating $\psi_t$.

For the purposes of the proof, we introduce two time-reparametrization operations on a Hamiltonian function $G_t$ which does not affect its time-1 map in the universal cover:

$$G_t^* := \beta'(2t)G_{\beta(2t)} \text{ and } G_t^{**} := \beta'(2t-1)G_{\beta(2t-1)}.$$

Here $\beta : \mathbb{R} \to [0,1]$ is a standard smooth cut-off so $\beta(t) = 0$ for $t \le 0$ and $\beta(t) = 1$ for $t \ge 1$; we require that $\beta'(t)$ is non-negative.

The significance of these operations is the following: $G_t^*$ is supported where $t \in (0, 1/2)$ while $G_t^{**}$ is supported where $t \in (1/2, 1)$.

Due to the fact that spectral invariants depend only on the time-1 map in the universal cover (and the average value of the Hamiltonian), it follows that for any homology class $\Pi$, we have:

$$c(S_a \# H_t; \Pi) = c(S_a^* \# H_t^{**}; \Pi).$$

Now introduce the piecewise smooth function $K_a = \min\{S_a, 0\}$, as shown in Figure 4. Since $K_a$ is pointwise less than $S_a$, it follows by a standard continuation argument that, for any class $\Pi$, we have

$$c(K_a^* \# H_t^{**}; \Pi) \le c(S_a^* \# H_t^{**}; \Pi).$$

We note that $K_a^*$ is not smooth, but nonetheless the spectral invariant $c(K_a^* \# H_t^{**}; \Pi)$ is well-defined via a limiting process, because of the Hofer continuity of spectral invariants.

The next stage of the argument is a deformation argument. Roughly speaking, the idea is to interpolate from $K_a$ to 0 while keeping track of the indices and actions of the orbits during the process. To make this precise, we argue in a slightly ad hoc fashion.

Fix $0 < \epsilon < a$ and introduce the $s$-dependent family of functions:

$$T_s = \max\{K_a, sK_a - \epsilon\},$$

Note that on the set $\{K_a \ge -\epsilon\} = \{S_a \ge -\epsilon\}$, $T_s = T_1$. First smooth $T_1$ on this set so as to make $T_1^* \# H_t^{**}$ have non-degenerate orbits on $\{K_a \ge -\epsilon/2\}$.

Then, for each $s$, smooth $T_s$ on $\{K_a \le -\epsilon/2\}$ so that the only orbits outside in $\{K_a \le -\epsilon/2\}$ are the Morse critical points of indices $\{0, 2, \dots, 2n-2\}$ located near the divisor $\{K_a = -a\}$.

(1) orbits contained entirely in the region where $T_s^* \# H_t^{**} = T_1^* \# H_t^{**}$,
(2) orbits whose Floer homology grading is in
$$\{0, 2, \dots, 2n-2\},$$
located near the divisor $\mathbb{C}P^{n-1}$.

Because the minimal Chern number of $\mathbb{C}P^n$ is $n+1$, it follows that the orbits of type (2) never appear in a linear combination representing $[M]$, which has degree $2n$. Consequently $c(T_s^*\#H_t^{**};[M])$ is independent of $s$, since it is valued in the $s$-independent nowhere dense spectrum of orbits of type (1). Note that $T_0^*\#H_t^{**}$ is $\epsilon$-close to $H_t^{**}$, and thus we conclude that:

$$c(K_a^*\#H_t^{**};[M]) = c(T_1^*\#H_t^{**};[M]) = c(H_t;[M]) + O(\epsilon).$$

Since $\epsilon$ was arbitrary we conclude $c(K_a^*\#H_t^{**};[M]) = c(H_t;[M])$.

Combining the two steps, with $\Pi = [M]$, we conclude:

$$c(S_a\#H_t;[M]) = c(S_a^*\#H_t^{**};[M]) \geq c(K_a^*\#H_t^{**};[M]) = c(H_t;[M]). \tag{10}$$

Next, we recall (8), so that:

$$c(S_a\#H_t;[M]) - c(S_a\#H_t;\Gamma) = c(S_a\#H_t;[M]) - c(H_t;[M]) + a \geq a,$$

where we use (10) in the final step. This completes the proof. □

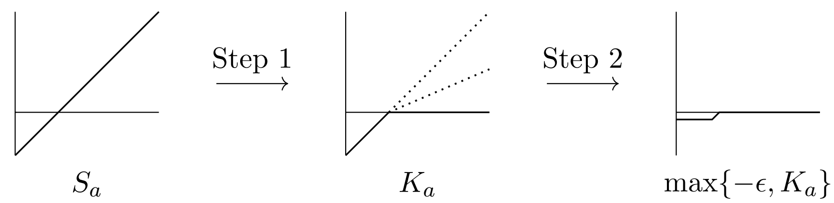


FIGURE 4. Deformation used in the proof of Lemma 2.11.

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