A DIMENSION FORMULA FOR RELATIVE SYMPLECTIC FIELD THEORY

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Abstract

This dissertation is devoted to proving virtual dimension formulas for the moduli spaces of holomorphic curves which appear in relative Symplectic Field Theory. The crucial ingredients are a generalization of the large antilinear deformation argument in [**Tau96**] and [**Ger18**] to the case when the domain of the curve has boundary and punctures, and an exponential convergence result generalizing the 3-dimensional results proved in [**Abb99**].

Acknowledgements

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Contents

Chapter 1

Introduction

Let (Y^{2n+1}, ξ, α) be a contact manifold with a contact form α. *Symplectic Field Theory* can be thought of as a Floer-type theory for the *contact action functional*:

$$\mathcal{A}(\gamma) = \int_{\mathbb{R}/\mathbb{Z}} \gamma^* \alpha,$$

defined on the infinite dimensional space of loops $C^\infty(\mathbb{R}/\mathbb{Z}, Y)$. *Relative Symplectic Field Theory* is the analog where we introduce a *Legendrian* submanifold $\Lambda^n \subset Y$ and consider the analogous action:

$$\mathcal{A}(c) = \int_0^1 c^* \alpha,$$

on the domain $C^\infty([0,1], Y, \Lambda)$, i.e., smooth paths joining Λ to itself. The data (Y, ξ, α) specifies a vector field R called the *Reeb* vector field, and it is easy to see that critical points of $\mathcal{A}$ are unparametrized flow lines for R.[1] These split into two classes: *orbits* and *chords*.

As explained in §1.3, there is a family of complex structures on $\mathbb{R} \times Y$ adapted to the contact form α which have the property that *holomorphic curves* defined on punctured Riemann surfaces are asymptotic to Reeb chords and orbits at their boundary and interior punctures, respectively. In this sense, the counts of holomorphic curves can be interpreted as some sort of Morse/Floer homology for the above action functional.

For such a choice of complex structure J, and a punctured Riemann surface (Σ, j) we can form the *moduli space* of holomorphic maps $\mathcal{M} = \mathcal{M}(\Sigma, j, J)$. Near a given $u \in \mathcal{M}(\Sigma, j, J)$, we construct a Sobolev manifold $W^{1,p,\delta}$, where $\delta > 0$ is a sufficiently small exponential weight, and a (smooth) non-linear operator $\bar{\partial}_J$ defined on $W^{1,p,\delta}$ so that $\mathcal{M}$ is identified with the inverse image $\bar{\partial}_J^{-1}(0)$. The associated linearized operator D_u is a *Cauchy-Riemann operator with non-degenerate asymptotics*. The operator D_u is a Fredholm operator, as explained in §6. When D_u is surjective, then $\mathcal{M}$ is a manifold whose dimension equals the Fredholm index D_u. In general, we say that the Fredholm index of D_u is the *virtual*, or *expected*, dimension of $\mathcal{M}$. The main result of this thesis is providing the general formula for the virtual dimension $d(u)$:

$$d(u) = (n+1)\mathrm{X}(\bar{\Sigma}) - n\,|\partial\Gamma_-| - n\,\left|\Gamma^{\mathrm{int}}\right| + M_{\mathfrak{s}} \cdot [u] + \sum_{\zeta \in \Gamma_+} \mu_{\mathrm{CZ}}(A_\zeta^{\mathfrak{s}}) - \sum_{\zeta \in \Gamma_-} \mu_{\mathrm{CZ}}(A_\zeta^{\mathfrak{s}}).$$

[1] In other words, if $c'(t) = f(t)R(t)$ for $f(t) \in \mathbb{R}$, then c will be a critical point for $\mathcal{A}$.

We briefly summarize the terms.

(i) $\mathrm{X}(\bar{\Sigma})$ is the Euler characteristic of the unpunctured surface,

(ii) $\partial\Gamma_-$, Γ^{int} are the negative boundary punctures and all interior punctures, respectively,

(iii) $M_{\mathfrak{s}} = \mathfrak{s}^{-1}(0)$ is the *relative Maslov Class* associated to a section $\mathfrak{s}$ of $\det_{\mathbb{C}}(\xi)^{\otimes 2}$ (see §1.3.3 for the requirements on $\mathfrak{s}$), and

(iv) $\mu_{\mathrm{CZ}}(A_\zeta^{\mathfrak{s}})$ is the *Conley-Zehnder index* of the non-degenerate asymptotic operator associated to the puncture ζ. The section $\mathfrak{s}$ specifies a unique way to extract this index, as explained in §1.3.3.

Both the homological intersection $M_{\mathfrak{s}} \cdot [u]$ and the Conley-Zehnder indices $\mu_{\mathrm{CZ}}(A_\zeta^{\mathfrak{s}})$ depends on the auxiliary section $\mathfrak{s}$ in a non-trivial fashion, however, the combination appearing in the formula of $d(u)$ is independent of $\mathfrak{s}$.

This thesis also contains a novel proof of the index formula for Cauchy-Riemann operators with non-degenerate asymptotic ends, based on the *large antilinear deformation* ideas of [**Tau96**], [**Ger18**], [**Wen20**]. A consequence of this argument is the invariance of a novel *relative Euler characteristic* for surfaces with boundary punctures (whose invariance is not a priori obvious).

The final main result of this thesis is a proof of an exponential decay result for holomorphic curves asymptotic to Reeb chords, which was previously only proven in the case $\dim Y = 3$, due to [**Abb99**], although experts certainly knew the result was true in all dimensions.

1.1. Motivation

The Reeb vector field appears throughout symplectic geometry:

(i) If Y is the unit tangent bundle of a Riemannian manifold (M, g), then the *geodesic flow* on Y is the Reeb flow for a certain contact form; see 1.1.1.

(ii) If Y is a convex hypersurface in $\mathbb{R}^{2n}$, then for appropriate contact forms on Y, the Reeb vector field directs the *characteristic foliation of* Y.[2]

The overarching goal of relative SFT is to define homological invariants generated by the unparametrized Reeb orbits and chords of Λ, and thereby prove *rigidity* type theorems. For instance, two open problems are:

Weinstein conjecture. For every (Y, ξ, α) with Y compact, the Reeb vector field R has a closed orbit.

[2]We recall that the characteristic foliation $\mathcal{F}$ is defined by the property that, for any Hamiltonian H so that Y is level set $H = c$, X_H points along $\mathcal{F}$.

Arnol'd chord conjecture. For every $(Y, \xi, \alpha, \Lambda)$ with Y compact, there is a Reeb chord of Λ.

These conjectures have been verified in various special cases. For instance, in [**HT11**] and [**HT10**], the authors solve the chord conjecture when $\dim(Y) = 3$, generalizing earlier work [**Abb99**]. In [**Moh01**], the author establishes the chord conjecture whenever (Y, ξ) is the boundary of a subcritical Stein manifold. One can think of these conjectures as contact versions of the famous symplectic Arnol'd conjectures.

Relative SFT is an umbrella term for any construction which associates algebraic objects (e.g., vector spaces, free algebras, etc) generated by Reeb chords and orbits, equipped with *differentials* defined by counting solutions to a certain holomorphic curve PDE with appropriate asymptotic conditions. The goal is to prove that the *homology* of the resulting object is stable under isotopy of the data needed to define the PDE, and is thus an invariant of the isotopy class of (Y, Λ, ξ).

Such invariants have been rigourously defined in various special cases. For instance, the Chekanov-Eliashberg DGA, defined in [**Che97**], [**EGH00**, §2.8], which associates a *differential graded algebra* to each Legendrian knot in $\mathbb{R}^3$, freely generated by the Reeb chords of the knot, and whose differential counts holomorphic disks with boundary on the knot. The study of the Chekanov-Eliashberg DGA, and its generalizations, have been a fruitful area of research. See for instance, [**Sab02**], [**Etn04**], [**EHK16**], and the references therein.

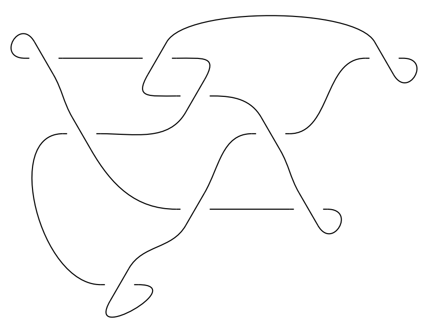


FIGURE 1. A Legendrian knot in $\mathbb{R}^3$.

1.1.1. *Legendrians, geodesic flow, and optics.* Let Y^{2n+1} be the unit tangent bundle of a Riemannian manifold (M^{n+1}, g), and pick a contact form α so that $\alpha_v = g(v, -)$. More precisely, in any local orthonormal frame X, there is $\theta : Y \to S^n$ so that $v = \theta(v) \cdot X(\mathrm{pr}(v))$. Then we set $\alpha = \sum_i \theta_i(v) \mathrm{pr}^* g(X_i, -)$.

It is an exercise in manipulating tensors to show that the Reeb flow for α is equal to the geodesic flow on Y. Moreover, it is clear, by construction, that each fiber Λ_p is a Legendrian. Thus we conclude that the Reeb chords from Λ_p to Λ_q are in bijection with the geodesics joining p to q.

In general, if $\Lambda(t) := \varphi_t(\Lambda_p)$ denotes the time t Reeb flow, then we can think of (the projection of) $\Lambda(t)$ as the evolving *wave-front* arising from a beacon of light emitted at p when $t = 0$. The theoretical basis used here is *Fermat's principle* which states that light travels along geodesics.

Basic theorems in contact geometry imply that $\Lambda(t)$ always remains Legendrian. Inspecting the contact form α, this means that the tangent space to $\mathrm{pr} \circ \Lambda(t)$ at x is orthogonal to all the unit vectors in $\Lambda(t) \cap \mathrm{pr}^{-1}(x)$. In other words, the velocity component of the wave-front is always normal to the position component of the wave-front. This latter property is equivalent to *Huygens' principle* in optics. Thus we see the equivalence of Fermat's and Huygens' principles.

1.2. Asymptotic Operators as Hessians

Asymptotic operators are linear first-order ODE operators of the form:

$$A = -J\partial_t - S(t),$$

where $S(t)$ symmetric $2n \times 2n$, and $J = \mathrm{diag}([\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}], \dots, [\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}])$. A priori, A is defined on $C^1(\mathbb{R}, \mathbb{R}^{2n})$. We typically restrict the domain to either $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ or $C^\infty([0,1], \mathbb{R}^{2n}, \mathbb{R}^n)$, and we call these two cases *orbits* and *chords* (of $\mathbb{R}^n \subset \mathbb{R}^{2n}$). In these cases, A is self-adjoint when using the obvious L^2 inner product.

Such operators have an immediate connection to symplectic geometry: the corresponding linear ODE, $\partial_t \eta = JS(t)\eta$, has fundamental solution valued in the symplectic group $\mathrm{Sp}(2n) := \{P : P^T J P = J\} \subset \mathrm{GL}_{2n}(\mathbb{R})$.

Asymptotic operators typically appear as the analogs of the *Hessian* when one tries to do Morse theory for symplectic action functionals. Asymptotic operators are said to be *non-degenerate* whenever $\ker A = 0$. This non-degeneracy is the analog of the Morse condition. Because they are self-adjoint *elliptic* operators, we can find orthonormal eigenbases consisting of eigenfuctions, i.e., u satisfying $Au = \lambda u$, where the set of eigenvalues is discrete. Unlike classical Morse theory, A will always have infinitely many positive and negative eigenvalues, and hence we cannot define the Morse index, because there is an infinite dimensional negative eigenspace.

One can associate an index to each non-degenerate asymptotic operator A called the *Conley-Zehnder index*, which has similar properties to index of a symmetric matrix in finite dimensions.

1.3. Geometric Preliminaries

We give a brief review of the geometric structures relevant to relative SFT.

1.3.1. *Contact manifolds, complex structures, and the symplectization.* Briefly, the geometric set-up is the following. Let $(Y, \alpha, \xi, J_\xi, \Lambda)$ be the data of:

(i) a $2n+1$ dimensional smooth manifold Y,

(ii) a 1-form α satisfying $\alpha \wedge \mathrm{d}\alpha^n > 0$, i.e., a *contact form*, which induces a *contact distribution* $\xi := \ker \alpha$; thus (Y, ξ) is a *contact manifold*,

(iii) a complex structure J_ξ on ξ compatible with $\mathrm{d}\alpha$, in the sense that $\mathrm{d}\alpha(-, J_\xi -)$ is an inner product on ξ,

(iv) an n-dimensional compact submanifold $\Lambda \subset Y$ so that $T\Lambda \subset \xi|_\Lambda$, i.e., a *Legendrian submanifold*.

This data induces a canonical Riemannian metric on Y given by $g := \alpha \otimes \alpha + \mathrm{d}\alpha(-, J_\xi -)$.

Associated to this, we define the *symplectization* of Y to be the data $(\mathbb{R} \times Y, \omega, J)$, where

(v) $\omega = \mathrm{d}(e^\sigma \mathrm{pr}^* \alpha)$, where $\sigma : \mathbb{R} \times Y \to \mathbb{R}$ and $\mathrm{pr} : \mathbb{R} \times Y \to Y$ are the coordinate projections, and

(vi) J is the unique almost complex structure satisfying $J\partial_\sigma = R$, $J|_\xi = J_\xi$, where R is the Reeb vector field defined above.

Each choice of J_ξ leads to an ω-compatible almost complex structure; such almost complex structures are called *admissible*.

1.3.2. *Asymptotic conditions for holomorphic curves in the symplectization.* It can be shown that holomorphic curves[3] u in the symplectization with boundary on $\mathbb{R} \times \Lambda$ have the property that, near any puncture, there is a Reeb chord or orbit c and unique numbers $T > 0$ and σ_0 so that[4]

(i) $\sigma \circ u(s,t) - Ts - \sigma_0 = o(1)$ as $|s| \to \infty$,

(ii) $\mathrm{dist}(\mathrm{pr} \circ u(s,t), c(t)) = o(1)$ as $|s| \to \infty$.

Note that if we switch coordinates $s \to -s$, then T will need to change sign in order to keep satisfying (i), and hence the requirement that $T > 0$ actually specifies a *sign* for each puncture, i.e., a canonical way of deciding whether s is valued in $[0, \infty)$ or $(-\infty, 0]$. Negative punctures are asymptotic to $\sigma = -\infty$, while positive punctures are asymptotic to $\sigma = +\infty$. The quantity T is the action of the chord or orbit.

[3]We require the curves to have finite energy. The precise notion of energy we mean is called the *Hofer* energy of a holomorphic curve, introduced in [**Hof93**]. See §9 for the precise definition.

[4]There are also removable singularities, namely those with $\mathrm{dist}(u(s,t), p) = o(1)$ as $|s| \to \infty$.

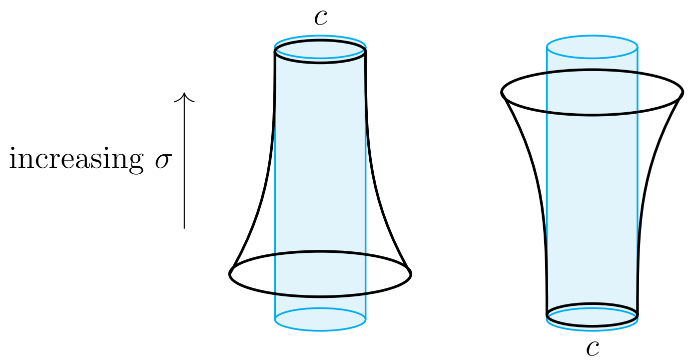


FIGURE 2. A punctured holomorphic disk (outlined in black) is asymptotic to a trivial cylinder over Reeb orbit c (shown in blue). There are two possibilities, either the σ coordinate converges to $+\infty$ (shown on the left) or the σ coordinate converges to $-\infty$.

1.3.3. *The canonical bundle, squared.* The *canonical bundle* is the complex determinant line bundle $K = \det_{\mathbb{C}}(\xi)$. When $\dim(Y) = 3$, we simply have $K = \xi$. Observe that along Λ there is an *injective* map $\det_{\mathbb{R}}(T\Lambda) \to K$, and hence defines a real-line subbundle of K. The orientability of this line bundle is equivalent to the orientability of Λ.

Introduce $K^2 = K \otimes K$, and for $p \in \Lambda$ let $\mathfrak{l}_p$ be the positive ray in K_p^2 spanned by:

$$(e_1 \wedge \cdots \wedge e_n) \otimes (e_1 \wedge \cdots \wedge e_n)$$

where $e_1, \ldots, e_n$ is *any* basis of $T\Lambda_p$. Similarly, let $-\mathfrak{l}_p$ denote the negative ray.

Definition 1.1. We say that a section $\mathfrak{s} \in K^2$ is *admissible* or *compatible* with $(Y, \alpha, J, \xi, \Lambda)$ provided that:

(i) $\mathfrak{s} \notin -\mathfrak{l}$ holds along Λ,

(ii) $\mathfrak{s}$ is non-vanishing along each Reeb chord of Λ,

(iii) $\mathfrak{s} = \mathfrak{c} \otimes \mathfrak{c}$ along each Reeb orbit, where $\mathfrak{c} \neq 0$. If such $\mathfrak{c}$ exists, then it is unique up to homotopy.

(iv) The zero set $\mathfrak{s}^{-1}(0)$ is cut transversally.

This section $\mathfrak{s}$ induces *canonical homotopy classes of asymptotic operators* at each Reeb chord and orbit, i.e., assigns canonical Conley-Zehnder indices to each chord and orbit. In other words, we should think of $\mathfrak{s}$ as a coherent way to pick Conley-Zehnder indices for each orbit or chord.

Roughly speaking, the asymptotic operator is given by linearizing the non-linear Reeb flow equation at the chord or orbit. To linearize, we require a coordinate system. Thus, let Φ_t, $t \in \mathbb{R}/\mathbb{Z}$ or $t \in [0, 1]$ be *admissible coordinates* around the orbit or chord, as defined in §3.1. Introduce $\varphi := \det(\mathrm{d}\Phi_t(0))1$, $\varphi \in K$.

For each orbit, use $\mathfrak{c}$ to select the homotopy class of coordinate charts can be homotoped so as to satisfy $g(\varphi, \mathfrak{c}) > 0$, where g is the induced Riemannian metric on $\det_{\mathbb{C}}(\xi)$.

For each chord, we use $\mathfrak{s}$ to select *two* homotopy classes of coordinate charts, namely those which can be homotoped so as to satisfy $g(\varphi^{\otimes 2}, \mathfrak{s}) > 0$. These charts differ by how they orient $T\Lambda_0$. It is proved in §3.4 that the asymptotic operators arising from the two homotopy classes of charts in the chord case have the same Conley-Zehnder index.

The relationship between coordinates and $\mathfrak{s}$ is explained in greater detail in §3.2.

1.3.4. *The relative Maslov class.* The zero set $M_{\mathfrak{s}} := \mathfrak{s}^{-1}(0)$ represents the *Maslov class* for the Legendrian Λ. The transversality hypothesis implies that this zero set is a compact cooriented codimension 2 submanifold of Y.

We should think of $M_{\mathfrak{s}}$ as representing a class in the relative cohomology group $H^2(Y, \Lambda)$. However, there are additional homotopical restrictions placed on $\mathfrak{s}$ by virtue of conditions (ii) and (iii) above. These additional constraints can be interpreted as a lift of $[M_{\mathfrak{s}}] \in H^2(Y, \Lambda)$ to $H^2(Y, \Lambda \cup \mathcal{R})$, where $\mathcal{R}$ is the set of Reeb orbits and chords.

1.3.5. *Relative Maslov numbers of maps.* Suppose that Σ is a potentially non-compact oriented surface and $u : \Sigma \to Y$ has topological boundary contained in $\Lambda \cup \mathcal{R}$, in the sense that for all $z_n \to \infty$, the limit points of $u(z_n)$ are contained in $\Lambda \cup \mathcal{R}$. Here $\mathcal{R}$ is a collection of Reeb chords and orbits. Then we can define the Maslov number of u as the homological intersection number with $M_{\mathfrak{s}}$, as follows.

Definition 1.2. The homological intersection number $M_{\mathfrak{s}} \cdot [u]$ is defined by perturbing u on a compact neighborhood of $u^{-1}(M_{\mathfrak{s}})$ so as to make u and $M_{\mathfrak{s}}$ transverse, and then counting the signed count of intersection points (using the orientation of Σ and the coorientation of $M_{\mathfrak{s}}$).

This number is invariant under homotopies of $\mathfrak{s}$ which remain *admissible*, since the inverse image $u^{-1}(M_{\mathfrak{s}})$ will remain in some compact set during such homotopies.

This number is also invariant under homotopies of u, provided that the homotopies $u(\pi, z)$ satisfy the property that the limit points of $u(\pi_n, z_n)$ lie in $\Lambda \cup \mathcal{R}$ whenever $z_n \to \infty$ (with no restriction on π_n). Here π should be thought of varying in $[0, 1]$.

1.4. The dimension formula for relative SFT

With these preliminaries out of the way, we can state a main result of this thesis.

Theorem 1.3. Let $\mathcal{M}$ be the moduli space of all parametrized holomorphic curves nearby u with the same underlying punctured domain Σ. Then the *virtual* (or *expected*) dimension of $\mathcal{M}$ is given by:

$$\dim(\mathcal{M}) = (n+1)\mathrm{X}(\bar{\Sigma}) - n\left|\partial\Gamma_-\right| - n\left|\Gamma^{\mathrm{int}}\right| + M_{\mathfrak{s}} \cdot [u] + \sum_{\zeta\in\Gamma_+} \mu_{\mathrm{CZ}}(A_\zeta^{\mathfrak{s}}) - \sum_{\zeta\in\Gamma_-} \mu_{\mathrm{CZ}}(A_\zeta^{\mathfrak{s}}).$$

where $\mathrm{X}(\bar{\Sigma})$ is the Euler characteristic of the unpunctured domain (i.e., the compactified domain), and n is the dimension of Λ. Here $\partial\Gamma_-$ are the negative boundary punctures, while Γ^{int} are the interior punctures.

Proof. This is proved in §4.3. □

Remark 1.4. By *virtual dimension*, we mean the Fredholm index of the linearized operator associated to u, as defined in §4.2. If one wishes to compute the dimension of curves with constrained asymptotic markers (as is the case in the definition of contact homology), one should subtract an additional $\left|\Gamma^{\mathrm{int}}\right|$. As explained in §4.3, our formula matches the one in [**BM04**, Proposition 4] which is stated for curves without boundary.

As a corollary, we obtain the following dimension formula in the special case when Y is the 1-jet space of an n-dimensional manifold, and Λ is a collection of 1-jet sections $\Lambda_{f_1}, \Lambda_{f_2}, \dots$ (see §3.6). For appropriate choices, Reeb chords $\Lambda_{f_i} \to \Lambda_{f_j}$ between 1-jet sections are in bijection with positive critical points of the function difference $f_j - f_i$. We require all the critical points c appearing as asymptotics to be *Morse*, and hence it makes sense to speak of the Morse index of c, denoted $\mu_{\mathrm{Mor}}(c)$. We then have:

Theorem 1.5. The dimension of the space of parametrized maps near u with the same underlying domain is:

$$\dim(\mathcal{M}) = (n+1)\mathrm{X}(\bar{\Sigma}) - n\left|\Gamma_+\right| + \sum_{\zeta\in\Gamma_+} \mu_{\mathrm{Mor}}(\zeta) - \sum_{\zeta\in\Gamma_-} \mu_{\mathrm{Mor}}(\zeta).$$

Proof. In §3.6.1 we explain how to pick the section $\mathfrak{s}$. For this choice we have

$$\mu_{\mathrm{CZ}}(A_\zeta^{\mathfrak{s}}) = \mu_{\mathrm{Mor}}(\zeta) - n,$$

as proved in §3.6.2.1. Moreover, the choice of $\mathfrak{s}$ is globally non-zero, hence $M_{\mathfrak{s}} = \emptyset$ and so there is no Maslov class term. We then substitute into the formula from Theorem 1.3. □

Some more dimension formulas involving this 1-jet example, including ones which allow variations of the domain, are explained in §4.3.1.

1.5. Outline of thesis and survey of literature

Special cases of the formula in Theorem 1.3 have appeared throughout the literature. In particular, the original SFT paper [**EGH00**] has a dimension formula in the case when $\partial\Sigma = \emptyset$. The paper [**CEJ10**] contains a dimension formula which applies in our geometric context, although it is stated in different terms (i.e., they do not define the class $M_{\mathfrak{s}}$ or the

Conley-Zehnder index for Reeb chords, which are the crucial terms in our formula). The equivalence of the formula in [**CEJ10**] and ours follows *a posteriori*, as they both compute the same quantity, although an *a priori* proof of their equivalence would certainly involve many of the results proved in this thesis.

As explained in §7, the Conley-Zehnder index is defined as the Fredholm index of a certain operator, similarly to [**Abo14**, §1] or [**Par19**, §2]. In §2.4, we show that this definition as a Fredholm index is equivalent to a definition as a *spectral flow*, agreeing with the definition given in [**Wen20**]. The Conley-Zehnder of a chord also can be expressed as the signed intersection number of a certain path of Lagrangians with the singular codimension 1 cycle in the Lagrangian Grassmannian introduced in [**Arn67**]. See [**BEE12**, Remark 2.1] for an approach to assigning integers to Reeb chords which uses this intersection theoretic interpretation of the Conley-Zehnder index for chords.

The main focus of §3 concerns the linearization the Reeb ODE at a chord; the linearization process leads asymptotic operator in the sense described above. See [**Wen20**] for a discussion of the linearization in the case Reeb orbits. A similar coordinate system approach in the case of linearizing the Hamiltonian ODE at an intersection point of between two Lagrangians is presented in [**RS01**] (in the case when the Hamiltonian is 0). In §3.1.3, we explain the analogous picture for Reeb orbits.

In the case when Y is the 1*-jet space of a smooth manifold*, the linearization can be computed explicitly. This is the topic of §3.6. See [**EES02**], [**EHK16**], [**Ekh07**], [**BEE12**], and [**EL17**] for research involving Legendrians in 1-jet spaces.

In §4, we explain how to linearize the holomorphic curve PDE in the symplectization, and then apply the index formula from §6 to prove the dimension formula (Theorem 1.3). This section was inspired by [**Wen10**], [**Wen20**], [**MS12**], [**BC07**], who implement similar linearization procedures. A crucial aspect is the exponential decay result in §10.

The Conley-Zehnder indices and Maslov classes of Legendrian knots in $\mathbb{R}^3$ are studied in §5. We present an algorithm for computing them via certain *crossing-moves*. One application of our framework is that we obtain canonical Conley-Zehnder indices for knots with rotation number zero which agree with those in [**Etn04**, §4.1] and [**EES02**, §2.3].

In §6, the main focus is the analysis of *asymptotically non-degenerate* Cauchy-Riemann operators on punctured Riemann surfaces with boundary. Briefly, asymptotically non-degenerate Cauchy-Riemann operators are those which are asymptotic to operators of the form

$$\partial_s + J\partial_t + S(t) = \partial_s - A,$$

in strip-like and cylindrical ends around the punctures, where the asymptotic operator A is non-degenerate. The main result is a proof of the Fredholm property for this class of operators, following the strategy introduced by [**Sal97**, §2.3] for the case of closed orbits. In §7 we determine how the Fredholm index depends on the asymptotic operators via a kernel-gluing argument, analogous to the one given in [**Sch95**, §3]. This leads to a natural definition of the Conley-Zehnder indices as the Fredholm indices of certain operators, as mentioned above. In §8, we generalize the "large-antilinear deformation" argument introduced by [**Tau96**], [**Ger18**], [**Wen20**], to compute an explicit formula for the Fredholm index for any asymptotically non-degenerate Cauchy-Riemann operator.

In the final two chapters, §9 and §10, we generalize the exponential convergence result for Reeb chords in [**Abb99**] to all dimensions, following similar strategies to those in [**RS01**]. The results in §9 I learned from [**Abb14**]. The relevance of exponential estimates for holomorphic curves with non-degenerate asymptotic conditions is well-studied in SFT, see [**Hof93**], [**HWZ96**], [**HWZ02**], [**BEH$^+$03**]. In this thesis, the exponential convergence result is inextricable linked with the desired dimension formulas (i.e., one cannot prove a virtual dimension formula without a priori exponential decay results). The phenomenon relating exponential decay and the Fredholm index of the linearized operator is explained in §4.2.2.

Chapter 2

Conley-Zehnder indices associated to an asymptotic operator

As explained in §7, the definition of the *Conley-Zehnder index* associated to an non-degenerate asymptotic operator $A = -J\partial_t - S(t)$ is as the Fredholm index of any operator $D = \partial_s - A_s$ acting on:

$$W^{1,p}(\mathbb{R} \times S; \mathbb{R}^{2n}, \mathbb{R}^n) \to L^p(\mathbb{R} \times S; \mathbb{R}^{2n}),$$

where $S = [0,1]$ or $S = \mathbb{R}/\mathbb{Z}$, and $A_s = -J\partial_t - S(s,t)$ satisfies $S(s,t)\xi = C\xi = \bar{\xi}$ for $s < 0$ and $A_s = A$ for $s > 1$. It is proved in §6 that any operator of this form is Fredholm. We denote the Conley-Zehnder index by $\mu_{\mathrm{CZ}}(A) = \mathrm{Index}(D)$.

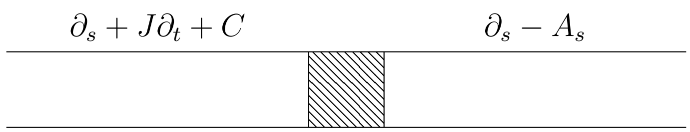


Figure 1. The Conley-Zehnder index is the Fredholm index of any Cauchy-Riemann operator on the infinite strip or cylinder which interpolates between the two asymptotic conditions. The matrix C represents complex conjugation.

Remark 2.1. Suppose that A is degenerate. It can be shown that any operator $\partial_s - A_a$ of the above form (i.e., $A_s = A$ for $s > 1$) is not Fredholm.

The first goal in this chapter is to compute the Conley-Zehnder indices for all *split and autonomous* asymptotic operators, as explained in §2.3.

2.1. Non-degeneracy as an integral condition

Write $A = -J\partial_t - S(t)$, and observe that A is non-degenerate if and only if the boundary value problem:

$$\partial_t \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = JS(t) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \text{ with } \begin{cases} \text{chord case: } y(0) = y(1) = 0, \\ \text{orbit case: } x(0) = x(1) \text{ and } y(0) = y(1), \end{cases}$$

has no non-zero solutions. We see that the non-degeneracy condition amounts to requiring:

$$\begin{cases} \text{chord case: } \mathbb{R}^n \cap \mathrm{F}(1)(\mathbb{R}^n) = 0, \\ \text{orbit case: } \mathrm{F}(1) \text{ does not have 1 as an eigenvalue,} \end{cases} \tag{2.1}$$

where $\mathbb{R}^n \subset \mathbb{C}^n$ is the real subspace and $\mathrm{F}(t)$ is the *fundamental solution* of the ODE, i.e., solves $\mathrm{F}(0) = 1$ and $\mathrm{F}'(t) = JS(t)\mathrm{F}(t)$ (so that any solution is of the form $x(t) = \mathrm{F}(t)x(0)$). Clearly $\mathrm{F}(1)$ depends continuously on the path $S(t)$, and if $-J\partial_t - S_\tau(t)$ is a family of *non-degenerate* operators, then $\mathrm{F}(1)\mathbb{R}^n$ will remain transverse to $\mathbb{R}^n$ (chord case), or will never have 1 as an eigenvalue (orbit case). This will be relevant in §2.5 when we give an intersection theoretic definition of the Conley-Zehnder index, using Arnol'd's singular cycle in the Lagrangian Grassmannian.

2.1.1. *The fundamental solution is valued in the symplectic group.* We observe that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{F}(t)^T J\mathrm{F}(t) = \mathrm{F}(t)^T S(t)^T J^T JF(t) + \mathrm{F}(t)^T JJS(t)\mathrm{F}(t) = 0,$$

precisely since $S(t)^T = S(t)$. Thus $\mathrm{F}(t)$ is valued in the symplectic group.

2.1.2. *Relationship between non-degeneracy for orbits and chords.* In general, let P be a symplectic matrix, and consider the *graph* of P, denoted $L(P)$, as a linear subspace in $\mathbb{R}^{4n}$, equipped with the symplectic form $\mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega$ where $\mathrm{pr}_1, \mathrm{pr}_2$ are the coordinate projections onto $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. This graph is always a Lagrangian. Moreover, this graph is transverse to the diagonal Lagrangian if and only if P does not have 1 as an eigenvalue. Thus, if $\mathrm{F}(t)$ is the fundamental solution of an orbit asymptotic operator A, then A is non-degenerate if and only of $L(\mathrm{F}(1))$ is transverse to the diagonal.

2.2. Input from the index formula

To compute the Conley-Zehnder indices, we require the following inputs from §6. This statement of the index formula allows one to fairly easily determine how a change of trivialization affects the Conley-Zehnder index.

Theorem 2.2. Let $E \to \mathbb{R} \times [0,1]$ be a rank n unitary bundle with totally real subbundle $F \to \mathbb{R} \times \{0,1\}$ defined along the boundary. Suppose:

- $X_1^+, \dots, X_n^+$ and $X_1^-, \dots, X_n^-$ form two sets of unitary frames which restrict to frames of F along the boundary, and
- $D : C^\infty(E,F) \to C^\infty(E)$ is a differential operator so that

$$D(uX^+) = (\partial_s u - A_s^+ u)X^+,$$
$$D(uX^-) = (\partial_s u - A_s^- u)X^-,$$

where $A_s^+ \to A^+$ as $s \to \infty$, and $A_s^- \to A^-$ as $s \to \infty$, for (non-degenerate) asymptotic operators $A^\pm$. Here $A_s^\pm = -J\partial_t - S_\pm(s,t)$.

Then $D : W^{1,p}(E, F) \to L^p(E)$ is Fredholm, and

$$\mathrm{Index}(D) = \mu_{\mathrm{Mas}}(E, F, X^{\pm}) + \mu_{\mathrm{CZ}}(A^+) - \mu_{\mathrm{CZ}}(A^-),$$

where $\mu_{\mathrm{Mas}}(E, F, X^{\pm})$ is the signed count of zeroes of a transverse section of $\det_{\mathbb{C}}(E)^{\otimes 2}$ which agrees with $(X_1^{\pm} \wedge \cdots \wedge X_n^{\pm})^{\otimes 2}$ as $s \to \pm\infty$ and which restricts to the canonical section of $\det_{\mathbb{R}}(F)^{\otimes 2}$ along the boundary. This is called the *Maslov number* associated to the data $(E, F, X^{\pm})$.

Proof. This is a direct consequence of the general index formula given in §6. □

Theorem 2.3. Let $A_{\mathrm{ref}}\xi = -J\partial_t - \bar{\xi} = -J\partial_t - C\xi$, where C is the matrix of complex conjugation. Then $\mu_{\mathrm{CZ}}(A_{\mathrm{ref}}) = 0$.

Proof. This follows from the fact that the operator $D = \partial_s - A_{\mathrm{ref}}$ is an isomorphism, see 6.20, hence $\mathrm{Index}(D) = 0$. However, by definition, the index of this operator computes $\mu_{\mathrm{CZ}}(A_{\mathrm{ref}})$. □

Remark 2.4. The $W^{1,p}$ topology is defined via the trivializations induced by $X^{\pm}$. More precisely, a section ξ is in $W^{1,p}$ if $\xi = u^{\pm}X^{\pm}$ and u^+ is in $W^{1,p}$ for $s > -1$ and u^- is in $W^{1,p}$ on $s < 1$. In the sense of 6.3.2, the pair (X^+, X^-) defines an *asymptotically Hermitian structure.*

Remark 2.5. If F is any real vector bundle, $\det(F) \simeq F^{\wedge \mathrm{top}}$ is a real line bundle, and hence $\det(F)^{\otimes 2}$ has a *canonical* orientation. If F also has a metric (e.g., one inherited from the unitary metric on E), then $\det(F)^{\otimes 2}$ has a canonical section (namely, the section $(X_1 \wedge \cdots \wedge X_n)^{\otimes 2}$ where $X_1, \ldots, X_n$ is an orthonormal frame).

One immediate corollary of the definition is:

Corollary 2.6. If A_r, $r \in [0, 1]$ is a continuous path in the space of *non-degenerate* asymptotic operators, then $\mu_{\mathrm{CZ}}(A_0) = \mu_{\mathrm{CZ}}(A_1)$.

Proof. The Fredholm index is unchanged under continuous paths in the space of Fredholm operators. So long as the asymptotics remain non-degenerate, the operators remain Fredholm. Since $\mu_{\mathrm{CZ}}(A)$ is defined as the Fredholm index of an operator $D = \partial_s - A_s$ which has $A_s = A_{\mathrm{ref}}$ for $s \to -\infty$ and $A_s = A$ as $s \to \infty$. We can therefore define a path of Fredholm operators $D_r = \partial_s - A_{s,r}$ by requiring that $A_{s,r} = A_r$ for s sufficiently positive and $A_{s,r} = A_{\mathrm{ref}}$ for s sufficiently negative. Then we compute

$$\mu_{\mathrm{CZ}}(A_0) = \mathrm{Index}(D_0) = \mathrm{Index}(D_1) = \mu_{\mathrm{CZ}}(A_1).$$

as desired. □

2.3. Computing the Conley-Zehnder indices for autonomous split operators

In this section we derive various facts about the Conley Zehnder indices *for chords*, culminating in a formula for the index whenever the asymptotic operator is *autonomous and split*. Recall that split operators are those which preserve the decomposition $(\mathbb{C}^n, \mathbb{R}^n) \simeq (\mathbb{C}, \mathbb{R}) \oplus \cdots \oplus (\mathbb{C}, \mathbb{R})$, which amounts to the matrix $S(t) = S$ having 2×2 blocks along the diagonal with respect to the coordinates $x_1, y_1, \ldots, x_n, y_n$.

We focus entirely on the chord case. We will return to the orbit case in §2.6.

2.3.1. *Chambers in the space of* 2×2 *symmetric matrices.* In the case when A acts on sections of the trivial complex line bundle, we can write $S = \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$ for real parameters a, b, c. Consider the space of 2×2 matrices as identified with $\mathbb{R}^3$ via the coordinates (a, b, c).

2.3.1.1. *Characterization of non-degeneracy in the case* $n = 1$. We have the following characterization of non-degeneracy:

Lemma 2.7. The operator $A = -J\partial_t - S$ is non-degenerate if and only if

(i) $a \neq 0$,

(ii) $\det(S) = ac - b^2 \notin (\pi\mathbb{Z})^2 \smallsetminus \{0\}$.

Proof. First observe that, for $A(x(t) + iy(t)) = 0$ if and only if

$$x' = -bx - cy \text{ and } y' = ax + by,$$

since $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right] = \left[\begin{smallmatrix} -b & -c \\ a & b \end{smallmatrix}\right]$. Thus, if $a = 0$, then there exist non-zero solutions $y = 0$, $x(t) = e^{-bt}$, and these satisfy the boundary condition $y(0) = y(1) = 0$. Thus the condition $a \neq 0$ is necessary for non-degeneracy.

Now, take second derivatives of the equation involving y' to obtain the following boundary value problem:

$$y'' + (ac - b^2)y = 0 \text{ and } y(0) = y(1) = 0 \tag{2.2}$$

It is well-known that this has non-zero solutions if and only if $ac - b^2 \in (\pi\mathbb{Z})^2 \smallsetminus \{0\}$, in which case:

$$y(t) = \omega^{-1} y'(0) \sin(\omega t) \text{ for } \omega^2 = ac - b^2.$$

This can be proved by setting $z = \omega^{-1} y'(t)$, and then observing that (z, y) solves the ODE $y' = \omega z$, $z' = -\omega y$, whose solution is:

$$\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} z(0) \\ y(0) \end{bmatrix}.$$

This only leads to a non-zero $y(t)$ when $z(0) \neq 0$ and $\omega \in \pi\mathbb{Z} \smallsetminus \{0\}$.

In particular, if $ac - b^2 \in (\pi\mathbb{Z})^2 \smallsetminus \{0\}$, then we can find non-zero solutions by first solving $y'' + \omega^2 y = 0$ and then solving the ODE for $x(t)$. Thus (ii) is necessary for non-degeneracy.

On the other hand, if $ac - b^2$ is not in $(\pi\mathbb{Z})^2 \smallsetminus \{0\}$, we know that y must be identically zero. If $a \neq 0$ then $y' = ax$ implies that x is also identically zero. Thus (i) and (ii) are sufficient to establish non-degeneracy. This completes the proof. □

2.3.1.2. *Definition of the chambers.* For each $k > 1$, define the "chamber"

$$U_{k,\pm} = \left\{(a,b,c) : k^2\pi^2 < ac - b^2 < (k+1)^2\pi^2 \text{ and } \pm a > 0\right\},$$

and, for $k = 0$, define

$$U_{0,\pm} = \left\{(a,b,c) : -\infty < ac - b^2 < \pi^2 \text{ and } \pm a > 0\right\}.$$

By §2.3.1.1, these open sets cover the space of matrices S which lead to non-degenerate asymptotic operators.

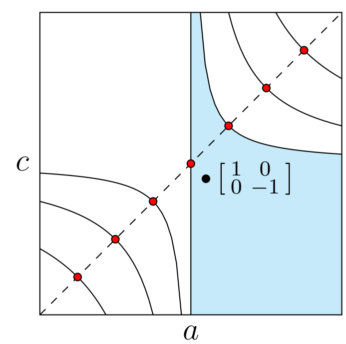


FIGURE 2. The chambers (shown is the plane $b = 0$). The shaded region contains those matrices with $\mu_{\mathrm{CZ}} = 0$. The red dots are matrices of the form $k\pi I$, for $k \in \mathbb{Z}$ (these correspond to degenerate asymptotic operators).

Lemma 2.8. Each chamber $U_{k,\pm}$ is path-connected.

Proof. First let us treat the case $k > 0$. It is clear that $U_{k,\pm} \cap \{b = 0\}$ is connected, as it consists of one branch of the region $C_1 < ac < C_2$ trapped between two hyperbolas (with $C_1 < C_2$ both positive). Thus it suffices to explain how to connect each $(a,b,c) \in U_{k,\pm}$ to a point of the form $(a', 0, c')$ while remaining in $U_{k,\pm}$.

Let $K = a(0)c(0) - b(0)^2$, which is a positive constant, and let $b(t) = (1-t)b(0)$. Evolve a, c in time according to the formula:

$$a(t) = \frac{\sqrt{K + b(t)^2}}{\sqrt{a(0)c(0)}} a(0) \text{ and } c(t) = \frac{\sqrt{K + b(t)^2}}{\sqrt{a(0)c(0)}} c(0).$$

This is well-defined since $a(0)c(0)$ is necessarily positive. Then we see that the quantity $a(t)c(t) - b(t)^2 = K$ remains constant, and hence the path remains in $U_{k,\pm}$, as desired.

The case when $k = 0$ is easier. Observe that $b(t) = e^t b(0)$, $a(t) = a(0), c(t) = c(0)$ defines a path which remains in $U_{0,\pm}$. In other words, we can make the b coordinate arbitrarily large, and thereby make $ac - b^2$ arbitrarily negative. Then, once b is sufficiently large, we can move a to either $+1$ or -1 and c to $-a$. Then we can send b back to 0. Thus any point in $U_{0,+}$ can be joined to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, and any point in $U_{0,-}$ can be joined to $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

This completes the proof. $\square$

It is clear that any path $S(r)$ which remains in a given chamber will satisfy

$$\mu_{\mathrm{CZ}}(-J\partial_t - S(r)) = \text{const.}$$

Since we know that $\mu(-J\partial_t - C) = 0$ where C is the matrix of complex conjugation, we conclude that $\mu_{\mathrm{CZ}}(-J\partial_t - S) = 0$ for any $S \in U_{0,+}$.

Corollary 2.9. Every 2×2 symmetric matrix S, with $A = -J\partial_t - S$ non-degenerate, can be joined to a matrix $S_k = (\pi k + 1)I$, $k \in \mathbb{Z}$, without leaving the chamber it started in. In other words, if S has $a > 0$ (resp., $a < 0$) and $\det(S) \in (\pi^2 k^2, \pi^2(k+1)^2)$, then S lies in the same chamber as S_k (resp., S_{-k}). $\square$

2.3.1.3. *Computing the Conley-Zehnder indices for the representatives.* Since the Conley-Zehnder index is constant on each chamber, it suffices to compute μ_{CZ} for each chamber representative $S_k = (\pi k + 1)I$. To compute this, we will use Theorem 2.2 to determine how the index changes with the trivialization.

Let X_0 be the standard Hermitian frame for $(\mathbb{C}, \mathbb{R}) \to (\mathbb{R} \times [0,1], \mathbb{R} \times \{0,1\})$ (i.e., $X_0 = 1$). Fix $A = -i\partial_t - S(t)$, and consider $D = \partial_s - A$, i.e., $D(uX_0) = (\partial_s u - Au)X_0$. By Theorem 2.2, we know that $\mathrm{Index}(D) = 0$. We will now recompute this index using a different set of asymptotic trivializations.

Let $X^- = X_0$ and $X^+ = e^{\pi i t} X_0$. We compute:

$$\begin{aligned} D(uX^-) &= D(uX_0) = (\partial_s u + i\partial_t u + S(t)u)X^-, \\ D(uX^+) &= D(ue^{\pi i t}X_0) = (\partial_s u + i\partial_t u + e^{-\pi i t} S(t) e^{\pi i t} u - \pi u)X^+. \end{aligned}$$

Thus we have $A^- = -i\partial_t u - S(t) = A$ and $A^+ = -i\partial_t - e^{-\pi i t} S(t) e^{\pi i t} + \pi$. The index formula then gives:

$$\mathrm{Index}(D) = \mu_{\mathrm{Mas}}(X^+, X^-) + \mu_{\mathrm{CZ}}(A^+) - \mu_{\mathrm{CZ}}(A^-) = 0.$$

Therefore, in order to compute the index difference, it suffices to compute the Maslov number associated to the change in trivialization. Let $\beta : \mathbb{R} \to [0,1]$ be a cut-off function which equals 0 for $s \leq 0$ and equals 1 for $s \geq 1$ and satisfies $\beta'(s) > 0$ for $s \in (0,1)$. Set $\sigma = (1 - \beta(s)) + \beta(s)e^{2\pi i t}$. It is clear that σ agrees with $X^- \otimes X^-$ for $s \leq 0$, agrees with $X^+ \otimes X^+$ for $s \geq 1$, and always equals 1 along the boundary (the canonical section of $\mathbb{R} \otimes \mathbb{R} \to \mathbb{R} \times \{0,1\}$). Moreover, it is clear that $\sigma(s,t) = 0$ if and only if $e^{2\pi i t} = -1$ and

$\beta(s) = 1/2$, in which case we have:

$$\mathrm{d}\sigma = \beta'(s)(e^{2\pi i t} - 1)\mathrm{d}s + 2\pi i \beta(s) e^{2\pi i t}\mathrm{d}t \implies \mathrm{d}\sigma = -2\beta'(s)\mathrm{d}s - \pi i \mathrm{d}t,$$

hence the single zero is transverse (since we assume $\beta'(s) > 0$ whenever $\beta(s) \in (0,1)$). Moreover, the linearization is orientation preserving. It follows that the signed count of zeros of σ is $+1$, and hence, by definition, $\mu_{\mathrm{Mas}}(X^+, X^-) = +1$. Thus we conclude that:

$$\mu_{\mathrm{CZ}}(A^+) - \mu_{\mathrm{CZ}}(A^-) = -1.$$

Now, let us apply this with $A^- = -i\partial_t - 1$, which has $\mu_{\mathrm{CZ}} = 0$, to obtain:

$$\mu_{\mathrm{CZ}}(-i\partial_t - 1 + \pi) = -1.$$

Repeating the argument, we conclude that $\mu_{\mathrm{CZ}}(-i\partial_t - 1 + k\pi) = -k$. Recalling that $S_k = (k\pi + 1)I$, we conclude that $\mu_{\mathrm{CZ}}(-i\partial_t - S_{-k}) = -k$ for $k \geq 0$. In a similar fashion, we conclude that $\mu_{\mathrm{CZ}}(-i\partial_t - 1 - k\pi) = k$, and thereby obtain:

$$\mu_{\mathrm{CZ}}(-i\partial_t - S_k) = k, \tag{2.3}$$

for all $k \in \mathbb{Z}$. See Figure 3.

Example 2.10. Consider an operator $D = \partial_s + i\partial_t - \delta\rho(s)$ where $\rho(s) \to \pm 1$ as $s \to \pm\infty$, acting on sections of the trivial line bundle over a strip. Here $0 < \delta < \pi$. Then:

$$\mathrm{Index}(D) = \mu_{\mathrm{CZ}}(-i\partial_t + \delta) - \mu_{\mathrm{CZ}}(-i\partial_t - \delta) = -1.$$

Such an operator appears, for instance, when one considers $\partial_s + i\partial_t$ acting on exponentially weighted Sobolev spaces.

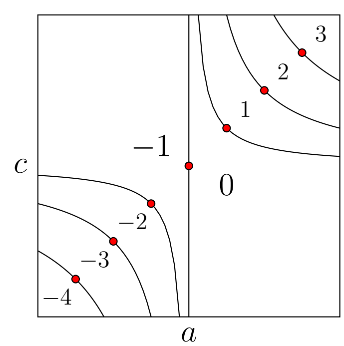


FIGURE 3. The chambers in the space of 2×2 matrices $\left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$ (shown is the plane $b = 0$) and the Conley-Zehnder indices of the associated asymptotic operators. The red dots are matrices of the form $k\pi I$, for $k \in \mathbb{Z}$. Recall that the chambers are partioned by equations of the form $ac = b^2 + \pi^2 k^2$, and so as b departs from 0 the chambers expand away from 0.

2.3.2. *Formula for the Conley-Zehnder index of an autonomous split operator.* Suppose that $A = -J\partial_t - S$ is an autonomous split non-degenerate asymptotic operator. Let $S = \mathrm{diag}(S_1, \dots, S_n)$ be the block decomposition, and let $S_i = \left[\begin{smallmatrix} a_i & b_i \\ b_i & c_i \end{smallmatrix}\right]$.

Theorem 2.11. The Conley-Zehnder index of A is given by

$$\sum_{i=1}^{n} \mu_{\mathrm{CZ}}(-J\partial_t - S_i),$$

where $\mu_{\mathrm{CZ}}(-J\partial_t - S_i)$ is equal to:

(i) $+k$ if $a_i > 0$ and $\det(S_i) \in (\pi^2 k^2, \pi^2(k+1)^2)$,

(ii) $-k-1$ if $a_i < 0$ and $\det(S_i) \in (\pi^2 k^2, \pi^2(k+1)^2)$,

(iii) 0 if $a_i > 0$ and $\det(S_i) < \pi^2$,

(iv) -1 if $a_i < 0$ and $\det(S_i) < \pi^2$.

Proof. The formulas for $\mu_{\mathrm{CZ}}(-J\partial_t - S_i)$ when S_i is a 2×2 block follow from the results of §2.3.1.1-2.3.1.3.

The fact that $A = -J\partial_t - S$ and $A_{\mathrm{ref}} = -J\partial_t - C$ are both split implies that we can find a split $D = \partial_s - A_s$ which interpolates from A_{ref} to A. It is well-known that the Fredholm index of a split operator is the sum of the Fredholm indices. Since

$$\mu_{\mathrm{CZ}}(A) = \mathrm{Index}(D),$$

the desired result follows. □

Example 2.12. Suppose that $S = \mathrm{diag}(\left[\begin{smallmatrix} a_1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} a_n & 0 \\ 0 & 0 \end{smallmatrix}\right])$, where all a_i are non-zero. Show that $\mu_{\mathrm{CZ}}(-J\partial_t - S)$ is equal to minus the number of a_i which are negative. This kind of asymptotic operators appear when one linearizes the Reeb chord equation in 1-jet space, see §3.6.2.1.

2.4. Conley Zehnder indices as a spectral flow

In this section we show that $\mu_{\mathrm{CZ}}(A_1) - \mu_{\mathrm{CZ}}(A_0)$ is a spectral flow, following [**Wen20**, §3.3], [**Flo89a**, pp. 595], and [**RS95**]. The arguments in this section work equally well for orbits and for chords. The main result is:

Theorem 2.13. If $\mu_{\mathrm{CZ}}(A_1) = \mu_{\mathrm{CZ}}(A_0)$, then A_1 is homotopic to A_0 within the space of non-degenerate operators.

Proof. We defer to §6 for any analytical concerns about Cauchy-Riemann operators on infinite strips and cylinders.

The index formula proves that $\mu_{\mathrm{CZ}}(A_1) - \mu_{\mathrm{CZ}}(A_0)$ equals the Fredholm index of an operator of the form $\partial_s - A_s$ where $A_s = A_0$ for s very negative and $A_s = A_1$ for s very positive.

We are free to make $A_s = -J\partial_t - S(s,t)$ with $S(s,t)$ symmetric, i.e., A_s always remains self-adjoint. By the arguments in [**Wen20**, Appendix C], a generic choice A_s will have the property that:

(i) There exists a smooth function $\lambda : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ and a smoothly varying eigenbasis $\varphi_k(s,t)$ so that $A_s\varphi_k(s,t) = \lambda(k,s)\varphi_k(s,t) =: \lambda_k(s)\varphi_k(s,t)$.

(ii) All eigenvalues are simple; i.e., the function $\mathbb{Z} \times \{s\} \to \mathbb{R}$ is injective for all s.

(iii) The function λ is transverse to 0.

Let us define the spectral flow $\mathfrak{sf}$ to be (minus) the number of preimages in $\lambda^{-1}(0)$ counted with sign. The goal is to prove that $\mathfrak{sf} = \mu_{\mathrm{CZ}}(A_1) - \mu_{\mathrm{CZ}}(A_0)$.

By the stabilized kernel gluing operation proved in §7 we know that $\mu_{\mathrm{CZ}}(A_1) - \mu_{\mathrm{CZ}}(A_0)$ behaves additively under a certain concatenation/gluing operation. Via an appropriate sequence of deformations of our operator, breaking it into a sequence of glued strips, we see that it suffices to prove $\mathfrak{sf} = \mu_{\mathrm{CZ}}(A_1) - \mu_{\mathrm{CZ}}(A_0)$ in the case when $\lambda^{-1}(0)$ consists of a single point, i.e., one of the eigenvalues crosses 0. Without loss of generality, let us suppose it is the $k = 0$ eigenvalue. Via another kernel gluing argument, if necessary, we may localize at the crossing and suppose that:

$$\lambda_0(s) \in (-\epsilon, \epsilon) \text{ and } \lambda_k(s) \notin [-\epsilon, \epsilon] \text{ for all } k \neq 0 \text{ and } s \in \mathbb{R}.$$

Now observe that $\mu_{\mathrm{CZ}}(A_1 \pm \epsilon) = \mu_{\mathrm{CZ}}(A_0 \pm \epsilon)$. This is because during the path from A_0 to A_1 no eigenvalue ever hits the lines $\lambda = \pm\epsilon$, and hence the path remains in the space of non-degenerate asymptotic operators. The Fredholm index of $D = \partial_s - A_s \pm \epsilon$ is 0, as it is homotopic to the translation invariant operator $\partial_s - A_1 \pm \epsilon$ (or $\partial_s - A_0 \pm \epsilon$).

Now, if $\lambda_0(-\infty) < 0 < \lambda_0(+\infty)$, then $A_1 + \epsilon$ is homotopic to A_1. Thus

$$\mu_{\mathrm{CZ}}(A_1) - \mu_{\mathrm{CZ}}(A_0) = \mu_{\mathrm{CZ}}(A_0 + \epsilon) - \mu_{\mathrm{CZ}}(A_0). \tag{2.4}$$

On the other hand, if $\lambda_0(-\infty) > 0 > \lambda_0(+\infty)$, then

$$\mu_{\mathrm{CZ}}(A_1) - \mu_{\mathrm{CZ}}(A_0) = \mu_{\mathrm{CZ}}(A_0 - \epsilon) - \mu_{\mathrm{CZ}}(A_0). \tag{2.5}$$

We will now show that (2.4) equals -1 and (2.5) equals $+1$. This will conclude the proof that Fredholm index, namely $\mu_{\mathrm{CZ}}(A_1) - \mu_{\mathrm{CZ}}(A_0)$, is the sum of the negative crossings minus the sum of the positive crossings, which is, by definition, $\mathfrak{sf}$.

The benefit of the reduction to the comparison of $A_0 \pm \epsilon$ and A_0 is that *the eigenfunctions remain constant during the linear homotopy* $A_s = A_0 + \beta(s)\epsilon$, where $\beta(s)$ is a cut-off function which increases from 0 to 1. Thus, let $\varphi_k(t)$ be such an eigenbasis.

A general solution to $\partial_s u - A_s u = 0$ can be written as

$$u(s,t) = \sum_{k\in\mathbb{Z}} u_k(s)\varphi_k(t),$$

where $u_k(s)$ are smooth solutions which decay exponentially in the ends, as can be proven by applying Lemma 6.19 applied in the ends where A_s is constant. This implies:

$$\partial_s u - A_s u = 0 \implies \sum_{k\in\mathbb{Z}} (u_k'(s) - \lambda_k(s)u_k(s))\varphi_k(t).$$

This means that $u_k'(s) - \lambda_k(s)u_k(s) = 0$. Let $\Lambda_k(s)$ be an anti-derivative for $\lambda_k(s)$, and thererby conclude that the derivative of $f_k(s) = e^{-\Lambda_k(s)}u_k(s)$ vanishes, hence $f_k(s)$ is constant. Clearly if $k \neq 0$, $|\lambda_k(s)|$ is uniformly bounded below by ϵ. hence $e^{-\Lambda_k(s)}$ has 0 as a limit point. In particular, if $f_k \neq 0$, then $u_k(s)$ is unbounded. Hence $f_k = 0$ and thus $u_k(s) = 0$. Thus the only possible solution is of the form $u(s,t) = u_0(s)\varphi_0(t)$, where $u_0(s) = ce^{\Lambda_0(s)}$. In order for this to be bounded we must have $\lambda_0(-\infty) > 0 > \lambda_0(+\infty)$, and in this case we have a 1-dimensional contribution to $\ker(\partial_s - A_s)$. This proves that (2.5) gives $+1$.

Let us now analyze the cokernel; i.e., let $v = \sum v_k(s)\varphi_k(t)$ be orthogonal to the image of $\partial_s - A_s$. Then integration by parts gives:

$$-\int (\partial_s v_k(s) + \lambda_k(s)v_k(s))u_k(s)\mathrm{d}s = 0,$$

and hence $e^{\Lambda_k(s)}v_k(s) = \text{const}$. The same argument shows that $v_k(s)$ must be 0 for $k \neq 0$, and $v_0(s) = ce^{-\Lambda_0(s)}\varphi_0(t)$ is only a solution when $\lambda_0(-\infty) < 0 < \lambda_0(+\infty)$. This proves that (2.4) gives -1.

Our goal is now to show that $\mathfrak{sf} = 0$ implies A_0 is homotopic to A_1 through the space of non-degenerate operators. We follow the argument from [**Wen20**, Theorem 3.53].

As we did above, break the cylinder into consecutive regions where each region has a single "up" crossing or "down" crossing. The signed count of crossings is zero. This means there must be an up crossing followed by a down crossing, or vice-versa. Consecutive pair of crossings with the opposite signs can be cancelled by a straightforward operation described below. We will clearly be able to iteratively cancel all the crossings, until there are none left. Then the resulting A_s will be a path from A_0, A_1 through the space of non-degenerate asymptotic operators.

To complete the proof, we explain the cancellation step. Suppose A_s has one down crossing followed by one up crossing. Thus $\lambda_0^{-1}((-\infty, 0])$ is a compact interval $[a,b]$. Pick a positive function h supported in $(a-\epsilon, b+\epsilon)$ so that $h + \lambda_0 > 0$ and $h + \lambda_{-1} < 0$ hold at all points. This can be achieved since $\lambda_{-1} < \lambda_0$ holds at all points on $[a-\epsilon, b+\epsilon]$. Then $A_s + h$ is a path from A_0 to A_1 which remains non-degenerate, as desired. $\square$

A nice corollary of Theorem 2.13 is that we obtain an intersection theoretic definition of the Conley-Zehnder index for chords.

2.5. Intersection theoretic definition of the Conley-Zehndex index for chords

Let $\mathrm{F}(t)$ be the fundamental solution of $z'(t) = JS(t)z(t)$, so that $z(t) = \mathrm{F}(t)z(0)$ solves the ODE with initial condition $z(0)$.

Consider $L(t) = \mathrm{F}(t)\mathbb{R}^n$ as a path of *Lagrangians* in $\mathbb{R}^{2n}$. This is possible since $\mathrm{F}(t)$ is valued in the symplectic group. Consider the special path defined as:

$$L^*(t) = \operatorname{diag}(e^{i(1-t)}, \dots, e^{i(1-t)})\mathbb{R}^n.$$

This starts at the rotated Lagrangian $\operatorname{diag}(e^i, \dots, e^i)\mathbb{R}^n$ (rotated by 1 radian in each $\mathbb{R}^2$ factor), and rotates clockwise back to $\mathbb{R}^n$ in time 1.

Let $\tilde{L}(t)$ be the (continuous) concatenation of $L^*(t)$ with $L(t)$. The non-degeneracy assumption implies that $\tilde{L}(1), \tilde{L}(0)$ are both transverse to $\mathbb{R}^n$.

Introduce the set M of Lagrangians L which are not transverse to $\mathbb{R}^n$ (in the Lagrangian Grassmannian $\mathcal{L} = U(n)/O(n)$, see §3.1.4). This set $M = M^1 \cup M^2 \cup \dots$ is stratified in a natural way by letting M^k be the space where $\dim L \cap \mathbb{R}^n = k$. This set M is well studied in symplectic topology, and we have the following two facts:

(i) M^k has codimension $k(k+1)/2$,

(ii) the velocity vector field of the circles $t \mapsto \operatorname{diag}(e^{i\pi t}, \dots, e^{i\pi t})L$ coorients M^1,

See [**Arn67**, Lemma 3.2.1, 3.5.1]. In particular, M^1 is codimension 1 and M^2 is codimension 3. Moreover, M^1 can be cooriented in a natural way. Standard results in transversality theory imply that the homological intersection number between $\tilde{L}(t)$ and M is invariant under arbitrary homotopies of which keep $L(1)$ disjoint from M. Let us denote this quantity $M(A)$.

Theorem 2.14. $M(A) = \mu_{\mathrm{CZ}}(A)$.

Proof. Let $L_i(t) = \mathrm{F}_i(t)\mathbb{R}^n$ be the path of Lagrangians associated to asymptotic operators A_i. By the integral condition for non-degeneracy in §2.1, if A_0 is homotopic to A_1 through the space of non-degenerate asymptotic operators, then $\tilde{L}_0(t)$ will be homotopic to $\tilde{L}_1(t)$ with endpoints remaining disjoint from M. Thus $M(A_0) = M(A_1)$. Theorem 2.13 implies that $\mu_{\mathrm{CZ}}(A_0) = \mu_{\mathrm{CZ}}(A_1)$ is a sufficient condition for the existence of a homotopy. Thus, it suffices to prove the theorem for the representative with $\mu_{\mathrm{CZ}}(A) = k$, namely the one with:

$$S = \operatorname{diag}(1 + k\pi, 1, \dots, 1).$$

We have $L(t) = \operatorname{diag}(e^{(1+k\pi)it}, e^{it}, \ldots, e^{it})$. Thus $\tilde{L}(t)$ is the path which starts at $L^*(0)$, rotates back by e^i, and then rotates forwards by e^i, and then continues rotating the first coordinate only for an extra $k\pi$ radians. Clearly, this path is homotopic with fixed endpoints to the path

$$L'(t) = \operatorname{diag}(e^i e^{k\pi it}, e^i, \ldots, e^i)\mathbb{R}^n.$$

This has k intersections with the top strata of M.

To analyze the signs of the intersection, observe that the velocity vectors of all the curves:

$$\operatorname{diag}(e^{i(t-t_0)\lambda_1}, \ldots, e^{i(t-t_0)\lambda_n})L'(t_0) \tag{2.6}$$

are transverse to M^1 provided $\lambda_1 > 0$ (assuming $L'(t_0) \in M$).

Since $L'(t) = \operatorname{diag}(e^{k\pi i(t-t_0)}, 1, \ldots, 1)L'(t_0)$, we can smoothly homotope the velocity at $t = t_0$ through vectors transverse to M^1 until it matches the vector field from (ii). This completes the proof. $\square$

See [**RS93**] for another approach to defining the Conley-Zehnder indices via intersection theory.

2.6. Conley Zehnder indices for orbits

In this section we compute the Conley-Zehnder indices for split and autonomous asymptotic operators on $\mathbb{R}/\mathbb{Z}$. It suffices to consider the line bundle case.

Interestingly, in this case, only operators with $\mu_{\mathrm{CZ}} \in \{0\} \cup (1 + 2\mathbb{Z})$ can be represented by *autonomous* asymptotic operators. The Conley-Zehnder indices for such operators are shown in Figure 4.

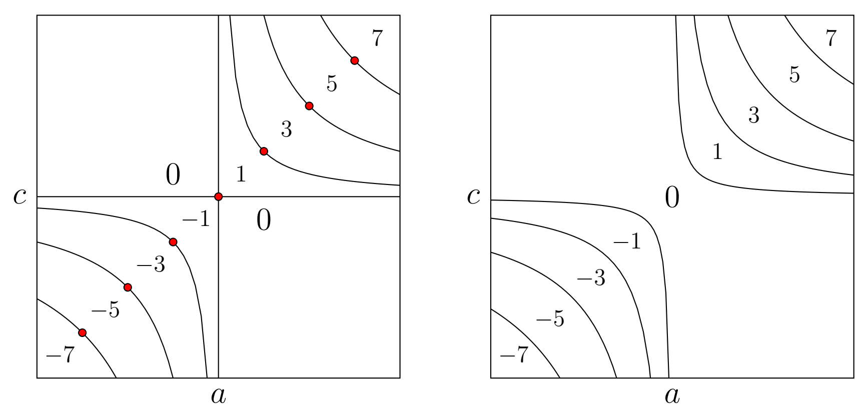


FIGURE 4. Chambers in the space of 2×2 symmetric matrices $\left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$. On the left is the plane $b = 0$, while on the right is when $b \neq 0$. The chamber walls are cut out by the equation $ac - b^2 \in (2\pi\mathbb{Z})^2$, and the red dots correspond to matrices in $2\pi\mathbb{Z}$. Operators with even and non-zero Conley-Zehnder index cannot be represented in this figure.

Our arguments do provide a full family of representatives. Indeed, we have:

Proposition 2.15. For all $\delta \neq 0$, the operator:

$$A_k := -J\partial_t - \pi k - \delta e^{-2\pi kit} C \tag{2.7}$$

has Conley-Zehnder index equal to k, where C is the 2×2 matrix of complex conjugation.

Proof. The proof is completed at the end of §2.6.0.3. □

Remark 2.16. For k odd, this operator can be deformed to an autonomous operator simply by letting $\delta \to 0$. When $k = 0$, the operator is already autonomous. However, when k is even and non-zero, this operator cannot be deformed to an autonomous one without becoming degenerate.

Remark 2.17. If we are working with $n > 1$, then, since $(1 + 2\mathbb{Z}) + (1 + 2\mathbb{Z}) = \mathbb{Z}$, we can represent all the Conley-Zehnder indices by autonomous split operators.

2.6.0.1. *Solving the autonomous ODE.* The argument is similar to the one given in §2.3.1.1; if $x(t) + iy(t)$ is 1-periodic and solves the autonomous ODE,

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = J \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \iff \begin{aligned} x'(t) &= -bx(t) - cy(t) \\ y'(t) &= ax(t) + by(t), \end{aligned}$$

then we can differentiate both sides of the above equation to conclude that

$$x''(t) + \omega^2 x(t) = 0 \text{ and } y''(t) + \omega^2 y(t) = 0,$$

where $\omega^2 = ac - b^2$. There are no non-zero solutions when $ac - b^2 < 0$.

This can be solved explicitly and we conclude that $x(t), y(t)$ are first order trigonometric polynomials with frequency ω. The requirement that x and y be 1-periodic implies that $\omega \in 2\pi\mathbb{Z}$ (or $x = y = 0$).

Thus we conclude:

Proposition 2.18. The asymptotic operator $A = -J\partial_t - \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$ on $\mathbb{R}/\mathbb{Z}$ is degenerate if and only if $ac - b^2 \in (2\pi\mathbb{Z})^2$. □

This implies that the chambers are as shown in Figure 4. It remains only to determine the Conley-Zehnder index of each chamber.

2.6.0.2. *The dependence of the Conley-Zehnder index on the trivialization.* Following the same argument as in §2.3.1.3, we deduce the following:

Proposition 2.19. For any non-degenerate asymptotic operator $A = -J\partial_t u - S(t)u$ on $\mathbb{C} \to \mathbb{R}/\mathbb{Z}$, we have $\mu_{\mathrm{CZ}}(e^{2\pi it} A e^{-2\pi it}) - \mu_{\mathrm{CZ}}(A) = 2$.

Proof. The argument is the same as §2.3.1.3. The crucial aspect being that for the non-standard asymptotic trivializations $X^- = X_0$ and $X^+ = e^{-2\pi it}X_0$, we have $D = \partial_s - A$ given by:

$$D(uX_-) = (\partial_s u + J\partial_t u + S(t)u)X^-$$
$$D(uX_+) = (\partial_s u + J\partial_t u + e^{2\pi it}S(t)e^{-2\pi it}u + 2\pi u)X^+.$$

Since Index$(D) = 0$ (as it is translation invariant), we conclude from the index formula that:

$$-\mu_{\text{Mas}}(X^+, X^-) = \mu_{\text{CZ}}(e^{2\pi it}Ae^{-2\pi it}) - \mu_{\text{CZ}}(A).$$

The Maslov number is easy to compute as the number of zeros of the section

$$\sigma(s,t) = (1 - \beta(s)) + \beta(s)e^{-4\pi it}.$$

There are two zeros, when $t = 0.25$, $t = 0.75$, and $\beta(s_0) = 1/2$, and they contribute with negative signs. This can be seen by computing linearization at the zeros:

$$-2\beta'(s)\mathrm{d}s - 4\pi ie^{-4\pi it}\beta(s)\mathrm{d}t = -2\beta'(s_0)\mathrm{d}s + 2\pi i\mathrm{d}t,$$

which is clearly orientation reversing. This completes the proof. □

Since $A = -J\partial_t - \delta C$ lies in the same chamber as $A = -J\partial_t - C$, for all $\delta \neq 0$, we conclude by repeated application of Proposition 2.19 that

$$A_{2k,\delta} = -J\partial_t - \delta e^{2\pi kit}Ce^{-2\pi kit} - 2\pi k$$

has Conley-Zehnder index equal to $2k$. Moreover, $A_{2k,\delta}$ is always non-degenerate for $\delta \neq 0$ (since $A_{0,\delta}$ is).

Remark 2.20. For odd k, consider

$$A_{k,\delta} = -J\partial_t - \delta e^{\pi kit}Ce^{-\pi kit} - \pi k.$$

We may interpret this as an operator on $C^\infty(\mathbb{R}/2\mathbb{Z})$ in an obvious fashion. This operator is conjugate to an operator on $C^\infty(\mathbb{R}/\mathbb{Z})$ by the formula

$$2A(u(0.5t))(2t) = -J\partial_t u - (2\delta e^{2\pi it}Ce^{-2\pi kit} + 2\pi k)u.$$

In particular, *the extension* of $A_{k,\delta}$ to $C^\infty(\mathbb{R}/2\mathbb{Z})$ is non-degenerate for all δ, and hence the original operator $A_{k,\delta}$ is also non-degenerate for all $\delta \neq 0$.

2.6.0.3. *Determining the Conley-Zehnder index when $S = \pi$.* Proposition 2.18 implies that $A = -J\partial_t - \pi$ is non-degenerate. The goal of this section is to compute this operator, as it is inaccessible, starting from A_{ref}, via the reparametrization trick from §2.6.0.2. The strategy is to use the *spectral flow* interpretation of the change in Conley-Zehnder index along a path of asymptotics operators. The proof that the spectral flow computes the Conley-Zehnder index is exactly the same as the case for chords explained in §2.4. Indeed, the main reference for that section, [**Wen20**], defines the Conley-Zehnder index for orbits as a spectral flow.

Consider the path given by:

$$A_\tau = -J\partial_t - \begin{bmatrix} \pi & 0 \\ 0 & \pi - \tau \end{bmatrix}$$

This becomes singular when $\tau = \pi$, and then enters the chamber containing $-J\partial_t - C$, which has $\mu_{\mathrm{CZ}} = 1$, by definition. The *eigenvalue equation* $A_\tau u = \lambda u$ reduces to

$$x'(t) = -(\pi + \lambda - \tau)y(t) \text{ and } y'(t) = (\pi + \lambda)x(t),$$

which is singular when $\lambda = \tau - \pi$ (and when $\lambda = -\pi$), and all the other eigenvalues remain far from 0 for $\tau \in [0, \pi + \epsilon)$. See the left part of Figure 4. Moreover, the 1-dimensional eigenspace when $\lambda = \pi - \tau$ is simply $y(t) = \text{const}$, $x(t) = 0$. Since this is a single *up-crossing*, i.e., the eigenvalue went from negative to positive as τ crossed the a-axis, we conclude from the spectral flow is decreased by 1 at the end of the path. Thus we conclude that

$$\mu_{\mathrm{CZ}}(-J\partial_t - \pi) = +1.$$

Then by applying the change of trivialization trick in 2.6.0.2, we conclude that

$$\mu_{\mathrm{CZ}}(-J\partial_t - k\pi) = k.$$

for all *odd* k. This completes the proof of Proposition 2.15 and Figure 4.

2.7. Conley-Zehnder indices relevant for exponential weights

Recall that when we place exponential weights on a positive puncture, the asymptotic operator changes to $A - \delta$, while at negative punctures the asymptotic weight changes to $A + \delta$. We are particularly interesting in the case when $A = -J\partial_t$ for $n = 1$, as this case is needed in §4.2.2. Let $d = \mu_{\mathrm{CZ}}(A + \delta)$. By inspecting the chambers in Figures 3 and 4 we conclude:

(i) negative boundary punctures have $d = 0$,

(ii) positive boundary punctures have $d = -1$,

(iii) negative interior punctures have $d = +1$,

(iv) positive interior punctures have $d = -1$,

The reader is referred to [**BM04**, pp. 129] for a similar discussion.

Chapter 3

Asymptotic operators of Reeb chords and orbits

let $\Lambda \subset (Y, \xi, \alpha, J)$ be Legendrian, and let c be a non-degenerate Reeb chord or orbit. The linearization of the Reeb flow ODE at c gives an asymptotic operator. However, we need coordinates to linearize. The choice of coordinates leads to interesting homotopical problem. In this chapter, we describe the relationship between choices of coordinates, the resulting asymptotic operators, and *admissible* sections $\mathfrak{s}$ of the square $K^{\otimes 2}$ of the canonical bundle.

3.1. Admissible coordinates

Let $\Phi_t : B(1)^{2n} \to Y^{2n+1}$ be coordinates so that:

(i) $t \mapsto \Phi_t(0)$ is some (positive) parametrization of c,

(ii) the time derivative Φ_t' is transverse to the space derivative $\mathrm{d}\Phi_t$,

(iii) $\mathrm{d}\Phi_t(0) : \mathbb{R}^{2n} \to \xi$ is a *unitary* or *symplectic* isomorphism, and

(iv) $\Phi_0(\mathbb{R}^n) = \Lambda_0$ and $\Phi_1(\mathbb{R}^n) = \Lambda_1$ (for chords), or

(v) $\Phi_t = \Phi_{1+t}$ (for orbits).

Comparing $\varphi = \mathrm{d}\Phi_t^1(0)^{-1}\mathrm{d}\Phi_t^0(0)$ yields $\mathrm{Sp}(2n)$ or $U(n)$-valued transition functions which take boundary values in $\mathrm{GL}_n(\mathbb{R}) \cap \mathrm{Sp}(2n)$ or $O(n)$.

Remark 3.1. By declaring two charts to be equivalent if they agree except with different radii $\delta_1 < \delta_2$, admissible charts should be considered as *germs*, although we will typically suppress this aspect in our explanations and notation. In this germ sense, we define a *homotopy* of admissible coordinate charts to be a homotopy Φ_t^τ of smooth maps $B(\delta) \to \mathbb{R}^{2n}$ which satisfy (i)-(v) for each τ, for some δ small enough.

Remark 3.2. For $p \in B(1)$, the time derivative is $\Phi_t'(p) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}_{\epsilon=0}\Phi_{t+\epsilon}(p)$. The space derivative satisfies $\mathrm{d}\Phi_t(p)v = \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}\Phi_t(p + \epsilon v)$. The condition that $\Phi_t(0) = c(t)$ constrains $\Phi_t'(0)$ to lie in the positive half-line determined by $R(\Phi_t(0))$. If we use a *constant speed* parametrization, then $\Phi_t'(0) = TR(\Phi_t(0))$ where T is the *action* of c.

3.1.1. *Symplectic versus unitary.* The inclusion of the space of unitary coordinates with constant speed parametrizations into the space of symplectic coordinates is a weak homotopy equivalence. Indeed:

Claim 3.3. Let Φ_t^π, $\pi \in D$, be a parametrized family of symplectic coordinates, so that Φ_t^π are constant speed unitary coordinates for $\pi \in \partial D$. Then Φ_t^π can be deformed relative ∂D so as to make it unitary and have constant speed everywhere.

Proof. The argument establishing this splits into two parts.

Let $c(\pi) := t \mapsto \Phi_t^\pi(0)$. Every $c(\pi)$ is *some* parametrization of the same chord or orbit c. By a smooth deformation replacing Φ_t^π by $\Phi_{t+\rho}^\pi$ for appropriate ρ, we may suppose that all $c(\pi)$ have constant speed. If Φ_t^π already had constant speed on the boundary, then we can do this deformation leaving Φ_t^π unchanged for $\pi \in \partial D$. Thus the inclusion of symplectic coordinates with constant speed into the space of symplectic coordinates is a weak homotopy equivalence.

Next we analyze the unitary versus symplectic aspect of the claim. There certainly exists some unitary frame for ξ along c, and we will use this as a reference. Then $\mathrm{d}\Phi_t^\pi(0)$ is represented by a symplectic matrix for each π. It is well-known that $\mathrm{Sp}(2n)$ deformation retracts onto $U(n)$, and hence there exists a smooth family of symplectic matrices $\varphi_t^\pi(s)$ so that $\varphi_t^\pi(0) = 1$ and $\mathrm{d}\Phi_t^\pi(0)\varphi_t^\pi(1)$ is unitary. If $\mathrm{d}\Phi_t^\pi(0)$ is already unitary for given t, π, then the construction we have in mind has $\varphi_t^\pi(s) = 1$ for all s. We interpret φ_t^π as a family of (linear) maps $B(1) \to \mathbb{R}^{2n}$.

Then, in the sense of germs, $\Phi_t^\pi \circ \varphi_t^\pi(s)$ is a homotopy of constant speed symplectic coordinates which is unitary when $s = 1$. This proves the claim. $\square$

3.1.2. *The space derivative classifies coordinates.* Fix an auxiliary unitary frame along c as a reference, and suppose that c spans $T\Lambda$ along the boundary.

Claim 3.4. The projection map $\Phi_t \mapsto (\Phi_t(0), \mathrm{d}\Phi_t(0))$ which sends each coordinate system onto the induced path/loop in $c \times \mathrm{Sp}(2n)$ (taking boundary values in $\mathrm{GL}_n(\mathbb{R}) \cap \mathrm{Sp}(2n)$) is a weak-homotopy equivalence.[1]

The argument establishing this is certainly well-known to experts.

Proof. Suppose that Φ_t^π is a family of coordinate systems, defined for $\pi \in \partial D$, and suppose that $(f_\pi(t), F_\pi(t))$ formally extends $(\Phi_t^\pi(0), \mathrm{d}\Phi_t^\pi(0))$ to D. The goal is to show that we can extend Φ_t^π so that the induced data $(\Phi_t^\pi(0), \mathrm{d}\Phi_t^\pi(0))$ is homotopic to $(f_\pi(t), F_\pi(t))$ relative ∂D.

Using f, F, we can define a coordinate system by the formula:

$$\Psi_t^\pi(x) = \mathrm{Exp}_{f_\pi(t)}(F_\pi(t) \cdot x \cdot X),$$

[1]Note that $\mathrm{GL}_n(\mathbb{R}) \cap \mathrm{Sp}(2n)$ deformation retracts onto $\mathrm{GL}_n(\mathbb{R}) \cap U(n) = O(n)$ when we use the retraction induced by polar decomposition.

where X was the chosen frame (and Λ is totally geodesic). Then Ψ_t^π and Φ_t^π agree to first order for $\pi \in \partial D$. In particular, there exist $h, k = O(x^2)$ so that:

$$\Psi_{t+h}^\pi(x+k) = \Phi_t^\pi(x).$$

We can extend h, k from $\partial D(1)$ to all of $D(1)$ using a cut-off function of a collar coordinate (keeping them $O(x^2)$ for each π, t). Thus we can extend $\Phi_t^\pi(x)$ to all of D. This extension even has the property that $\Phi(0), \mathrm{d}\Phi(0) = f, F$. □

3.1.3. *Path component classification of coordinates around an orbit.* Let γ be an embedded orbit, and let X be an auxiliary unitary frame for $\gamma^*\xi$. Consider the map which assigns to each orbit coordinate system (centered on any cover of γ) the loops $\Phi_t(0) \in \gamma$ and $\det \mathrm{d}\Phi_t(0) \in S^1$. Consider this map as valued in the space of pairs $(f(t), d(t))$ where $f(t)$ has positive covering degree. Then the map is a bijection on π_0. Thus the path components of the space of coordinates with fixed geometric orbit γ are in bijection with $\mathbb{N} \times \mathbb{Z}$. The map to $\mathbb{Z}$ is non-canonical.

3.1.4. *Path component classification of coordinates around a chord.* We define the Lagrangian Grassmannian $\mathcal{L}$ by the fiber sequence $O(n) \to U(n) \to \mathcal{L}$. The following well-known argument classifies based loops in $\mathcal{L}$.

Proposition 3.5. $\pi_1(\mathcal{L}, \mathbb{R}^n) \simeq \pi_0([0,1]; U(n), O(n)) \simeq \mathbb{Z}$ via $\det^2$.

Proof. Apply the five-lemma to the following diagram, where the rows are exact:

$$\begin{array}{ccccccccc}
\pi_1 O(n) & \longrightarrow & \pi_1 U(n) & \longrightarrow & \pi_1(\mathcal{L}) & \longrightarrow & \pi_0(O(n)) & \longrightarrow & \pi_0(U(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_1 G & \longrightarrow & \pi_1 U(n) & \xrightarrow{\det^2} & \pi_1(S^1) & \longrightarrow & \pi_0(G) & \longrightarrow & \pi_0 U(n),
\end{array}$$

This completes the proof. See [**Arn67**]. □

Theorem 3.6. Let c be a Reeb chord from Λ_0, p_0 to Λ_1, p_1. The map which assigns each coordinate system the data of:

(i) the c-valued map $f(t) = \Phi_t(0)$, joining p_0 to p_1 (with positive action),

(ii) the $\mathbb{Z}/2$-valued sign induced by whether $\mathrm{d}\Phi_0(0) : \mathbb{R}^n \to T\Lambda_0$ preserves orientation,

(iii) the S^1-valued *loop* $\det^2 \mathrm{d}\Phi_t(0)$ (since $\det_{\mathbb{C}}$ maps $O(n)$ into ± 1),

is an isomorphism on π_0. Observe that the data of (i) is trivial if c does not lie on a Reeb orbit. Otherwise the path component data in (i) is determined by the action, and can be identified with $\mathbb{N}$.

Proof. The previous analysis shows that $f(t), \mathrm{d}\Phi_t(0)$ captures the full homotopy groups of the space, where $\mathrm{d}\Phi_t(0)$ is thought of as a $U(n)$-valued function taking boundary values in $O(n)$. The above long-exact sequence argument shows that if $\deg\det^2 \mathrm{d}\Phi_t(0) = 0$, then $\mathrm{d}\Phi_t(0)$ can be homotoped to a constant map in $O(n)$. The $\mathbb{Z}/2$ valued sign distinguishes the two components of $O(n)$. The details are left to the reader. $\square$

3.2. Choosing coordinates and the square of the canonical bundle

Recall the definition of the canonical bundle $K = \det_{\mathbb{C}}(\xi, J)$ from §1.3.3. The isomorphism class of $\det_{\mathbb{C}}(\xi)$ is independent of J, but in the following it will be useful to consider K as a fixed unitary line bundle over Y (rather than as an isomorphism class).

Since Λ is a Legendrian there is a canonical map $\det_{\mathbb{R}} T\Lambda \to \det_{\mathbb{C}}(\xi)|_\Lambda$, namely, the map induced by inclusion. To see why, observe that $\mathrm{d}\alpha|_{\Lambda\otimes\Lambda} = 0$ implies that $J\Lambda$ is orthogonal to Λ using the metric g. This implies that any basis of Λ induces a complex basis for ξ.

The real-line bundle $\det_{\mathbb{R}} T\Lambda$ may or may not be orientable. However, we can always define the oriented ray $\mathfrak{l} \in K^2$ as $\mathfrak{l} = (e_1 \wedge \cdots \wedge e_n)^{\otimes 2}$ where $e_1, \dots, e_n$ is any basis. Let $-\mathfrak{l}$ denote the negative ray $(-\infty, 0]$.

Let Φ_t be admissible coordinates centered at an orbit or chord c. Write $\varphi(t) = \det_{\mathbb{C}} \mathrm{d}\Phi_t(0)1$, where $1 = e_1 \wedge \cdots \wedge e_n$ is the standard basis of $\det_{\mathbb{C}}(\mathbb{C}^n)$. This definition makes sense even if Φ_t is only symplectic, as we can write:

$$\varphi = \mathrm{d}\Phi_t(0)e_1 \wedge \cdots \wedge \mathrm{d}\Phi_t(0)e_n,$$

and the symplectic condition will guarantee this is nowhere zero.

3.2.1. *The chord case.* Clearly, in the chord case, $\varphi(t)^{\otimes 2} \in \mathfrak{l}$ for $t = 0, 1$. If $\mathfrak{s}$ is another section of K^2 which is non-vanishing along c, and satisfies $\mathfrak{s} \notin -\mathfrak{l}$ on the boundary, then we can compare φ^2 and $\mathfrak{s}$ and get an integer valued winding number, analogous to the relative winding numbers requiring that φ^2 and $\mathfrak{s}$ both lie in $\mathfrak{l}$ along the boundary, see 4.2.

Then we simply require that the relative winding number between φ^2 and $\mathfrak{s}$ is zero. This determines the homotopy class of $\det_{\mathbb{C}} \mathrm{d}\Phi_t(0)^2$. By Theorem 3.6, we conclude that $\mathfrak{s}$ specifies *two* homotopy classes of coordinates, differening by how $\mathrm{d}\Phi_t(0)$ orients $T\Lambda_0$. In particular, we can represent both homotopy classes by Φ_t and $\Phi_t \circ \rho$ where $\rho = \mathrm{diag}(-1, 1, \dots, 1)$. In §3.4 we show that the asymptotic operators computed in either homotopy class have the same Conley-Zehnder index.

3.2.1.1. *Digression on a winding number.* The homotopy classes of non-vanishing sections $\mathfrak{t}$ satisfying $\mathfrak{t} \notin -\mathfrak{l}$ at both endpoints is non-canonically identified with $\mathbb{Z}$ via a relative winding number. The action of $\mathbb{Z}$ on this space via $\mathfrak{t} \mapsto e^{2\pi i t k}\mathfrak{t}$ acts freely and transitively on the set of homotopy classes.

A geometric explanation is as follows: if $\mathfrak{t}_1, \mathfrak{t}_2$ are two sections, first deform $\mathfrak{t}_1, \mathfrak{t}_2$ (remaining disjoint from $-\mathfrak{l}$) so that $\mathfrak{t}_1, \mathfrak{t}_2$ take boundary values in the right half plane (as determined by $\mathfrak{l}$). Then define $\theta = \mathfrak{t}_1/\mathfrak{t}_2$, and observe that θ is non-zero takes boundary values away from $(-\infty, 0]$. By perturbing θ slightly, we can define an integer winding number by counting the number of preimages of $(-\infty, 0]$. It is straightforward to see that (a) this does not depend on the choices mode (since $\mathbb{C} \setminus (-\infty, 0]$ is contractible), and (b) this winding number classifies the homotopy classes of sections $\mathfrak{t}$, as desired.

3.2.2. *The orbit case.* In this case, a section $\mathfrak{s}$ of K^2 determines *at most* a unique homotopy class of coordinates Φ_t, via the requirement that $\mathfrak{s}$ is homotopic to φ^2. If we require that $\mathfrak{s} = \mathfrak{c} \otimes \mathfrak{c}$ where $\mathfrak{c}$ is a non-vanishing section of K along c, then the homotopy class of coordinates induced by $\mathfrak{s}$ is non-empty (otherwise it is empty).

3.3. The asymptotic operator as a linearization

The asymptotic operator is defined by linearizing the Reeb flow equation at a solution c. To do so, fix an admissible symplectic coordinate chart Φ.

Let Φ^*TY denote the bundle over $S \times B(1)$ whose fiber at (t, p) is $TY_{\Phi_t(p)}$. Let Π be the section of $\mathrm{Hom}(\Phi^*TY, \mathbb{R}^{2n}) \to S \times B(1)$ defined by:

$$\Pi_t(p) \cdot \mathrm{d}\Phi_t(p) = \mathrm{id} \text{ and } \Pi_t(p)(R(\Phi_t(p))) = 0.$$

It is clear by property (iv) that any path between the same Legendrians which is sufficiently C^1 close to c will be of the form $t \mapsto \Phi_t(\eta(t))$ after reparametrization, where $\eta(0), \eta(1) \in \mathbb{R}^n$. Similarly, by (v), any loop sufficiently C^1 close to c will be of the form $t \mapsto \Phi_t(\eta(t))$. The degree of C^1 closeness necessary is that the path should be positively transverse to the contact distribution.

This follows the Reeb flow if and only if $\Pi_t(\eta(t))\frac{\partial}{\partial t}\Phi_t(\eta(t)) = 0$ for all t, which expands to:

$$\Pi_t(\eta(t))(\mathrm{d}\Phi_t(\eta(t))\eta'(t) + \Phi_t'(\eta(t))) = 0 \implies \eta'(t) + F_t(\eta(t)) \cdot \eta(t) = 0, \tag{3.1}$$

where $F_t(x) \cdot x = \Pi_t(x)\Phi_t'(x)$ for all $x \in B(1)$; this factorization is possible since the right hand side vanishes when $x = 0$.

Definition 3.7. The linearized Reeb flow operator associated to the admissible coordinate system Φ_t is defined to be:

$$JA(\eta) = \eta'(t) + F_t(0) \cdot \eta(t). \tag{3.2}$$

The operator A is called the *asymptotic operator* associated to Φ_t. Clearly (3.2) is the linearization of (3.1). It is clear that smooth homotopies of admissible coordinate systems yield smooth homotopies of asymptotic operators.

Lemma 3.8. We have $F_t(0) = -JS(t)$ for a family of symmetric matrices $S(t)$, where J is the standard complex structure on $\mathbb{R}^{2n}$. As a consequence, $A = -J\partial_t - S(t)$.

Proof. It is clear that in the standard basis for $TB(1) = \mathbb{R}^{2n}$ we have

$$F_t(0)\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i}(\Pi_t(x_1,\dots,x_{2n})\Phi_t'(x_1,\dots,x_{2n})). \tag{3.3}$$

This derivative is a bit hard to compute, since Π_t and Φ_t' are not sections of trivial bundles, but rather sections of $\mathrm{Hom}(\Phi^*TY, \mathbb{R}^{2n})$ and Φ^*TY, respectively. Thus we will introduce an auxiliary connection, denoted by ∇, on Φ^*TY. This connection will be pulled back from a special connection on TY, which we now describe.

Let ∇ be a *symmetric* connection on TY which satisfies $\nabla_R \mathrm{d}\alpha = 0$ and $\nabla R = 0$. Such a connection exists as it can be constructed in local coordinates which are well-adapted to the Reeb flow, and then glued together using a partition of unity. More precisely, if ∇^k, $k = 1, 2, \dots, N$, are connections on open sets $U_1, \dots, U_N$ which cover Y, then, for a partition of unity $\rho_1, \dots, \rho_N$ subordinate to the cover, the formula

$$\nabla(X) = \sum \nabla^k(\rho_k X)$$

defines a connection on all of TY. Moreover, if each ∇^k satisfies $\nabla^k R = 0$, $\nabla^k_R \mathrm{d}\alpha = 0$ and $\nabla^k_X Y - \nabla^k_Y X = [X, Y]$ (for vector fields supported in U_k), then so will ∇. The verification of these assertions is left to the reader.

Let ∇ denote the pull back connection on Φ^*TY, which we extend to connections on $\mathrm{Hom}(\Phi^*TY, \mathbb{R}^{2n})$ by requiring that $\mathrm{d}(A \cdot v) = \nabla A \cdot v + A \cdot \nabla v$ for all sections v.

Taking the derivative of (3.3) at $x = 0$ yields:

$$\frac{\partial}{\partial x_i}(\Pi_t(x)\Phi_t'(x)) = \nabla_{x_i}\Pi_t(0) \cdot \Phi_t'(0) + \Pi_t(0) \cdot [\nabla_{x_i}\Phi_t'](0). \tag{3.4}$$

Recalling that the connection was symmetric and Φ_t' is the time-derivative of the smooth map $(t, x) \mapsto \Phi_t(x)$, we conclude that $\nabla_{x_i}\Phi_t'(0) = \nabla_t[\mathrm{d}\Phi_t(0)]\frac{\partial}{\partial x_i}$. For the other term, observe that:

$$\Pi_t(x) \cdot R(\Phi_t(x)) = 0 \implies \nabla_{x_i}\Pi_t(x) \cdot R(\Phi_t(x)) = 0,$$

where we have used the fact that $\nabla R = 0$. We know that $\Phi_t'(0)$ is proportional to $R(\Phi_t(0))$, thus (3.4) simplifies to

$$\frac{\partial}{\partial x_i}(\Pi_t(x)\Phi_t'(x)) = \Pi_t(0)[\nabla_t \mathrm{d}\Phi_t(0)]\frac{\partial}{\partial x_i}.$$

It follows that $F_t(0) = \Pi_t(0)\nabla_t \mathrm{d}\Phi_t(0)$. We will now show that this is $-JS(t)$ for a symmetric matrix $S(t)$.

Let ω denote the symplectic form on $\mathbb{R}^{2n}$. Recall that $\mathrm{d}\Phi_t(0) : (\mathbb{R}^{2n}, \omega) \to (\xi, \mathrm{d}\alpha)$ was a symplectic linear transformation, and hence $\Pi_t(0) : (\xi, \mathrm{d}\alpha) \to (\mathbb{R}^{2n}, \omega)$ is also symplectic (as it is the inverse to $\mathrm{d}\Phi_t(0)$). Thus for $v, w \in \mathbb{R}^{2n}$ we have

$$\omega(v, \Pi_t(0)\nabla_t \mathrm{d}\Phi_t(0)w) = \mathrm{d}\alpha(\mathrm{d}\Phi_t(0)v, \nabla_t \mathrm{d}\Phi_t(0)w),$$

where we use that $\mathrm{d}\alpha(X_1, \mathrm{d}\Phi_t(0)\Pi_t(0)X_2) = \mathrm{d}\alpha(X_1, X_2)$ since it holds for $X_2 \in \xi$ and for $X_2 \in R\mathbb{R}$. Then, using that $\nabla_R \mathrm{d}\alpha = 0$, and Φ maps ∂_t proportionally to R along $x = 0$, we conclude:

$$\mathrm{d}\alpha(\mathrm{d}\Phi_t(0)v, \nabla_t \mathrm{d}\Phi_t(0)w) = -\mathrm{d}\alpha(\nabla_t \mathrm{d}\Phi_t(0)v, \mathrm{d}\Phi_t(0)w) = -\omega(\Pi_t(0)\nabla_t \mathrm{d}\Phi_t(0)v, w).$$

Thus $\omega(v, F_t(0)w) = -\omega(F_t(0)v, w)$. Recalling that $\omega(v, w) = v^T J w$, we conclude that

$$v^T J F_t(0) w = -v^T F_t(0)^T J w \implies J F_t(0) + F_t(0)^T J = 0,$$

which implies that $JF_t(0)$ is symmetric, as desired. □

3.3.0.1. *Geometric criteria for non-degeneracy.* Another important property is that non-degeneracy of the asymptotic operator is equivalent to non-degeneracy of the Reeb orbit or chord.

Lemma 3.9. Suppose that c is a non-degenerate Reeb orbit or Reeb chord from Λ_0 to Λ_1, in the sense that

(i-c) $\mathrm{d}\varphi_*^T(T\Lambda_0) \oplus T\Lambda_1 = \xi_{c(1)}$,

(i-o) $\mathrm{d}\varphi_*^T : \xi_0 \to \xi_0$ does not have 1 as an eigenvalue,

where φ denotes the Reeb flow and T is the action of c. Then the asymptotic operator defined in (3.2) is non-degenerate.

Proof. We continue with the notation introduced in the proof of the previous lemma. Suppose that $\eta(t)$ lies in the kernel of A. Then

$$\eta'(t) = -\Pi_t(0)\nabla_t \mathrm{d}\Phi_t(0)\eta(t).$$

Let $\mu(t) = \mathrm{d}\Phi_t(0)\eta(t)$, considered as a section of $c^*\xi$. We compute

$$\nabla_t \mu(t) = \mathrm{d}\Phi_t(0)\eta'(t) + \nabla_t \mathrm{d}\Phi_t(0)\eta(t) = 0.$$

Let φ^τ denote the Reeb flow by time τ. For any $v \in \xi_{c(0)}$, the vector field $\mathrm{d}\varphi^\tau(v)$ defined along $c(S)$ satisfies $\nabla_R \mathrm{d}\varphi^\tau(v) = 0$. This is easy to prove; first extend v to a vector field V on some transverse slice Σ passing tangent to ξ at $c(0)$, and then extend V to a neighborhood of $c(0)$ by requiring that $[V, R] = 0$. Then we can extend V to a neighborhood of a distant point $\varphi^\tau(c(0))$ by requiring that $\mathrm{d}\varphi^\tau \circ V \circ \varphi^{-\tau} = V$. It is clear that $[V, R] = 0$ still holds on this neighborhood. By symmetry of the connection and the fact that $\nabla R = 0$ we have:

$$\nabla_R V = \nabla_V R + [R, V] = 0.$$

In particular we conclude that $\nabla_R \mathrm{d}\varphi^\tau(v) = 0$ holds along the Reeb flow line starting at $c(0)$. Thus, if $\mu(t)$ is any section defined along a Reeb flow line satisfying $\nabla_R \mu = 0$, we must have $\mu(\varphi^\tau(c(0))) = \mathrm{d}\varphi^\tau(\mu(0))$.

Thus $\mu(0)$ and $\mu(1)$ are related by $\mathrm{d}\varphi^T$. However,

(i) in the chord case, $\mu(0)$ lies in the tangent space to Λ_0 while $\mu(1)$ lies in the tangent space to Λ_1, and

(ii) in the orbit case, $\mu(0)$ is an eigenvector of $\mathrm{d}\varphi^T$ with eigenvalue 1.

Thus, in either case, non-degeneracy implies non-degeneracy of the asymptotic operator. The converse implication is proved in a similar fashion. This completes the proof. □

We will require one further fact about the time derivative Φ'_t for a later computation.

Lemma 3.10. The derivative of $x \mapsto \alpha(\Phi'_t(x))$ vanishes at $x = 0$.

Proof. We use the same connection to take the derivative; we have

$$\frac{\partial}{\partial x_i}(\alpha(\Phi'_t(x))) = \langle \nabla_{x_i}\alpha, \Phi'_t(x)\rangle + \langle \alpha, \nabla_{x_i}\Phi'_t(x)\rangle.$$

It is clear that $\langle \nabla_{x_i}\alpha, R(\Phi_t(x))\rangle = 0$, since $\nabla R = 0$, hence $\langle \nabla_{x_i}\alpha, \Phi'_t(0)\rangle = 0$. Thus it remains to show the second term vanishes at $x = 0$. We use symmetry of the connection to obtain:

$$\nabla_{x_i}\Phi'_t(x) = \nabla_{\partial_t}\mathrm{d}\Phi_t(x)\partial_{x_i}.$$

Thus

$$\langle \alpha, \nabla_{x_i}\Phi'_t(x)\rangle = \frac{\partial}{\partial t}\langle \alpha, \mathrm{d}\Phi_t(x)\partial_{x_i}\rangle - \langle \nabla_{\partial_t}\alpha, \mathrm{d}\Phi_t(x)\partial_{x_i}\rangle.$$

The first term vanishes at $x = 0$. The second term also vanishes at $x = 0$ since ∂_t is mapped to a multiple of R when $x = 0$. This completes the proof. □

3.4. Conley-Zehnder index associated to a Reeb orbit or chord

Let c be a non-degenerate Reeb orbit or chord of Λ, and let $\mathfrak{s}$ be an admissible section of K^2.

The *Conley-Zehnder index* associated to this data is defined to be the Conley-Zehnder index of the asymptotic operator A, using any symplectic coordinate system Φ_t in the homotopy class(es) determined by $\mathfrak{s}$, as explained in 3.2.

3.4.1. *Independence on the choice of orientation.* In the chord case, there are two homotopy classes of coordinates determined by $\mathfrak{s}$, and they differ by whether or not $\mathrm{d}\Phi_0(0) : \mathbb{R}^n \to \Lambda_0$ preserves or reverses orientation. However, if $\rho \in O(n)$ reverses orientation, then $\Phi_t(\rho x)$ has non-linear operator equal to:

$$\eta'(t) + \rho^{-1}\Pi_t(\rho\eta(t))\Phi_t'(\rho\eta(t)) = 0.$$

When we linearize this at $\eta = 0$, we obtain the asymptotic operator:

$$A_\rho = -J\partial_t + \rho^{-1}S(t)\rho.$$

If $S(t)$ is split with respect to $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$, then it is easy to see that $A_\rho = A$ for suitable choice of ρ, e.g., $\operatorname{diag}(-1, 1, \dots, 1)$. In general, since the Conley-Zehnder index is defined as the Fredholm index of an operator of the form $D = \partial_s - A_s$ on the strip, where $A_s = A_{\text{ref}}$ for $s < 0$ and $A_s = A$ for $s > 1$, and A_{ref} *is split*, we conclude that the Conley-Zehnder index of A_ρ equals the Conley-Zehnder index of A, even for non-split A (we can just globally conjugate D by $\operatorname{diag}(-1, 1, \dots, 1)$ which does not change the Fredholm index).

Thus we conclude that the Conley-Zehnder index does not depend on the orientation assigned to Λ_0.

3.5. Independence of the Conley-Zehnder index on the complex structure

Recall that the Conley-Zehnder index relied on a choice of section $\mathfrak{s}$ of $\det_{\mathbb{C}}(\xi, J)^{\otimes 2}$. This bundle a priori depends on the choice of complex structure J. However, the space of almost complex structures compatible with $\mathrm{d}\alpha$ is contractible. Along any path J_τ of $\mathrm{d}\alpha$-compatible almost complex structures, the bundles $\det_{\mathbb{C}}(\xi, J_\tau)^{\otimes 2}$ form a complex line bundle over $[0, 1] \times Y$. Parallel transport allows us to map any section $\mathfrak{s}$ of the first bundle to a section of the second bundle.

Now suppose that $\mathfrak{s}$ is defined for $\det_{\mathbb{C}}(\xi, J_0)$ and $\mathfrak{s} \not\in -\mathfrak{l}$ holds along the Lagrangian. We apply parallel transport to obtain a new section $\mathfrak{s}'$ for $\det_{\mathbb{C}}(\xi, J_1)^{\otimes 2}$ which is still disjoint from $-\mathfrak{l}$ along Λ (since parallel transport is linear). In this fashion, we obtain an identification between the homotopy classes of sections $\mathfrak{s}'$ which work for (J_1, Λ) and the homotopy classes of $\mathfrak{s}$ which work for (J_0, Λ_0).

This identification respects the homotopy class of admissible coordinate systems, i.e., a coordinate chart is $\mathfrak{s}$-admissible if and only if it is $\mathfrak{s}'$-admissible.

3.6. Short Reeb chords in 1-jet spaces

In this section we describe how to construct admissible coordinate charts around Reeb chords in 1-jet spaces, write down the linearized operators in these charts, and then compute their Conley-Zehnder indices using the results of Chapter 2.

Let $Y = J^1(\Lambda)$ be the 1-jet space of an n-dimensional smooth manifold Λ. Recall that every function $f : \Lambda \to \mathbb{R}$ determines a 1-jet extension to a section Λ_f of $Y \to \Lambda$. We will typically identify Λ (the base of the fibration $Y \to \Lambda$) with the canonical section Λ_0.

Recall that Y has a contact form $\alpha = \mathrm{d}z - \lambda_{\mathrm{can}}$ so that $\Lambda_f^* \lambda_{\mathrm{can}} = \mathrm{d}f$, and $\Lambda_f^* z = f$.[2] Clearly in this contact structure any Legendrian section of $Y \to \Lambda$ must be the 1-jet of some function.

The Reeb chords joining Λ_{f_0} to Λ_{f_1} are in bijection with the critical points of $f_1 - f_0$ with positive critical values. The action (length) of each chord is equal to the critical value. Let us call such critical points *positive* critical points.

For any Reeb chord c joining two germs of Legendrians Λ_0, Λ_1 *which have non-singular projection* onto Λ, there are unique germs f_0, f_1 of smooth functions defined near $\mathrm{pr}(p) \in \Lambda$ so that $\Lambda_0 = \Lambda_{f_0}$ and $\Lambda_1 = \Lambda_{f_1}$, and p is a positive critical point of $f_1 - f_0$. We do not consider Legendrians which have singular projection onto Λ at the endpoints of Reeb chords in this section.

Lemma 3.11. The Reeb chord $\Lambda_{f_0} \to \Lambda_{f_1}$ associated to a positive critical point p is non-degenerate if and only if $f_1 - f_0$ is *Morse* at p.

Proof. In canonical coordinates near p, i.e., z, x, y so that $\alpha = \mathrm{d}z - y\mathrm{d}x$, with x valued in $D(1) \subset \mathbb{R}^n$, y valued in $\mathbb{R}^n$, and $x(p) = 0$, we have

$$\varphi^\tau \Lambda_{f_0} = \begin{cases} y = \mathrm{d}f_0, \\ z = f_0 + \tau. \end{cases}$$

The Reeb chord will be non-degenerate if and only if the intersection at $\tau = f_1 - f_0$ is transverse. By linearizing in the x and τ directions, it is easy to see that we can span any vector satisfying

$$z = \text{arbitrary and } y = \nabla \mathrm{d}f_0 \cdot h \text{ and } x = h.$$

Similarly when we linearize Λ_{f_1} with respect to x we can solve any vector of the form

$$z = \mathrm{d}f_1 \cdot k \text{ and } y = \nabla \mathrm{d}f_1 \cdot k \text{ and } x = k.$$

The Reeb chord will be non-degenerate, by definition, if and only if we can solve for every y, x as function of h, k:

$$y = \nabla \mathrm{d}f_0 \cdot h + \nabla \mathrm{d}f_1 \cdot k \text{ and } x = h + k.$$

This is solvable if and only if $y - \nabla \mathrm{d}f_0 \cdot x = (\nabla \mathrm{d}f_1 - \nabla \mathrm{d}f_0) \cdot k$ is solvable, which is precisely the Morse condition on $f_1 - f_0$. This completes the proof. □

[2]Here the notation Λ_f^* means that we pull back from the total space of the fibration $Y \to \Lambda$ back to the base via the section Λ_f.

3.6.1. *Choosing $\mathfrak{s}$ in 1-jet space.* Pick a Riemannian metric g on Λ. This induces a Levi-Civita connection on $T^*\Lambda$, which induces a splitting $TT^*\Lambda = T\Lambda \oplus T^*\Lambda$. There is a projection short exact sequence:

$$0 \to \mathbb{R}\partial_z \to TY \to T\Lambda \oplus T^*\Lambda \to 0,$$

which identifies ξ with $T\Lambda \oplus T^*\Lambda$. Extend J to all of ξ by requiring that the projection $\xi \to T\Lambda \oplus T^*\Lambda$ is holomorphic for the complex structure whose restriction to $T\Lambda$ acts by the isomorphism $g_* : T\Lambda \to T^*\Lambda$. It follows from work in [**CC22**] that $-\mathrm{d}\lambda_{\mathrm{can}}(-, J-)$ is the diagonal Riemannian metric on $T\Lambda \oplus T^*\Lambda$, and hence the projection $\xi \to T\Lambda \oplus T^*\Lambda$ is unitary when the former is given the metric $\mathrm{d}\alpha(-, J-)$.

The complex linear identification $\xi \simeq T\Lambda \oplus T^*\Lambda$ yields a real line subbundle

$$\det_{\mathbb{R}}(T\Lambda)^{\otimes 2} \subset \det_{\mathbb{C}}(\xi)^{\otimes 2}$$

which has a canonical unit length section $\mathfrak{s}(g)$.

Lemma 3.12. If Λ_f is a 1-jet section, then $g(\mathfrak{l}_f, \mathfrak{s}) > 0$ where g is the unitary metric induced by $\mathrm{d}\alpha(-, J-)$, and $\mathfrak{l}_f$ is the canonical unit length section in $\det_{\mathbb{R}}(\Lambda_f)^{\otimes 2}$.

Proof. With respect to the splitting the tangent space to Λ_f is spanned by

$$e_i + \nabla \mathrm{d} f \cdot e_i \in T\Lambda \oplus T^*\Lambda$$

where $e_1, \ldots, e_n$ form a local g-orthonomal frame. Then

$$\mathfrak{l}_f = [(e_1 + \nabla \mathrm{d} f \cdot e_1) \wedge \cdots \wedge (e_n + \nabla \mathrm{d} f \cdot e_n)]^{\otimes 2},$$

and $\mathfrak{s} = [e_1 \wedge \cdots \wedge e_n]^{\otimes 2}$. Recalling the conventions for the metric on the top wedge product we have that

$$g(\mathfrak{s}, \mathfrak{l}'_f) = \det{}^2 g(e_i, e_j + \nabla \mathrm{d} f \cdot e_j) = \det{}^2 g(e_i, e_j) = 1,$$

where we have used the orthogonality of $T\Lambda$ and $T^*\Lambda$ in the penultimate step. In general, we need to rescale $\mathfrak{l}'_f$ down to make it have unit length, as

$$g(\mathfrak{l}'_f, \mathfrak{l}'_f) = \det{}^2(g(e_i, e_j) + g(\nabla \mathrm{d} f \cdot e_i, \nabla \mathrm{d} f \cdot e_j)) \geq 1.$$

However, this scaling will preserve $g(\mathfrak{s}, \mathfrak{l}_f) > 0$. This completes the proof. □

It follows, in particular, that $\mathfrak{s} \notin -\mathfrak{l}_f$, and hence it is compatible.

3.6.2. *Admissible coordinate systems in 1-jet space.* Let c be a non-degenerate Reeb chord joining Λ_{f_0} to Λ_{f_1}. Throughout this section, introduce canonical coordinates z, x, y so that $x(c) = 0$. Since c is non-degenerate, we may pick our canonical coordinate so that $f(x) := f_1(x) - f_0(x) = a + \frac{1}{2} \sum \lambda_i x_i^2$ where each λ_i is ± 1.

We propose a symplectic coordinate system via the formula:

$$\Phi_t(x,y) = \begin{bmatrix} f_0(x) + tf(x) \\ x \\ \mathrm{d}f_0(x) + t\mathrm{d}f(x) + y. \end{bmatrix} \tag{3.5}$$

We proceed to check that Φ_t satisfies the required properties of admissible coordinates from §3.1. It is clear that $\Phi_t(0)$ is the time 1 constant speed parametrization of the Reeb chord correponding to the critical point $\mathrm{d}f(0) = 0$. Next, we note that, by construction $\Phi_0(\mathbb{R}^n \times \{0\}) = \Lambda_{f_0}$ and $\Phi_1(\mathbb{R}^n \times \{0\}) = \Lambda_{f_1}$. This verifies properties (i) and (iv) of the definition.

Next, we compute the time and space derivatives:

$$\Phi_t' = \begin{bmatrix} f(x) \\ 0 \\ \mathrm{d}f(x) \end{bmatrix} \text{ and } \mathrm{d}\Phi_t = \begin{bmatrix} \mathrm{d}f_0(x) + t\mathrm{d}f(x) & 0 \\ 1 & 0 \\ \nabla\mathrm{d}f_0 + t\nabla\mathrm{d}f & 1 \end{bmatrix} \tag{3.6}$$

It is clear that $\Phi_t'(0)$ and $\mathrm{d}\Phi_t(0)$ are transverse whenever $f(x) \neq 0$ (in particular, this holds on a neighborhood around $x = 0$ since c corresponds to a positive critical value). Thus property (ii) holds.

Next, observe that $x \mapsto \Phi_t(x,0)$ *always* parameterizes some Legendrian, and hence the linearization of $x \mapsto \Phi_t(x,0)$ is tangent to ξ. Similarly, $y \mapsto \Phi_t(0,y)$ always points in the directions tangent to ∂_y, which are tangent to ξ. Hence $\mathrm{d}\Phi_t(0)(\mathbb{R}^{2n}) = \xi_{\Phi_t(0)}$. Recall that the symplectic form in canonical coordinates is given by $\sum \mathrm{d}x_i \wedge \mathrm{d}y_i$. After pulling back by Φ_t we conclude that:

$$\mathrm{d}y_i \cdot \mathrm{d}\Phi_t(0) = \mathrm{d}y_i + \textstyle\sum_j A_{ij}\mathrm{d}x_j \text{ and } \mathrm{d}x_i \cdot \mathrm{d}\Phi_t(0) = \mathrm{d}x_i,$$

where A_{ij} *is symmetric*, and hence

$$\textstyle\sum \mathrm{d}x_i \cdot \mathrm{d}\Phi_t(0) \wedge \mathrm{d}y_i \cdot \mathrm{d}\Phi_t(0) = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i + \sum_{i,j} A_{ij}\mathrm{d}x_i \wedge \mathrm{d}x_j = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Thus property (iii) holds.

Since $\mathrm{d}\Phi_t(0)(\mathbb{R}^n)$ is the tangent space to the Legendrian Λ_{f_0+tf}, we conclude that the unit length section of $\det(\xi)^{\otimes 2}$ induced by $\varphi = \det \mathrm{d}\Phi_t(0)1$, satisfies $g(\mathfrak{s}, \varphi^2) > 0$ for all $t \in [0,1]$. Thus the coordinate system (3.5) is an $\mathfrak{s}$-admissible chart, and can be used to compute the linearized operator and Conley Zehnder indices.

3.6.2.1. *Computing the linearized operator in this coordinate system.* In order to define the linearized operator, recall that we use the projection $\Pi_t \in \mathrm{Hom}(\Phi^*TY, \mathbb{R}^{2n})$ satisfying $\Pi_t(x)(R(\Phi_t(x))) = 0$ and $\Pi_t(0)\mathrm{d}\Phi_t(0) = \mathrm{id}$. Using the canonical coordinates z, x, y as a frame for TY, we can write Π_t as a family of 2×3 matrices:

$$\Pi_t(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\nabla \mathrm{d} f_0 - t\nabla \mathrm{d} f & 1 \end{bmatrix}.$$

Recalling the formula for $\Phi_t'(x)$, we see that the non-linear Reeb flow operator, for $\eta = x(t) + Jy(t)$, is

$$0 = \eta'(t) + \Pi_t(\eta(t))\Phi_t'(\eta(t)) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{d} f(x) \end{bmatrix}.$$

Linearizing this operator at $x = 0$, $y = 0$, yields

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \nabla \mathrm{d} f(x) & 0 \end{bmatrix},$$

and hence, recalling $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, we have the asymptotic operator:

$$A = -J\partial_t - \begin{bmatrix} -\nabla \mathrm{d} f(x) & 0 \\ 0 & 0 \end{bmatrix} =: -J\partial_t - S.$$

This decomposition is with respect to the splitting $x_1, \dots, x_n, y_1, \dots, y_n$.

Recall that we had $f(x) = a + \frac{1}{2}\sum \lambda_i x_i^2$, so that $\nabla \mathrm{d} f(x) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Thus, with respect to the splitting $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ we have

$$S(t) = \mathrm{diag}(\left[\begin{smallmatrix} -\lambda_1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} -\lambda_n & 0 \\ 0 & 0 \end{smallmatrix}\right]).$$

It follows from Example 2.12 that

$$\mu_{\mathrm{CZ}}(A) = -\text{number of positive eigenvalues} = \text{Morse index}(f) - n. \tag{3.7}$$

Chapter 4

Dimensions of moduli spaces

The main goal of this chapter is to prove Theorem 1.3, namely, the formula for the expected dimension of the space of parametrized holomorphic maps nearby a given map u. At the end of the section we provide dimension formulas for counting holomorphic curves with boundary on 1-jet sections in 1-jet space, and explain how the formulas suggest one can define higher algebraic structures on Morse homology in the framework of relative SFT. The relationship between Morse theory and relative SFT is well-known in the literature; see [**Ekh07**], [**EL17**], [**BEE12**].

4.1. Outline of the strategy

The overarching strategy is to prove that the linearized operator D_u naturally has the structure of an asymptotically non-degenerate Cauchy-Riemann operator, as defined in §6, and then apply the index formula stated in §6.1. In §4.2, we give a general explanation of how to linearize the holomorphic curve equation. In order to apply the index formula, we require u^*TW to have an *asymptotically Hermitian structure*, as defined in §6.3.2, and this is explained in §4.2.1. There are some idiosyncracies involving exponential weights when working with holomorphic curves defined on punctured domains which lead to contributions to the dimension formula; this is discussed in §4.2.2. In §4.2.3, we describe the relationship between the results from §3 about Reeb chords and the linearization of the holomorphic curve equation; this section contains the technical computation of the linearized operator. Finally, in §4.3 and §4.3.1 we complete the proofs of Theorems 1.3 and 1.5.

4.2. Digression on the linearization of the holomorphic curve equation

In general, let (W^{2n}, J) be an almost complex manifold and $u : \Sigma \to W$ be a J-holomorphic map. The *linearization* of u is defined to be the Cauchy-Riemann operator D_u on u^*TW determined by the following local property. To set the stage, let $U \subset \Sigma$ be an open region with holomorphic coordinates s, t, and $\Psi_{s,t} : B(1) \to W$ a family of open embeddings defined along U, close to u in the sense that $u(s,t) = \Psi_{s,t}(x(s,t))$ can be solved for $x(s,t)$. Then, clearly, $\mathrm{d}\Psi_{s,t}(x)$ identifies $\mathbb{R}^{2n}$ with $TW_{\Psi_{s,t}(x)}$. Let $\mathrm{P}_{s,t}(x)$ denote the inverse map

$TW_{\Psi_{s,t}(x)} \to \mathbb{R}^{2n}$, and define the non-linear operator associated to $u = \Psi(x)$ by:

$$\partial_\Psi(\eta) = \mathrm{P}_{s,t}(x+\eta)[\frac{\partial}{\partial s}\Psi_{s,t}(x+\eta) + J(\Psi_{s,t}(x+\eta))\frac{\partial}{\partial t}\Psi_{s,t}(x+\eta)].$$

Note that ∂_Ψ maps sections of $\mathbb{R}^{2n}$ to sections $\mathbb{R}^{2n}$, and hence can be linearized in an obvious way. Let D_Ψ denote its linearization. We define:

$$D_u(w) = (\mathrm{d}s - i\mathrm{d}t) \otimes \mathrm{d}\Psi_{s,t}(x) \cdot D_\Psi \cdot \mathrm{P}_{s,t}(x)w. \tag{4.1}$$

This is a Cauchy-Riemann operator. Remarkably, the section $D_u(w)$ of $\Lambda^{0,1} \otimes u^*TW$ is independent of the choices made, and hence these local coordinate descriptions glue to define a global Cauchy-Riemann operator on u^*TW. Morally, the reason why this construction is independent of the choices is the same reason why the linearization of a section of a vector bundle is well-defined *at the zeros* of the section. We prove this invariance in §4.4.

Remark 4.1. The independence of the linearized operator on the choice of $\Psi_{s,t}$ gives us a lot of flexibility in how we choose to linearize the holomorphic curve equation. Two standard choices are:

(i) $\Psi_{s,t} : B(1) \to W$ is the Riemannian exponential map $x \mapsto \mathrm{Exp}_{u(s,t)}(\sum x_i X_i)$ where X_i is some travelling frame. This has the property that $\Psi_{s,t}(0) = u(s,t)$, which is sometimes useful.

(ii) $\Psi_{s,t} : B(1) \to W$ satisfies $\Psi_{s,t} = \Psi$ for a fixed embedding. This is a popular choice. Oftentimes one requires that $\mathrm{d}\Psi_{s,t}$ is complex linear at the origin.

Remark 4.2. The admissible coordinates Φ_t from §3.1 induce a family of $\Psi_{s,t}$ which can be used to linearize finite energy holomorphic curves near Reeb chords. Let

$$\Psi_{s,t}(\sigma,\tau,x) = \text{flow of } \Phi_t(x) \text{ by } (\sigma + Ts)\partial_\sigma + \tau R \text{ for time } 1,$$

where T is the action of the Reeb chord. The family $\Psi_{s,t}$ defines a family of open embeddings of a $2n+2$ dimensional balls of the form $\mathbb{R} \times (-\epsilon, \epsilon) \times B(1)$. Note that we can always extend $\Phi_t(x)$ for $t < 0$ and $t > 1$ by extending the time derivative $\Phi_t'(x)$ (which is a priori defined on a submanifold with boundary in Y).

Suppose that $u(s,t)$ is defined on a strip $[a,b] \times [0,1]$ and $t \mapsto \mathrm{pr} \circ u(s,t)$ is sufficiently close to the Reeb chord $\Phi_t(0)$ in C^1. Then we can solve:

$$u(s,t) = \Psi_{s,t}(\sigma(s,t), \tau(s,t), x(s,t)).$$

Moreover, as proved in §10, τ, x both converge to zero, and σ converges to a constant σ_0. The same sort of coordinate system also works for Reeb orbits. Indeed, the only meaningful difference is that τ need not converge to 0, but could converge to a non-zero constant τ_0.

Remark 4.3. For holomorphic curves which are not near Reeb chords/orbits, a useful coordinate system is obtained by flowing by the Reeb vector field R, as follows. Let φ parametrize a disk transverse to R in Y, and define

$$\Psi(\sigma,\tau,x) = \text{flow of } \varphi(x) \text{ by } \sigma\frac{\partial}{\partial\sigma} + \tau R \text{ for time } 1.$$

We restrict the range of τ around $\tau = 0$ so that the flow is an embedding. Coordinates of this type obviously cover Y.

Lemma 4.4. Let $W = \mathbb{R}\times Y$ be the symplectization of a contact manifold with an admissible complex structure (in the SFT sense). Suppose that $u : \Sigma \to W$ is a holomorphic curve. Let $\Pi_\xi : u^*TW \to u^*\xi$. Then $D_\xi = \Pi_\xi D_u|_{u^*\xi}$ is a Cauchy-Riemann operator.

Proof. This is trivial. Let f be a real-valued function, and let η be a section of $u^*\xi$. Then:

$$D_u(f\eta) = \mathrm{d}f \otimes \eta + \mathrm{d}f \circ j \otimes J_\xi\eta + fD_u(\eta).$$

Now project both sides onto ξ, via Π_ξ, whereby we obtain

$$D_\xi(f\eta) = \mathrm{d}f \otimes \eta + \mathrm{d}f \circ j \otimes J_\xi\eta + fD_\xi(\eta),$$

as desired. □

4.2.1. *Asymptotic trivializations of u^*TW.* To apply the index formula in 6.1, we require that the bundle under consideration has an *asymptotically Hermitian structure*. Briefly, such a structure is:

(i) a collection of strip/cylindrical ends biholomorphic to $[0,\infty) \times S$, $S = [0,1]$ or $\mathbb{R}/\mathbb{Z}$, whose complement is a compact set, and

(ii) a *unitary frame* $X_1,\dots,X_n$ on the restriction of u^*TW to each end, spanning $u^*T\Lambda$ along the boundary.

Under a truncation operation, we can think of such frames as being germs of frames at infinity. We say two frames be *commensurate* if they differ by a smooth transition function $\theta : [0,\infty) \times S \to U(n)$ (taking boundary values in $O(n)$) which satisfies $\left|\partial_s^k\theta\right| = o(1)$ for all $k \geq 1$. There are obvious modifications for negative punctures. A particular frame is called an *asymptotic trivialization*, and we define an asymptotically Hermitian structure to be a commensurability class of asymptotic trivializations.

One can consider homotopy classes of commensurate asymptotic trivializations at a fixed puncture, and by results in §3, we see that:

(chord) the set of path components forms a $\mathbb{Z} \times \mathbb{Z}/2$-torsor, classified by the (relative) winding number of $(X_1 \wedge \cdots \wedge X_n)^{\otimes 2}$ and the orientation $X_1 \wedge \cdots \wedge X_n$ on $T\Lambda_0$.

(orbit) At an interior puncture the set of path components forms a $\mathbb{Z}$-torsor, classified simply by the relative winding number of $X_1 \wedge \cdots \wedge X_n$, see 3.1.3.

By appealing to convergence results in §10, we will show that every holomorphic curve u asymptotic to a Reeb chord/orbit has a canonical asymptotically Hermitian structure on $u^*TW, u^*T\Lambda$. Moreover, if one chooses a homotopy class of admissible symplectic coordinates for the limit Reeb chord/orbit, then $u^*TW, u^*T\Lambda$ has a canonical homotopy class of asymptotic trivializations.

The construction is fairly straightforward, and we describe it in the case of Reeb chords, leaving the analogous orbit case to the reader. Pick a constant speed unitary coordinate chart Φ around c in the desired homotopy class, and use Remark 4.2 to obtain the coordinate system $\Psi_{s,t}(\sigma, \tau, x) = \mathrm{F}\Psi_t(x)$ around $\mathbb{R} \times c$ inside $\mathbb{R} \times Y$.

By Theorem 10.1, $u(s,t)$ can be written as $\Psi_{s,t}(\sigma, \tau, x)$ where τ and $\sigma - \sigma_0$ and their derivatives decay exponentially with rate $\delta(c)$. Then

$$\mathrm{d}\Psi_{s,t}(x) = \begin{bmatrix} 1 & (1 - \Pi_\xi)\mathrm{dFd}\Phi_t(x) \\ 0 & \Pi_\xi \mathrm{dFd}\Phi_t(x) \end{bmatrix} : \mathbb{C} \oplus \mathbb{R}^{2n} \to \mathbb{C} \oplus u^*\xi \simeq u^*TW$$

converges exponentially to a unitary isomorphism (namely $\mathrm{diag}(1, \mathrm{d}\Phi_t(0))$). There is a unique commensurability class, resp., homotopy class, of asymptotic trivializations which is commensurate, resp., homotopic, to the frame induced by the above isomorphism. More precisely, if $X_1, \ldots, X_n$ is a unitary frame of $(u^*\xi, u^*T\Lambda)$ and $e_1, \ldots, e_n$ is the standard $\mathbb{R}^n$ frame, and

$$\|\Pi_\xi \mathrm{dFd}\Phi_t(x)e_i - X_i\|_{C^k(s)} = o(e^{-\delta|s|}) \text{ as } |s| \to \infty,$$

then we say that X_i is *asymptotically* Φ-*standard*. The canonical commensurability class, resp., homotopy class, is represented by the frame $X_1, \ldots, X_n$.

Proposition 4.5. Asymptotically Φ-standard frames always exist.

Proof. This is slightly non-trivial, since $\Pi_\xi \mathrm{dFd}\Phi_t(x)e_i$ may not be a unitary frame. By construction, we know that $\Pi_\xi \mathrm{dFd}\Phi_t(x)e_i$ lies in $u^*T\Lambda$ when $t = 0, 1$, so that is good. By applying the Gram-Schmidt process to $\Pi_\xi \mathrm{dFd}\Phi_t(x)e_i$ we can make it orthogonal for the Hermitian metric. Moreover, this process does not leave $u^*T\Lambda$ along the boundary. The resulting frame is called X_i (and $X_1, J_\xi X_1, \ldots$ forms a basis at each point). Simple estimates of the change induced by Gram-Schmidt process imply that the resulting frame is asymptotically standard, and the details are left as exercise for the reader. See the proof of Lemma 4.7 for further discussion. □

4.2.2. *Digression on exponential weights.* See [**Wen10**, §2], [**BM04**, Proposition 4], for discussion in the case of interior punctures. See [**BC07**, §4.4], [**CEJ10**] for similar arguments.

Naively, the moduli space of holomorphic maps defined on Σ valued some almost complex manifold W (with boundary conditions on a totally real submanifold), is analyzed by considering the moduli space as embedded in some Sobolev manifold of all maps $W^{1,p}(\Sigma, W)$. One can do this in the case $W = \mathbb{R} \times Y$, with totally real submanifold $\mathbb{R} \times \Lambda$ for Λ a Legendrian.

There are other Sobolev manifolds we could consider. For instance, Theorem 10.1 implies that exponentially weighted Sobolev spaces $W^{1,p,\delta}$ will also contain any relevant holomorphic curves, provided δ is sufficiently small (i.e., smaller than any $\delta(c)$ appearing in the curve under consideration).[1]

This exponential decay result is fortuitous, because if we linearize the holomorphic curve equation in the unweighted space $W^{1,p}$ we obtain a *non-Fredholm* linearization. We digress for a moment to explain this phenomenon.

As stated in Equation (4.3) below, the linearization is a Cauchy-Riemann operator on u^*TW, and the linearization is asymptotically diagonal for the splitting $u^*TW = \mathbb{C} \oplus u^*\xi$, where the $\mathbb{C}$ factor is spanned by ∂_σ and R. In the strip-like ends, the linearization on the $\mathbb{C}$-factor takes the form:

$$D_{\mathbb{C}}(\eta) = \partial_s \eta + J\partial_t \eta,$$

This means that the asymptotics are degenerate. It can be shown that this implies D is not Fredholm. Roughly, the argument is that we can construct a sequence of sections η_n taking values the $\mathbb{C}$ factor with $\|\eta_n\|_{W^{1,p}} = 1$, but with $D(\eta_n)$ tending to 0 in L^p. We should construct η_n so that it has its L^p norm spread out over a large region, and thus has a small derivative, and so that its support lies in the region $|s| > s_n$ with $s_n \to \infty$. Lemma 4.7 implies that this construction is possible. *If* D were Fredholm, then standard linear compactness results would imply that η_n converges to a non-zero solution of $\partial_s \eta_\infty + J\partial_t \eta_\infty = 0$ lying in $W^{1,p}$. Briefly, the argument is that $D + \Pi : W^{1,p} \to L^p \oplus V$ is a *closed linear embedding* where Π is some map $W^{1,p} \to V$ onto a *finite dimensional* vector space. Thus, if η_n is bounded, then a subsequence of the projection $\Pi\eta_n$ converges. Since the other summand, $D : W^{1,p} \to L^p$, is a closed map and $D(\eta_n)$ converges to 0, we conclude that a subsequence of η_n converges in $\eta_\infty \in W^{1,p}$ (by applying the closed linear embedding property). However, there are no non-zero solutions of $\partial_s \eta_\infty + J\partial_t \eta_\infty = 0$ in $W^{1,p}(\mathbb{R} \times [0,1]; \mathbb{C}, \mathbb{R})$. This contradiction proves that D cannot be Fredholm.

However, if we instead use the $W^{1,p,\delta}$ space, the linearization (via conjugation with the Banach isomorphism $\eta \mapsto e^{-\delta s}\eta$), is described by the map $\eta \in W^{1,p} \mapsto e^{\delta s} D(e^{-\delta s}\eta)$, which is easy to compute as:

$$D_\delta(\eta) = \partial_s \eta + J\partial_t \eta + S(s,t)\eta - \delta\eta.$$

[1] Technically we lied in this sentence, as the different curves can have different constants σ_0, τ_0, but let us ignore this issue for the moment.

This was assuming we were working at a positive puncture. If we were instead working at a negative puncture, then we would have:

$$D_\delta(\eta) = \partial_s\eta + J\partial_t\eta + S(s,t)\eta + \delta\eta$$

These are Fredholm operators.

One can define a global linear isomorphism which conjugates $(D, W^{1,p,\delta}, L^{p,\delta})$ to $(D_\delta, W^{1,p}, L^p)$ where D_δ has its asymptotics given by the above formulas (one simply needs to multiply by an appropriate $\mathbb{R}$-valued function).

Assuming δ is sufficiently small compared to the spectral properties of A, we conclude that:

$$\operatorname{Index}(D_\delta) = \operatorname{Index}(D_\delta|_{\mathbb{C}}) + \operatorname{Index}(D_\xi).$$

The index formula from §6.1 can be immediately applied to $(D_\delta|_{\mathbb{C}})$ (i.e., it does not depend on the topology of ξ). We compute:

$$\begin{aligned}\operatorname{Index}(D_\delta|_{\mathbb{C}}) &= \mathrm{X}(\Sigma,\Gamma_\pm) + \sum_{\Gamma_+}\mu_{\mathrm{CZ}}(-J\partial_t\eta + \delta\eta) - \sum_{\Gamma_-}\mu_{\mathrm{CZ}}(-J\partial_t - \delta\eta)\\ &= \mathrm{X}(\Sigma,\Gamma_\pm) - |\Gamma_+| - \left|\Gamma_-^{\mathrm{int}}\right|\end{aligned}$$

where we have used the computation of the Conley-Zehnder indices relevant to exponential weights from §2.7. We therefore conclude that:

$$\operatorname{Index}(D^\delta) = \mathrm{X}(\Sigma,\Gamma_\pm) - |\Gamma_+| - \left|\Gamma_-^{\mathrm{int}}\right| + \operatorname{Index}(D_\xi),$$

where D_ξ is the restriction of the linearization to $u^*\xi$.

Unfortunately, as alluded to above, there is the problem that not all nearby holomorphic curves will lie in the $W^{1,p,\delta}$ charts centered on a fixed curve u, because two u_1, u_0 can have $\sigma\circ u_1 - \sigma\circ u_0$ converging to a non-zero constant.

Remark 4.6. At an interior puncture, we can also have $\tau\circ u_1 - \tau\circ u_0$ converging to a non-zero constant, where τ is an $\mathbb{R}/\mathbb{Z}$-valued coordinate parametrizing the asymptotic Reeb orbit. Typically in SFT one fixes a base-point on the underlying simple Reeb orbit and one requires that *asymptotic markers* converge to this base-point. See [**Wen20**], [**Par19**] for more discussion. This implies that any curve nearby u in the moduli space will be forced to have $\tau\circ u_1 - \tau\circ u_0$ convergent to 0.

To fix the issue with the σ and τ coordinate, we define:

$$W^{1,p,\delta,\mathrm{st}} \subset W^{1,p}_{\mathrm{loc}}$$

to be the image of $(u,\rho) \in W^{1,p,\delta} \oplus \mathbb{R}^\Gamma \oplus (i\mathbb{R})^{\Gamma^{\mathrm{int}}} \mapsto u+\rho \in W^{1,p}_{\mathrm{loc}}$ where $\mathbb{R}^\Gamma$ and $(i\mathbb{R})^{\Gamma^{\mathrm{int}}}$ are finite-dimensional families of sections, so that the section 1 corresponding to $\zeta\in\Gamma$ is supported in the strip-like end corresponding to ζ and converges exponentially, with rate at least δ, to 1 in the trivial $\mathbb{C}$ factor (i.e., ∂_σ). For interior punctures, we also require that the section i corresponding to ζ converges to i in the trivial $\mathbb{C}$ factor (i.e., R) with rate δ.

The set $W^{1,p,\delta,\mathrm{st}}$ is independent of the precise choice of finite-dimensional family, and the isomorphisms with $W^{1,p,\delta} \oplus \mathbb{R}^\Gamma \oplus (i\mathbb{R})^{\Gamma^{\mathrm{int}}}$ give a well-defined Banach space topology.

The linearized operator $D^{\delta,\mathrm{st}} : W^{1,p,\delta,\mathrm{st}} \to L^{1,p,\delta}$ is well-defined. Moreover, the Banach manifold locally modeled on $W^{1,p,\delta,\mathrm{st}}$ is large enough to contain all the holomorphic maps $\Sigma \to \mathbb{R} \times Y$ under consideration. We have:

$$\operatorname{Index}(D^{\delta,\mathrm{st}}) = \mathrm{X}(\Sigma, \Gamma_\pm) + |\partial\Gamma_-| + \left|\Gamma^{\mathrm{int}}\right| + \operatorname{Index}(D_\xi). \tag{4.2}$$

This quantity should be thought of as the expected dimension for the space of *parametrized* holomorphic maps nearby u. If one wishes to restrict curve via *asymptotic markers*, in the SFT sense, one should subtract $\left|\Gamma^{\mathrm{int}}\right|$ from this formula.

4.2.3. *The Cauchy-Riemann operator D_ξ.* The index formula for D_ξ depends on certain topological quantities associated to ξ, Λ, and the asymptotic Reeb chords and orbits.

Suppose that u is a holomorphic map with boundary on $\mathbb{R} \times \Lambda$ for a Legendrian Λ and assume u is asymptotic to the set $\mathcal{R}$ of Reeb chords of Λ at its boundary punctures, and the set $\mathcal{O}$ of Reeb orbits at its interior punctures.

Pick an admissible section $\mathfrak{s}$ of $\det(\xi)^{\otimes 2}$ as in §1.3.3. For each chord or orbit, pick unitary charts Φ_t compatible with $\mathfrak{s}$. Let also assume that the charts traverse each Reeb chord with constant speed in time 1.

The exponential decay estimate from Theorem 10.1 implies that every holomorphic strip with boundary on Λ_0, Λ_1 will eventually enter the domain of the chosen coordinate charts, as explained in Remark 4.2. We will now compute the linearization of the holomorphic curve equation in these coordinates. The computation is a bit technical.

Lemma 4.7. Let Φ_t be an $\mathfrak{s}$-admissible unitary coordinate chart centered at the non-degenerate Reeb chord (resp., orbit) c, so that $\Phi_t(0)$ traverses c with constant speed in time 1. Suppose that u is a holomorphic strip (resp., cylinder) asymptotic to c, and let $X_1, \dots, X_n$ be an asymptotically Φ-standard unitary frame of $u^*\xi$, as defined in §4.2.1. Let X_0 be the standard frame of the $\mathbb{C}$-summand of u^*TW, namely $X_0 = \partial_\sigma \circ u$ and $JX_0 = R \circ u$. Let $A^{\mathfrak{s}} = -J_0\partial_t - S(t)$ be the asymptotic operator for c, as computed by Φ following §3.3. Then for $\eta X = \eta_{\mathbb{C}} X_0 + \sum_k \eta_{\xi,k} X_k$, we have:

$$D_u(\eta X) = \begin{bmatrix} X_0' & X_1', \dots, X_n' \end{bmatrix} \begin{bmatrix} \partial_s + i\partial_t & \Delta_1 \\ \Delta_2 & \partial_s + J_0\partial_t + S(t) + \Delta_3 \end{bmatrix} \begin{bmatrix} \eta_{\mathbb{C}} \\ \eta_\xi \end{bmatrix}, \tag{4.3}$$

where Δ_i are zeroth order terms which converge to 0 as $|s| \to \infty$. Here we use the notation $X_k' = (\mathrm{d}s - i\mathrm{d}t) \otimes X_k$ to denote the induced frame for $\Lambda^{0,1} \otimes u^*TW$.

Proof. We use the coordinate system $\Psi_{s,t}(\sigma,\tau,x)$ introduced in Remark 4.2. If we let $\mathrm{F}_s(\sigma,\tau)$ be the time 1 flow by $(\sigma+Ts)\partial_\sigma+\tau R$, then $\Psi_{s,t}(\sigma,\tau,x)=\mathrm{F}_s(\sigma,\tau)\Phi_t(x)$. Throughout the argument, we will often drop subscripts, etc, from the notation (in order to fit the formulas), i.e., we will write F instead of $\mathrm{F}_s(\sigma,\tau)$, etc. We make one immediate reduction; by suitably changing our coordinates by a rotation, we may suppose that τ converges to 0 (i.e., we can set $\tau_0=0$).

As explained in Remark 4.2, we may suppose that $u(s,t)=\Psi_{s,t}(\sigma(s,t),\tau(s,t),x(s,t))$. Then we compute:

$$\partial_s\Psi_{s,t}(\sigma,\tau,x)=\mathrm{dFd}\Phi_t(x)\frac{\partial x}{\partial s}+\partial_\sigma\circ\Psi(\frac{\partial\sigma}{\partial s}+T)+R\circ\Psi\frac{\partial\tau}{\partial s}.$$

Next we compute the other term appearing in the non-linear equation:

$$J\partial_t\Psi_{s,t}(\sigma,\tau,x)=J\mathrm{dFd}\Phi_t(x)\frac{\partial x}{\partial t}+R\circ\Psi\frac{\partial\sigma}{\partial t}-\partial_\sigma\circ\Psi(\frac{\partial\tau}{\partial t}+T)+J(\mathrm{dF}\Phi_t'(x)-TR).$$

Adding these together gives a concise expression for the non-linear operator:

$$\partial_s\Psi+J\partial_t\Psi=\mathrm{dFd}\Phi\frac{\partial x}{\partial s}+J\mathrm{dFd}\Phi\frac{\partial x}{\partial t}+\partial_\sigma[\frac{\partial\sigma}{\partial s}-\frac{\partial\tau}{\partial t}]+R[\frac{\partial\tau}{\partial s}+\frac{\partial\sigma}{\partial t}]+J(\mathrm{dF}\Phi_t'(x)-TR).$$

Recall that, in order to linearize the PDE as prescribed by §4.2, we need to apply $\mathrm{P}_{s,t}=\mathrm{d}\Psi_{s,t}^{-1}$ to both sides and then linearize the resulting operator on $\mathbb{C}\oplus\mathbb{R}^{2n}$.

It is clear that $\mathrm{P}_{s,t}:u^*TW\to\mathbb{C}\oplus\mathbb{R}^{2n}$ maps ∂_σ to 1, R to i, and maps u^*TW onto $\mathbb{R}^{2n}$ via $\Pi_t(x)\mathrm{dF}^{-1}$, where $\Pi_t(x)\mathrm{d}\Phi_t(x)=\mathrm{id}$ and $\Pi_t(x)R=\Pi_t\partial_\sigma=0$, as prescribed by §3.3 and §3.1.3. We conclude that, for $z=\sigma+i\tau$,

$$\mathrm{P}(\partial_s\Psi+J\partial_t\Psi)=\begin{bmatrix}\partial_s z+i\partial_t z\\ \partial_s x+J_0\partial_t x\end{bmatrix}+(\begin{bmatrix}0&0\\0&J_0\end{bmatrix}-\mathrm{P}J\mathrm{dFd}\Phi)\frac{\partial x}{\partial t}+\mathrm{P}J(\mathrm{dF}\Phi_t'(x)-TR).$$

The first term is linear, indeed, it is the standard Cauchy-Riemann operator on sections of $\mathbb{C}\oplus\mathbb{R}^{2n}$. We will now proceed to estimate the other terms. Let $w=(z,x)$, and let $E(w)=A\cdot w\cdot w+B\cdot w\cdot\mathrm{d}w+C\cdot\mathrm{d}w\cdot\mathrm{d}w$ be an error term, where A,B,C are smooth tensor valued functions of w and $\mathrm{d}w$. We also suppose that E doesn't depend σ_0 or τ_0. We treat the $E(w)$ notation similarly to the "little o" notation, i.e., it acts as a sort of garbage collector term.

Observe that:

$$(\begin{bmatrix}0&0\\0&J_0\end{bmatrix}-\mathrm{P}J\mathrm{dFd}\Phi)\frac{\partial x}{\partial t}=E(w),$$

since the quantity in front of $\partial x/\partial t$ vanishes when $x=\tau=0$ (since dF acts trivially and $\mathrm{d}\Phi_t(0)$ is supposed to be unitary). Moreover, observe that

$$(\mathrm{P}J-J_0\mathrm{P})(\mathrm{dF}\Phi_t'(x)-TR)=E(w),$$

this is because $\mathrm{P}J - J_0\mathrm{P}$ vanishes when $x = \tau = 0$. Thus we conclude that

$$\mathrm{P}(\partial_s\Psi + J\partial_t\Psi) = \begin{bmatrix} \partial_s z + i\partial_t z \\ \partial_s x + J_0\partial_t x \end{bmatrix} + J_0\mathrm{P}(\mathrm{dF}\Phi_t'(x) - TR) + E(w).$$

Now we compute:

$$J_0\mathrm{P}(\mathrm{dF}\Phi_t'(x) - TR) = \begin{bmatrix} -\alpha(\Phi_t'(x) - TR) \\ J_0\Pi_t(x)\Phi_t'(x) \end{bmatrix}.$$

As proved in §3.3, $J_0\Pi_t(x)\Phi_t'(x) = S(t)x + E(w)$ for the family of symmetric matrices $S(t)$ appearing in Definition 3.7. Lemma 3.10 implies that $-\alpha(\Phi_t'(x) - TR) = E(w)$ (as it vanishes to second order in x). Thus

$$\mathrm{P}(\partial_s\Psi + J\partial_t\Psi) = \begin{bmatrix} \partial_s + i\partial_t & 0 \\ 0 & \partial_s + J_0\partial_t + S(t) \end{bmatrix}\begin{bmatrix} z \\ x \end{bmatrix} + E(w).$$

The first term is linear. Thus, when we linearize, we obtain

$$D_\Psi = \begin{bmatrix} \partial_s + i\partial_t & 0 \\ 0 & \partial_s + J_0\partial_t + S(t) \end{bmatrix} + D_E.$$

The crucial observation is that D_E is a first order differential operator *whose coefficients converge to* 0 as $s \to \infty$. This is because $E(w)$ is quadratic in w and $\mathrm{d}w$, and we know $w = (z, x)$ and its derivatives converge to 0 (recall that σ does not influence E, so the potentially non-zero limit $\sigma \to \sigma_0$ can be ignored).

The invariant linearized operator associated to $u = \Psi(w)$ is the conjugation of D_Ψ via P and $\mathrm{d}\Psi$:

$$D_u = (\mathrm{d}s - i\mathrm{d}t) \otimes \mathrm{d}\Psi \cdot D_\Psi \cdot \mathrm{P},$$

Let us focus on the $\mathrm{d}\Psi \cdot D_\Psi \cdot \mathrm{P}$ part. The key idea needed to manipulate this formula is to introduce the *asymptotically standard frames* on ξ, as defined at the end of §4.2.1.

Let $e_1, \dots, e_n$ be the standard unitary frame for $\mathbb{R}^{2n}$ (whose real span is $\mathbb{R}^n$), and let e_0 be the standard frame for $\mathbb{C}$ (whose real span is $\mathbb{R}$), so that $e_0, e_1, \dots, e_n$ forms a unitary frame for $(\mathbb{C} \oplus \mathbb{R}^{2n}, \mathbb{R} \oplus \mathbb{R}^n)$.

Then $\mathrm{d}\Psi(w)(e_0) = X_0$, $\mathrm{d}\Psi(w)(ie_0) = JX_0$, and, for $k \geq 1$,

$$\mathrm{d}\Psi(w)(e_k) = \mathrm{dFd}\Phi_t(x)e_k.$$

Now let $X_1, \dots, X_n$ be the asymptotically standard frame for $\Psi^*\xi$ obtained by applying Gram-Schmidt to $\Pi_\xi\mathrm{dFd}\Phi_t(x)e_k$, $k = 1, \dots, n$.

It is clear that $X_k = \Pi_\xi\mathrm{dFd}\Phi_t(0)e_k = \mathrm{d}\Phi_t(0)e_k$ when $x = \tau = 0$, because $\mathrm{d}\Phi_t(0)e_k$ is unitary (so Gram-Schmidt does nothing), and dF and Π_ξ act identically. Thus

$$\mathrm{P}(X_k - \Pi_\xi\mathrm{dFd}\Phi_t(x)e_k) = G(w) \cdot w, \tag{4.4}$$

treating $G(w) \cdot w$ as an error term where $G(w)$ is a smooth tensor valued function. Moreover, observe that

$$\mathrm{P}(\Pi_\xi \mathrm{dFd}\Phi_t(x)e_k) - \mathrm{P}(\mathrm{dFd}\Phi_t(x)e_k) = \mathrm{P}(\Pi_\xi \mathrm{dFd}\Phi_t(x)e_k) - e_k = G(w) \cdot w, \tag{4.5}$$

because the terms on the left are equal when $x = 0$, and where we use that $\mathrm{PdFd}\Phi_t(x)$ acts identically on the $\mathbb{R}^{2n}$ factor.

Let $\eta X = \eta_{\mathbb{C}} X_0 + \sum_{k=1}^n \eta_{\xi,k} X_k$, so $(\eta_{\mathbb{C}}, \eta_\xi)$ is $\mathbb{C} \oplus \mathbb{R}^{2n}$ valued. We have

$$D_u(\eta X) = \mathrm{d}\Psi \cdot D_\Psi(\eta_k \mathrm{P} X_k) = \mathrm{d}\Psi \cdot D_\Psi(\eta_k e_k) + \mathrm{d}\Psi \cdot D_\Psi(\eta \cdot G(w) \cdot w).$$

Let $L(\eta) = D_\Psi(\eta \cdot G(w) \cdot w)$, and observe that L is a first order operator whose coefficients decay to 0 as $|s| \to \infty$. This is because w and its derivatives decay to zero.

Next, observe that:

$$D_\Psi(\eta_k e_k) = \begin{bmatrix} \partial_s + i\partial_t & 0 \\ 0 & \partial_s + J_0\partial_t + S(t) \end{bmatrix} \begin{bmatrix} \eta_{\mathbb{C}} \\ \eta_\xi \end{bmatrix} + D_E(\eta_k e_k).$$

Clearly $\eta \mapsto D_E(\eta_k e_k)$ is another first order operator with the same properties as L, hence we can combine terms and conclude that:

$$D_u(\eta X) = \mathrm{d}\Psi \cdot \begin{bmatrix} \partial_s + i\partial_t & 0 \\ 0 & \partial_s + J_0\partial_t + S(t) \end{bmatrix} \begin{bmatrix} \eta_{\mathbb{C}} \\ \eta_\xi \end{bmatrix} + \mathrm{d}\Psi \cdot L(\eta).$$

We are almost done. The above equation implies that:

$$D_u(\eta X) = (\partial_s \eta_{\mathbb{C}} + i\partial_t \eta_{\mathbb{C}})X_0 + (\partial_s \eta_\xi + J_0 \partial_t \eta_\xi + S(t)\eta_\xi)_k \cdot \mathrm{d}\Psi(e_k) + \mathrm{d}\Psi \cdot L(\eta).$$

The next step is to replace $\mathrm{d}\Psi(e_k)$ by the unitary frame X_k of $u^*\xi$. Recall that we had $\mathrm{d}\Psi(e_k) = \mathrm{dFd}\Phi_t(x)e_k$, and so $X_k - \mathrm{d}\Psi(e_k)$ is of class $\mathrm{d}\Psi \cdot G(w) \cdot w$ (by applying $\mathrm{d}\Psi$ to the estimates in (4.4) and (4.5)). Thus we can update L to conclude that:

$$D_u(\eta X) = (\partial_s \eta_{\mathbb{C}} + i\partial_t \eta_{\mathbb{C}})X_0 + (\partial_s \eta_\xi + J_0 \partial_t \eta_\xi + S(t)\eta_\xi)_k \cdot X_k + \mathrm{d}\Psi \cdot L(\eta).$$

Since $L(\eta)$ is a first order operator $\mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ whose smooth coefficients decay to zero, and $\mathrm{d}\Psi$ is approximately unitary, we conclude that:

$$D_u(\eta X) = (\partial_s \eta_{\mathbb{C}} + i\partial_t \eta_{\mathbb{C}} + L_{\mathbb{C}}(\eta))X_0 + (\partial_s \eta_\xi + J_0 \partial_t \eta_\xi + S(t)\eta_\xi + L_\xi(\eta))_k X_k,$$

for first order operators $L_{\mathbb{C}}$, L_ξ whose smooth coefficients decay to zero as $s \to \infty$.

Now the crucical observation is that $D_u(\eta X)$ and

$$D_u^{\text{approx}}(\eta X) = (\partial_s \eta_{\mathbb{C}} + i\partial_t \eta_{\mathbb{C}})X_0 + (\partial_s \eta_\xi + J_0 \partial_t \eta_\xi + S(t)\eta_\xi)_k X_k,$$

are both Cauchy-Riemann operators (i.e., have the same symbol). See §4.4 for the proof that D_u is a Cauchy-Riemann operator.

Thus their difference, which is simply $L(\eta) = L_{\mathbb{C}}(\eta)X_0 + L_\xi(\eta)_k X_k$, *must be a zeroth order operator*. Thus we conclude the desired result: for this frame, we have

$$D_u(\eta X) = \begin{bmatrix} X_0' & X_1', \dots, X_n' \end{bmatrix} \begin{bmatrix} \partial_s + i\partial_t + \Delta_0 & \Delta_1 \\ \Delta_2 & \partial_s + J_0\partial_t + S(t) + \Delta_3 \end{bmatrix} \begin{bmatrix} \eta_{\mathbb{C}} \\ \eta_\xi \end{bmatrix},$$

where Δ_i are zeroth order terms which converge to 0 as $s \to \infty$.

For the final step, we will show that $\Delta_0 = 0$, which will complete the proof. To do so, consider the coordinate system:

$$\Psi_{s,t}(\sigma, \tau, x) = \mathrm{F}_{\sigma,\tau}(\varphi(x))$$

introduced in Remark 4.3. Suppose that $u = \Psi_{s,t}(\sigma, \tau, x)$, as above. In this simpler coordinate system, it is easy to compute the non-linear operator as:

$$\mathrm{P}(\partial_s \Psi + J\partial_t \Psi) = \begin{bmatrix} \partial_s z + i\partial_t z + \alpha(\mathrm{d}\varphi(x))\partial_t x \\ \partial_s x + J'(x,\tau)\partial_t x \end{bmatrix}$$

for suitable complex structure $J'(x, \tau)$ (here $z = \sigma + i\tau$ again). This coordinate system has the property that P and $\mathrm{d}\Psi$ act identically on the $\mathbb{C}$ summand, and hence we conclude by linearizing the above formula with respect to z that $\Delta_1 = 0$. This completes the proof. □

As a corollary of this result, we conclude the Cauchy-Riemann operator D_ξ takes the form $\partial_s - A^{\mathfrak{s}} + \Delta$, where $A^{\mathfrak{s}}$ is in the homotopy class of asymptotic operators computed using the homotopy class of chart determined by $\mathfrak{s}$, and $\Delta(s, t) \to 0$ as $|s| \to \infty$. Applying the index formula §6.1, we obtain:

$$\mathrm{Index}(D_\xi) = n\mathrm{X}(\Sigma, \Gamma_\pm) + M_{\mathfrak{s}} \cdot [u] + \sum_{\zeta \in \Gamma_+} \mu_{\mathrm{CZ}}(A^{\mathfrak{s}}_\zeta) - \sum_{\zeta \in \Gamma_-} \mu_{\mathrm{CZ}}(A^{\mathfrak{s}}_\zeta). \tag{4.6}$$

Here we use the fact that the signed count of zeros of $\mathfrak{s} \circ u$ is equal to the Maslov number of the pair $(u^*\xi, u^*T\Lambda)$ with the asymptotic trivializations induced by $\mathfrak{s}$. Clearly the signed count of zeros of $\mathfrak{s} \circ u$ is also the signed count of intersections of u with $M_{\mathfrak{s}}$.

4.3. The dimension of the space of parametrized holomorphic curves

Combining (4.6) with (4.2), we obtain the following dimension formula for the space of parametrized holomorphic curves nearby u. Namely, if $d = \mathrm{Index}(D^{\delta,\mathrm{st}})$, then

$$d = (n+1)\mathrm{X}(\Sigma, \Gamma_\pm) + |\partial\Gamma_-| + \left|\Gamma^{\mathrm{int}}\right| + M_{\mathfrak{s}} \cdot [u] + \sum_{\zeta \in \Gamma_+} \mu_{\mathrm{CZ}}(A^{\mathfrak{s}}_\zeta) - \sum_{\zeta \in \Gamma_-} \mu_{\mathrm{CZ}}(A^{\mathfrak{s}}_\zeta).$$

This index is independent of the choice of $\mathfrak{s}$, although the individual terms involving $\mathfrak{s}$ do depend on it. If one wishes to fix *asymptotic markers* at each interior puncture, one should subtract the $\left|\Gamma^{\mathrm{int}}\right|$ term.

We can also rewrite $\mathrm{X}(\Sigma, \Gamma_\pm) = \mathrm{X}(\bar{\Sigma}) - \left|\Gamma^{\mathrm{int}}\right| - |\partial\Gamma_-|$, and one obtains

$$d = (n+1)\mathrm{X}(\bar{\Sigma}) - n\,|\partial\Gamma_-| - n\left|\Gamma^{\mathrm{int}}\right| + M_{\mathfrak{s}} \cdot [u] + \sum_{\zeta\in\Gamma_+} \mu_{\mathrm{CZ}}(A^{\mathfrak{s}}_\zeta) - \sum_{\zeta\in\Gamma_-} \mu_{\mathrm{CZ}}(A^{\mathfrak{s}}_\zeta).$$

This completes the proof of Theorem 1.3.

Remark 4.8. The formula from [**BM04**, Proposition 4] computes a similar dimension of the space of (parametrized) curves without boundary, with unconstrained asymptotic markers. The formula they give is:

$$d = (n+1)\mathrm{X}(\bar{\Sigma}) + 2c_1^{\mathfrak{s}}(u) + \sum_{\Gamma_+}(\mu_{\mathrm{CZ}}(A^{\mathfrak{s}}_\zeta) - n) - \sum_{\Gamma_-}(\mu_{\mathrm{CZ}}(A^{\mathfrak{s}}_\zeta) + n).$$

which agrees with ours, since $2c_1^{\mathfrak{s}}(u) = M_{\mathfrak{s}} \cdot [u]$ when there is no Legendrian.

4.3.1. *Examples of disks in* 1*-jet space.* In 1-jet space, we take the globally non-vanishing $\mathfrak{s}$ described in §3.6.1. Thus the Maslov class is zero. Moreover, every Reeb chord $\Lambda_0 \to \Lambda_1$ corresponds to a critical point of the local function difference $f_1 - f_0$, and there are no Reeb orbits. Assuming the chord is non-degenerate, the Conley-Zehnder index of this Reeb chord is minus the number of positive eigenvalues of the Hessian, i.e., $\mu_{\mathrm{CZ}} = \mu_{\mathrm{Mor}} - n$. This was proved in §3.6.2.1.

Let Σ be a disk with boundary punctures, so $\mathrm{X}(\bar{\Sigma}) = 1$. Appealing to the formula for the relative Euler characteristic from Lemma 6.6 we can simplify the general formula from §4.3 to obtain:

$$\mathrm{Index}(D^{\delta,\mathrm{st}}) = n(1 - |\Gamma_+|) + 1 + \sum_{\zeta\in\Gamma_+} \mu_{\mathrm{Mor}}(\zeta) - \sum_{\zeta\in\Gamma_-} \mu_{\mathrm{Mor}}(\zeta).$$

Here are a few special cases:

(i) When $|\Gamma_+| = |\Gamma_-| = 1$, then the dimension is $1 + \mu_{\mathrm{Mor}}(\zeta_+) - \mu_{\mathrm{Mor}}(\zeta_-)$. This is to be expected, as there is an $\mathbb{R}^2$ action on the space of parametrized strips (one $\mathbb{R}$ action is by translation in the codomain and the other $\mathbb{R}$ action is by reparametrization in the domain). Thus, in order to get rigid counts, we should have

$$d = 1 + \mu_{\mathrm{Mor}}(\zeta_+) - \mu_{\mathrm{Mor}}(\zeta_-) = 2,$$

which is the usual "index difference 1" condition between the positive and negative ends.

(ii) When $|\Gamma_+| = 1$ and $|\Gamma_-| = 2$, then the dimension is

$$d = 1 + \mu_{\mathrm{Mor}}(\zeta_+) - \mu_{\mathrm{Mor}}(\zeta_-^1) - \mu_{\mathrm{Mor}}(\zeta_-^2).$$

In this case there is no reparametrization action (as a thrice punctured disk is stable), but there is still the $\mathbb{R}$ action by translation on the codomain. Thus in order to get rigid counts we should have $d = 1$.

(iii) When $|\Gamma_+| = 2$ and $|\Gamma_-| = 1$, then the dimension is

$$d = 1 - n + \mu_{\mathrm{Mor}}(\zeta_+^1) + \mu_{\mathrm{Mor}}(\zeta_+^2) - \mu_{\mathrm{Mor}}(\zeta_-).$$

Once again, the condition to get rigid counts is $d = 1$ (there is no reparametrization action).

(iv) For $|\Gamma_+| + |\Gamma_-| > 3$, then the space of punctured disks has non-trivial moduli. There are two approaches one can take to deal with moduli; either fix the domain and obtain rigid counts for that fixed domain (and perhaps analyze how these counts vary with the domain), or allow the domain to vary in its moduli space and obtain rigid counts in the parametric sense. Since the moduli space of punctured disks varies in a $|\Gamma_+| + |\Gamma_-| - 3$ dimensional moduli space, the expected dimension of the parametric moduli space is:

$$d = n(1 - |\Gamma_+|) + |\Gamma_-| + |\Gamma_+| - 2 + \sum_{\zeta \in \Gamma_+} \mu_{\mathrm{Mor}}(\zeta) - \sum_{\zeta \in \Gamma_-} \mu_{\mathrm{Mor}}(\zeta).$$

To extract rigid counts we require $d = 1$, again because of the translation action on the domain. For instance, when $\Gamma_+ = 1$ and $\Gamma_- = k$, we have

$$d = k - 1 + \mu_{\mathrm{Mor}}(\zeta^+) - \textstyle\sum \mu_{\mathrm{Mor}}(\zeta_i^-).$$

The rigidity condition is then

$$2 - k = \mu_{\mathrm{Mor}}(\zeta^+) - \sum \mu_{\mathrm{Mor}}(\zeta_i^-).$$

Remark 4.9. The rigid counts when $\mu_{\mathrm{Mor}}(\zeta^+) - \sum \mu_{\mathrm{Mor}}(\zeta_i^-) = 2 - k$, when counting disks with 1 positive puncture, is an indication that we can put a differential of the form $d = d_1 + d_2 + \ldots$ on a free tensor algebra generated by critical points of Morse functions, with grading shifted *down* by 1 (i.e., $|c| = \mu_{\mathrm{Mor}}(c) - 1$). Some care needs to be taken when picking exactly which Morse functions should appear. See [**EL17**] for results in this vein.

4.4. Invariance of the linearized operator

In this section we prove that the linearized operator associated to a holomorphic curve $u : \Sigma \to (W, J)$ is independent of the choices.

Recall the approach introduced in §4.2: over an open set $U \subset \Sigma$ with holomorphic coordinates $s + it$, we pick a family of open embeddings $\Psi_{s,t} : B(1) \to W$, close enough to u that the equation $u(s,t) = \Psi_{s,t}(x(s,t))$ can be solved for $x : U \to B(1)$. Also recall that $\mathrm{d}\Psi_{s,t}(x)$ denotes the space-derivative, i.e., the derivative of $h \mapsto \Psi_{s,t}(x+h)$, and $\mathrm{P}_{s,t}(x)$ is the inverse to $\mathrm{d}\Psi_{s,t}(x)$.

We define the non-linear operator associated to Ψ and u via the formula:

$$\partial_\Psi(\eta) = \mathrm{P}_{s,t}(x+\eta)[\frac{\partial}{\partial s}\Psi_{s,t}(x+\eta) + J(\Psi_{s,t}(x+\eta))\frac{\partial}{\partial t}\Psi_{s,t}(x+\eta)]. \tag{4.7}$$

This is an operator mapping sections of $\mathbb{R}^{2n}$ to sections of $\mathbb{R}^{2n}$, and, crucially, $\partial_\Psi(0) = 0$, hence ∂_Ψ be linearized by the formula $D_\Psi(\eta) = \lim_{\epsilon\to 0}\epsilon^{-1}\partial_\Psi(\epsilon\eta)$.

Recall that we define the linearized operator associated to u by the formula:

$$D_u(w) = (\mathrm{d}s - i\mathrm{d}t) \otimes [\mathrm{d}\Psi_{s,t}(x) \cdot D_\Psi(\mathrm{P}_{s,t}(x)w)]. \tag{4.8}$$

Our goal in this appendix is to explain why (4.8) is independent of the choice of Ψ and coordinate $s + it$.

First let us remove the dependence on Ψ. Let $\Psi^0_{s,t}$ and $\Psi^1_{s,t}$ be two coordinate systems. Let $u = \Psi^1(x^1) = \Psi^0(x^0)$. For η^1 sufficiently small, there is a unique $\eta^0 = F_{s,t}(\eta^1)$ so that:

$$\Psi^1_{s,t}(x^1(s,t) + \eta^1(s,t)) = \Psi^0_{s,t}(x^0(s,t) + \eta^0(s,t)). \tag{4.9}$$

Indeed, we have:

$$F_{s,t}(\eta^1) = (\Psi^0_{s,t})^{-1}(\Psi^1_{s,t}(x^1(s,t) + \eta^1)) - x^0(s,t). \tag{4.10}$$

It is apparent from (4.7) and (4.9) that

$$\mathrm{d}\Psi^0_{s,t}(x^0 + \eta^0)\partial_{\Psi^0}(\eta^0) = \mathrm{d}\Psi^1_{s,t}(x^1 + \eta^1)\partial_{\Psi^1}(\eta^1).$$

In particular, we have

$$\mathrm{P}^1_{s,t}(x^1 + \eta^1)\mathrm{d}\Psi^0_{s,t}(x^0 + F_{s,t}(\eta^1))\partial_{\Psi^0}(F_{s,t}(\eta^1)) = \partial_{\Psi^1}(\eta^1).$$

The strategy now is to linearize both sides at $\eta^1 = 0$. The linearization is relatively easy to compute because $\partial_{\Psi^0}(0) = \partial_{\Psi^1}(0) = 0$. Because of this, taking the linearization yields:

$$\mathrm{P}^1_{s,t}(x^1)\mathrm{d}\Psi^0_{s,t}(x^0)\lim_{\epsilon\to 0}\frac{\partial_{\Psi^0}(F_{s,t}(\epsilon\eta^1))}{\epsilon} = \lim_{\epsilon\to 0}\frac{\partial_{\Psi^1}(\epsilon\eta^1)}{\epsilon} = D_{\Psi^1}(\eta^1).$$

By the chain-rule we have

$$\lim_{\epsilon\to 0}\frac{\partial_{\Psi^0}(F_{s,t}(\epsilon\eta^1))}{\epsilon} = D_{\Psi^0}(\lim_{\epsilon\to 0}\frac{F_{s,t}(\epsilon\eta^1)}{\epsilon}) = D_{\Psi^0}(\mathrm{d}F_{s,t}(0)\eta^1).$$

It is clear from the formula (4.10) that the space derivative $\mathrm{d}F_{s,t}$ can be computed as

$$\mathrm{d}F_{s,t}(0) = \mathrm{P}^0_{s,t}(x^0)\mathrm{d}\Psi^1_{s,t}(x^1).$$

We obtain:

$$\mathrm{P}^1_{s,t}(x^1)\mathrm{d}\Psi^0_{s,t}(x^0)D_{\Psi^0}(\mathrm{P}^0_{s,t}(x^0)\mathrm{d}\Psi^1_{s,t}(x^1)\eta^1) = D_{\Psi^1}(\eta^1).$$

Given a section w of u^*TW, set $\eta^1 = \mathrm{P}^1_{s,t}(x^1)w$ in the above equation (and rearrange slightly) to conclude:

$$\mathrm{d}\Psi^0_{s,t}(x^0)D_{\Psi^0}(\mathrm{P}^0_{s,t}(x^0)w) = \mathrm{d}\Psi^1_{s,t}(x^1)D_{\Psi^1}(\mathrm{P}^1_{s,t}(x^1)w)$$

Comparing with (4.8), we see that we have proved the desired invariance of D_w on the choice of Ψ.

Next we remove the dependence on the holomorphic coordinate $s + it$. We may as well assume that we use Ψ which is independent of the point on U.

Observe that for any smooth map $\varphi : U \to W$ we have

$$(\mathrm{d}s - i\mathrm{d}t) \otimes (\frac{\partial \varphi}{\partial s} + J\frac{\partial \varphi}{\partial t}) = \mathrm{d}\varphi + J \cdot \mathrm{d}\varphi \cdot j.$$

Appling this with $\varphi(s,t) = \Psi(x(s,t) + \eta(s,t))$, and comparing with (4.8), we have

$$(\mathrm{d}s - i\mathrm{d}t) \otimes \partial_\Psi(\eta) = \mathrm{P}(x+\eta)[\mathrm{d}\varphi + J \cdot \mathrm{d}\varphi \cdot j].$$

The right hand side is independent of the coordinates used! Linearizing this at $\eta = 0$ proves that the combination $(\mathrm{d}s - i\mathrm{d}t) \otimes D_\Psi(\eta)$ is also independent of $s + it$. Setting $\eta = \mathrm{P}(x)w$ and composing with $\mathrm{d}\Psi(x)$ yields the desired result.

To complete the section, we record the fact that D_u is actually a Cauchy-Riemann operator on $(u^*TW, J(u))$.

Proposition 4.10. Let $f : \Sigma \to \mathbb{R}$ be a smooth function. Then

$$D_u(fw) = (\mathrm{d}f + i \cdot \mathrm{d}f \cdot j) \otimes w + fD_u(w),$$

i.e., D_u is a Cauchy-Riemann operator $(u^*TW, J(u)) \to \Lambda^{0,1} \otimes u^*TW$.

Proof. Left as exercise for the reader. □

4.4.1. *The linearized operator for non-holomorphic maps.* Suppose that $u : \Sigma \to W$ is a non-holomorphic map. It is easy to see that the linearized operator is depends of the choice of Ψ and coordinate $s + it$ in a non-trivial way. One can define non-canonical ways to extract a linearized operator (e.g., by using Riemannian exponential maps). In this section we explain how the quantity:

$$d(u) = (n+1)\mathrm{X}(\Sigma) - n\,|\partial\Gamma_-| - n\left|\Gamma^{\mathrm{int}}\right| + M_{\mathfrak{s}} \cdot [u] + \sum_{\zeta \in \Gamma_+} \mu_{\mathrm{CZ}}(A^{\mathfrak{s}}_\zeta) - \sum_{\zeta \in \Gamma_-} \mu_{\mathrm{CZ}}(A^{\mathfrak{s}}_\zeta),$$

is independent of $\mathfrak{s}$, assuming that u is holomorphic in a neighborhood of its punctures. We also assume that u has finite Hofer energy. These assumptions imply that u has well-defined asymptotics chords and orbits (which appear in $d(u)$).

By linearizing the holomorphic curve equation near the punctures, we obtain a Cauchy-Riemann operator D defined only on a neighborhood of the punctures. Perform the same modifications described above to obtain the operator $D^{\delta,\mathrm{st}}$ with non-degenerate asymptotics. Extend $D^{\delta,\mathrm{st}}$ to all of Σ arbitrarily. This is always possible as the space of Cauchy-Riemann operators is affine. Then the Fredholm index of $D^{\delta,\mathrm{st}}$ is $d(u)$. The construction of $D^{\delta,\mathrm{st}}$ is indepedent of $\mathfrak{s}$, and this proves the desired invariance.

Chapter 5

Legendrian Knots in $\mathbb{R}^3$

Using the framework established in the previous sections, we define the Maslov class and Conley-Zehnder indices for a Legendrian knot in $\mathbb{R}^3$. For connected Legendrians with rotation number zero, we define canonical integer gradings for Reeb chords agrees with the one [**Etn04**, §4.1]. We give a simple algorithm for computing the gradings in terms of *chord crossing rules* for when the Maslov class crosses a Reeb chord.

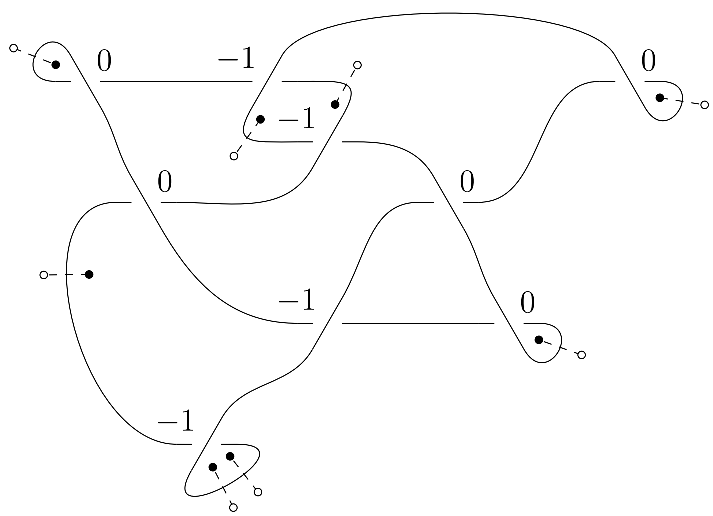


FIGURE 1. One of the Chekanov-Eliashberg knots. The Maslov class is a collection of linking circles. The Conley-Zehnder indices are computed using the $\mathfrak{s}$ which arises from surgery at the vertical tangencies (as explained in this chapter).

5.1. Review of Legendrian knots

Let L be a Legendrian knot in $\mathbb{R}^3$ with the standard contact structure $\mathrm{d}z - y\mathrm{d}x$. Via a Legendrian isotopy, let us suppose that $y > 0$, and $z|_L$ is a Morse function. Since $y > 0$, the critical points of z are the *vertical tangencies* Lagrangian projection $(x, y, z) \to (x, y)$ of the knot.

Recall that we define the *Maslov class* of the Legendrian knot to be the zero locus of any section $\mathfrak{s}$ of $\det_{\mathbb{C}}(\xi)^{\otimes 2}$ which points in the direction of $TL \otimes TL$ along L. More precisely, we should require that $\mathfrak{s}$ never lies in the negative ray $-TL^{\otimes 2} \simeq (-\infty, 0]$, using the canonical orientation of $TL^{\otimes 2}$.

In this non-compact case, we should impose an asymptotic condition on $\mathfrak{s}$: we require that $\mathfrak{s} = \partial_x \otimes \partial_x$ at infinity, using the identification of ξ with $\mathbb{R}^2$ via the x, y projection. It is straightforward to see that the relative winding number between ∂_x and TL is equal to zero if and only if we can choose $\mathfrak{s}$ to be globally non-zero. Such Legendrians are said to have zero *rotation number*.

As we will explain momentarily, for certain choice of $\mathfrak{s}$ (obtained from $\partial_x \otimes \partial_x$ by a linear surgery near the vertical tangencies), the Maslov class will be represented by a collection of linking circles around the knot, one located near each vertical tangency. We also recall $\mathfrak{s}$ determines Conley-Zehnder indices for each Reeb chord, assuming that $\mathfrak{s} \neq 0$ on the Reeb chords.

5.1.1. *Morse type crossings.* A crossing of the Lagrangian projection is of *Morse type* if the upper and lower strands are 1-jets and the local function difference between the upper and lower strand is $f(x) = a + \lambda x^2 + x^3 g(x)$ where $\lambda \neq 0$, and $a > 0$ is the action. Here we use a translated coordinate system where the crossing occurs at $x = 0$.

5.1.1.1. *Quadratic local models.* By deforming the local generating functions on a compact subset of the endpoints of the Reeb chord, we may assume that the lower function has a constant (positive) derivative y_0, i.e., equals $c_0 + y_0 x$, and the upper function equals $c_0 + a + y_0 x + \lambda x^2$, on small enough open sets.

To do this, one observes that the lower function needs to be strictly increasing (since the Legendrian is in the region $y > 0$), and hence we can deform it relative its endpoints to make it have constant positive derivative. We simultaneously perform this compactly supported deformation to the upper function. Thus, the x coordinate of the Reeb chord between the lower and upper strands is fixed during the deformation. By picking the open set on which we do the deformation sufficiently small, we can ensure the y coordinate varies an arbitrarily small amount. Thus the deformation introduces no new Reeb chords.

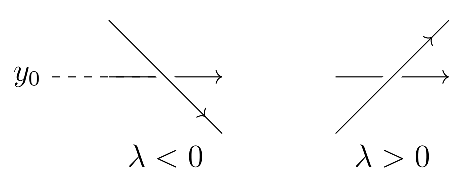


FIGURE 2. Local models for a crossing in the x, y plane. The number λ represents the second derivative of the local function difference, and is also the slope of the upper strand. We draw the orientation induced by the identification with 1-jets. We do not require this orientation extends to all of L (indeed, none of our constructions require an orientation of the knot).

5.1.2. *Conley-Zehnder indices.* Let c denote a Morse type crossing. Recall from 3.2.1 that a section $\mathfrak{s}$ which is non-vanishing and satisfies $\mathfrak{s} \notin -TL^{\otimes 2}$ at both endpoints determines a Conley-Zehnder index. If $\mathfrak{s}$ is homotopic to $\partial_x \otimes \partial_x$ along c relative the endpoints,[1] we have the following formula:

$$\mu_{\mathrm{CZ}}(c, \mathfrak{s}) = \begin{cases} -1 \text{ if } \lambda > 0 \\ \ \ 0 \text{ if } \lambda < 0 \end{cases}$$

This follows immediately from §3.6.2.1. The formula tells us how to compute the Conley-Zehnder indices *when we use* $\mathfrak{s} = \mathfrak{s}_0 = \partial_x \otimes \partial_x$. However, in general, such $\mathfrak{s}$ will not be compatible with the global structure of L.

If $\mathfrak{s}$ is compatible with L and is non-vanishing on the Reeb chords, then $\mathfrak{s}$ induces a relative Maslov class $M_s = \mathfrak{s}^{-1}(0)$ which is disjoint from L and all the Reeb chords. If we perturb $\mathfrak{s}$ in such a way that $M_{\mathfrak{s}}$ crosses a Reeb chord, then the Conley-Zehnder index of that Reeb chord, as determined by $\mathfrak{s}$, will change.

5.2. Canonical Conley-Zehnder indices for Maslov zero knots

Suppose that there exists $\mathfrak{s}$, compatible with L, which is globally nonzero. As explained above, the existence of such an $\mathfrak{s}$ is equivalent to the rotation number of L being zero.

We can assign canonical Conley-Zehnder indices to each *contractible* Reeb chord, via the following non-constructive geometric argument. First recall that a Reeb chord is contractible if it is as a smooth path with boundary on L. Clearly every Reeb chord on a connected Legendrian knot (in $\mathbb{R}^3$) is contractible.

To define the canonical Conley-Zehnder index, pick $\mathfrak{s}_0, \mathfrak{s}_1$ which are globally non-zero and compatible with L, and agree with $\partial_x^{\otimes 2}$ at infinity. We will show that $\mathfrak{s}_0, \mathfrak{s}_1$ assign the same Conley-Zehnder indices to each contractible Reeb chord (and hence we obtain canonical indices).

In $[0,1] \times \mathbb{R}^3$, the relative Euler class of $\mathfrak{s}_0, \mathfrak{s}_1$ is a compact surface Σ without boundary. Here the relative Euler class Σ is the zero set of a generic extension $\mathfrak{s}$ interpolating between $\mathfrak{s}_0$ and $\mathfrak{s}_1$, which remains compatible along L.

Let c be a Reeb chord. By a generic perturbation, we may suppose that

$$d = \Sigma \cap ([0,1] \times c)$$

is a finite collection of transverse points all contained in the interior. Let c_t be a contraction of c, i.e., $c_0 = c$ and c_1 is a constant on L. Then $[0,1] \times c_t$ can be thought of as a smooth family of maps. Since $[0,1] \times c_1$ has zero homological intersection number with Σ, we conclude that $d = 0$ *homologically*, i.e., the signs of the points in d add up to 0.

[1]Here relative means $\mathfrak{s} \notin -TL^{\otimes 2}$ holds at the endpoints during the homotopy.

By standard cancellation results, we can perturb the section $\mathfrak{s}$ near $[0,1]\times c$, relative its boundary, so as to make it non-vanishing. In this fashion, we may suppose that $\mathfrak{s}$ is non-vanishing along each Reeb chord during the homotopy from $\mathfrak{s}_0$ to $\mathfrak{s}_1$. In particular, by continuity of the Conley-Zehnder index, we conclude that $\mathfrak{s}_0$ and $\mathfrak{s}_1$ assign the same Conley-Zehnder indices.

5.3. Maslov class and vertical tangencies

Suppose that L is a Legendrian knot with Morse type crossings.

Let $\xi \simeq \mathrm{pr}^*\mathbb{R}^2$, equipped with the unitary structure so that dpr is a unitary isomorphism. Here pr is the Lagrangian projection. Let $\mathfrak{s}_0 = \partial_x \otimes \partial_x$ be considered as a section of $\xi^{\otimes 2}$.

Recall that $\mathfrak{s}_0$ is compatible with a Legendrian knot if $\mathfrak{s}_0 \notin -TL^{\otimes 2}$ holds everywhere. Standard properties of Hermitian metrics implies that this is equivalent to $g(\partial_x, TL) \neq 0$ holding everywhere. Thus, we see that the failure of compatibility with $\mathfrak{s}_0$ is located precisely at the vertical tangencies of L. We will now explain how to do a local surgery to $\mathfrak{s}_0$ near the vertical tangencies, returning a new section $\mathfrak{s}$ which is compatible with L.

We will analyze this problem by picking local unitary frames for ξ near a vertical tangency so that TL is identified with $\mathbb{R}$. In this local chart, the vertical tangencies satisfy $g(v,1)=0$, where v is the representation of ∂_x in the local unitary frame. See §5.3.2 for more details.

5.3.1. *A linear surgery argument.* In general, suppose that $v:\mathbb{R}^3\to\mathbb{C}^\times$ satisfies

$$f(x,y,z) := g(v(x,y,z),1),$$

has a transverse zero set which intersects $(0,0,0)$ and is transverse to ∂_y. In other words, the map v crosses the $i\mathbb{R}$ axis at $(0,0,0)$. Compact perturbations of v through the space of $\mathbb{C}^\times$ valued functions will never be able to kill the zero locus of f, and $f(0,y,0)$ will always have a zero for some value of y. Indeed, this follows from the intermediate value theorem, since f changes sign.

However, when one passes to the tensor product, i.e., considering $\mathfrak{s} = v\otimes v$ rather than v, then we *can* perturb $\mathfrak{s}$ on arbitrarily small compact subsets of $(0,0,0)$ so that $\mathfrak{s}\notin(-\infty,0]$. This will come at the expense of a non-empty zero set $M=\mathfrak{s}^{-1}(0)$ which links $\{0\}\times\mathbb{R}\times\{0\}$.

First we set-up a local model. Using that $i\mathbb{R}$ is the orthogonal complement to 1, we may suppose that

$$v(x,y,z) = e^{\pm iy} i$$

for $(x,y,z)\in[-\delta,\delta]^3$. This is possible by a simple deformation, and replacing v by $-v$ if necessary. We will now define a replacement $\tilde{v}$.

Let us focus on the + case. The definition of the replacement $\tilde{v}(0,y,0)$ is summarized in Figure 3. There $ay+b$ is the unique linear function on $[-\delta,\delta]$ which equals $-\delta$ when $y=-\delta$ and equals $-\pi+\delta$ when $y=\delta$. (note that $ie^{i\delta-i\pi}=-ie^{i\delta}$ is the *negative* of the endpoint of v). It is clear that $ay+b$ is strictly bounded in $(-\pi/2,0)$ on its domain, and hence $ie^{i(ay+b)}$ is never orthogonal to 1.

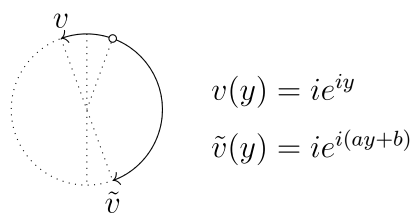


FIGURE 3. v versus $\tilde{v}$. Note that $\tilde{v}$ is never orthogonal to 1, and crosses i positively.

By construction, $\mathfrak{s}_0 = v\otimes v$ and $\mathfrak{s}_1 = \tilde{v}\otimes\tilde{v}$ have the same restrictions to the planes $y\in\{-\delta,\delta\}$, and $\mathfrak{s}_1$ is never in $(-\infty,0]$. This means that $\mathfrak{s}_1$ is compatible along the y-axis.

Introduce the surgered section:

$$\mathfrak{s} = (1-\beta(x^2+z^2))\mathfrak{s}_0 + \beta(x^2+z^2)\mathfrak{s}_1,$$

where β equals 1 on $[0,r_0]$, and equals 0 on $[r_1,\infty)$, with $0<r_0<r_1<\delta$. We suppose that β' is strictly negative on (r_0,r_1). Then $\mathfrak{s}$ agrees with $\mathfrak{s}_0$ on the boundary of the box $[-\delta,\delta]^3$. Since $\{0\}\times\mathbb{R}\times\{0\}$ is contained in the region where $\beta=1$, we know that $\mathfrak{s}$ is never in $(-\infty,0]$ along the y-axis, as desired. However, $\mathfrak{s}$ will have a zero set contained in the region $x^2+z^2\in(r_0,r_1)$.

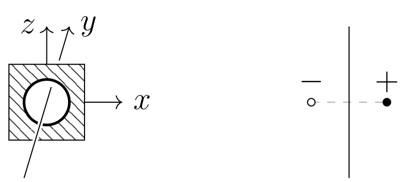


FIGURE 4. The zero set of $\mathfrak{s}$ is a cooriented linking circle. The left is the projection to the x,y plane. The signs are determined by the natural coorientation of the zero set, discussed below.

To see why the zero set is a circle, we simply compute:

$$-\mathfrak{s} = (1-\beta)e^{2iy} + \beta e^{2i(ay+b)} \implies -e^{-2iy}\mathfrak{s} = (1-\beta) + \beta e^{2i(cy+b)}.$$

Here $cy+b$ is the unique linear function which equals 0 when $y=-\delta$ and $-\pi$ when $y=+\delta$. In particular, after a linear reparametrization of the y coordinate to take time $[0,1]$, we have that:

$$-e^{-2iy}\mathfrak{s} = (1-\beta) + \beta e^{-2\pi i t}.$$

This has a zero if and only if $t = 0.5$ and $\beta = 0.5$. By our assumption on β, the set $\beta = 0.5$ is a circle $x^2 + z^2 = r$ in the plane centered at $y = 0$, for some $r \in (r_0, r_1)$.

5.3.1.1. *Coorientation of the zero set.* As $\mathfrak{s}$ is a section of a complex line bundle, its zero set inherits a natural coorientation. This coorientation can be computed by linearizing $\mathfrak{s}' = -e^{-2iy}\mathfrak{s}$ when $t = 0.5$ and $\beta = 0.5$, whereby we obtain

$$\mathrm{d}\mathfrak{s}' = -2\beta'\mathrm{d}r + 2\pi i\beta\mathrm{d}t = a\mathrm{d}r + ib\mathrm{d}y \text{ where } a, b > 0.$$

This implies that (∂_r, ∂_y) forms an oriented basis for the normal plane to the zero locus. Here ∂_r points radially away from the y-axis.

In the $-$, when v crosses $i\mathbb{R}$ negatively, we have $\mathfrak{s}' = (1-\beta) + \beta e^{2\pi i t}$, and the linearization becomes $\mathrm{d}\mathfrak{s}' = a\mathrm{d}r - ib\mathrm{d}y$ where $a, b > 0$. In particular, $(-\partial_r, \partial_y)$ forms an oriented basis.

5.3.2. *The Maslov class via surgery near the vertical tangencies.* In the standard coordinate system, there are two possibilities for a vertical tangency, either $x'' > 0$ or $x'' < 0$.

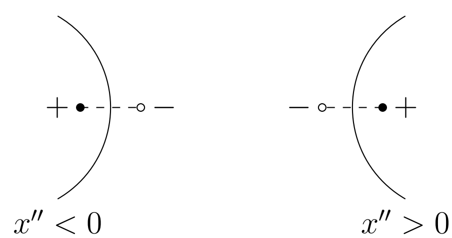


FIGURE 5. Two kinds of vertical tangencies. We draw the link as a dashed line. The signs determine the natural coorientation of the link.

Let p be the location of the vertical tangency. We apply the argument in the previous section. By picking a travelling oriented orthonormal frame for ξ nearby p, we may suppose that 1 spans TL and agrees with ∂_y at p. The vector field v which is ∂_x in the old frame is now non-constant in this frame.

However, we know that $g(v, 1)$ either decreases from positive to negative ($x'' < 0$) or increases from negative to positive ($x'' > 0$). In both cases, v crosses $-i$ (since $J\partial_x = \partial_y$ holds in the standard coordinate system, so $\partial_x = -J\partial_y$).

The surgery argument implies that we can deform $\mathfrak{s} = v \otimes v$ in an arbitrarily small neighborhood of p so that it satisfies $\mathfrak{s} \notin (-\infty, 0]$ on L (near p). This comes at the expense of adding a single link to the Maslov class $M_{\mathfrak{s}}$. The resulting section $\mathfrak{s}$ is now compatible at p.

In the $x'' < 0$ case, $(-\partial_r, \partial_y)$ forms an oriented basis for the normal bundle to the linking circle. On the other hand, in the $x'' > 0$ case, (∂_r, ∂_y) forms an oriented basis. This explains the signs appearing in Figure 5.

The choice of $\mathfrak{s}$ on the left of Figure 6 is equal to $\partial_x \otimes \partial_x$ outside of a small neighborhood of the vertical tangencies, and hence it is easy to compute the Conley-Zehnder indices (they are either -1 or 0). On the left we have replaced $\mathfrak{s}$ by a globally non-vanishing one by cancelling the links in the Maslov class; during this process, the Conley-Zehnder index will change to 1. This is the canonical Conley-Zehnder index. To deduce the canonical Conley-Zehnder index, we use the chord crossing moves from §5.4, and the following cancellation result.

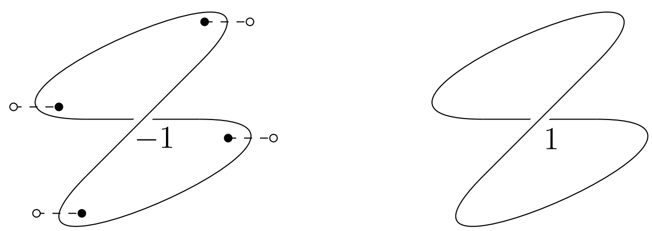


FIGURE 6. The Maslov class $M_{\mathfrak{s}} = \mathfrak{s}^{-1}(0)$ is a collection of linking circles with coorientations. As above, the black dots signify positive signs.

5.3.3. *Cancellation of oppositely oriented links.* In this section, we describe another linear surgery argument, which allows us to "cancel" nearby components of the Maslov class, provided their orientations are opposite. Let us suppose that L is locally aligned with the x-axis in a neighborhood of two oppositely oriented Maslov links.

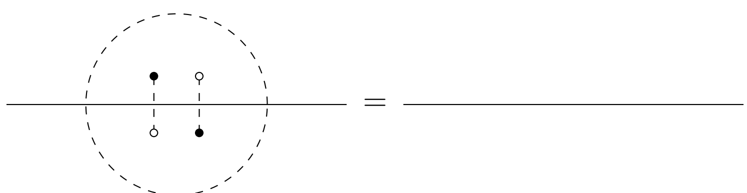


FIGURE 7. Cancellation of adjacent Maslov class links.

Via a smooth isotopy remaining disjoint from L and the Reeb chords, we can move the nearby links into standard position, which we suppose is:

$$M_- = \left\{x = -\delta, y^2 + z^2 = \delta^2\right\} \text{ and } M_+ = \left\{x = \delta, y^2 + z^2 = \delta^2\right\}.$$

This isotopy can be achieved through Maslov classes, i.e., zero sets of $\mathfrak{s}$.

Consider the sphere $S = \{x^2 + y^2 + z^2 = 4\delta^2\}$. Clearly this sphere contains M_-, M_+, and $S \cap L$ consists of the points $(\pm 2\delta, 0, 0)$. Without loss of generality, let us rescale the figure so $\delta = 1$. Let $\mathfrak{t} = \mathfrak{s}|_S$.

We use the identification of $\det(\xi)$ with $\mathbb{C}$ sending $\partial_x \otimes \partial_x$ onto 1. We can require that $\mathfrak{s} = 1$ along the x-axis, which can be achieved by an obvious deformation (recalling that $\mathfrak{s}$ is not allowed to point in the -1 direction).

Let H be the upper half-plane $\{(x, y, 0) : y > 0\}$, and note that the union of the links $M_\pm$ have zero homological intersection with H. Consider a generic ray $R = e^{i\theta}[0, \infty)$, $\theta \notin 2\pi\mathbb{Z}$. Then $\mathfrak{s}^{-1}(R)$ is a cobordism from $\mathfrak{s}^{-1}(0)$ (the two links) and $\mathfrak{t}^{-1}(R)$. It follows that $\mathfrak{t}^{-1}(R)$ has zero homological intersection with the arc $H \cap S$, which joins $(-2, 0, 0)$ to $(2, 0, 0)$. A standard cancellation of zeros argument implies that $\mathfrak{t}$ can be homotoped through non-vanishing sections so that the image of $\mathfrak{t}$ is disjoint from R. Clearly $\mathbb{C} \setminus R$ deformation retracts on $\{1\}$, and hence $\mathfrak{t}$ can be homotoped to the constant 1 through non-vanishing sections *which stay* 1 *on* $(\pm 2, 0, 0)$.

Let S_1, S_2, S_3 be three concentric spheres with $S = S_3$. Let B_i be the corresponding balls, and suppose that $M_\pm \in B_1$. Between S_1 and S_2, do a homotopy from $\mathfrak{s}|_{S_1}$ to the constant 1 (this exists by the same argument we gave for $\mathfrak{t}$). Then, between S_2 and $S = S_3$, do a homotopy from 1 to $\mathfrak{t} = \mathfrak{s}|_{S_3}$. This agrees with $\mathfrak{s}$ on the boundary of the shell region, and hence we can simply replace $\mathfrak{s}$ on the shell-region by the new section without affecting the zero set. Thus we may suppose that $\mathfrak{s}$ equals 1 on S_2. Now 1 and $\mathfrak{s}$ agree on the boundary $\partial B_2 = S_2$. We simply replace $\mathfrak{s}|_{B_2}$ with 1, which cancels the zero set, as desired.

This may seem like a discontinuous process, but it is not. Indeed, each time we do a "replacement" we should imagine performing a straight-line homotopy. During this process, a zero set may form (and then cancel), but, crucially, these changes in the zero set are constrained to the region we do the replacement. Moreover, by construction, these straight-line homotopies will remain valid Maslov classes for L.

5.4. Chord crossing moves for the Maslov class

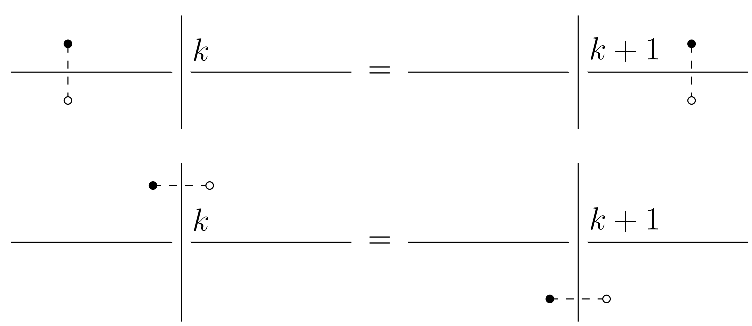


FIGURE 8. Chord crossing moves for the linking Maslov class, and the resulting change in the Conley-Zehnder index.

In this section we determine how the Conley-Zehnder index of a chord c, as determined by $\mathfrak{s}$, changes during homotopies where $\mathfrak{s}$ is allowed to vanish on c. We will encode the results as *crossing moves* in the Lagrangian projection, as shown in Figure 8.

There are two types of crossings we need to consider; depending on whether the link travels along the *upper* or *lower* strands. We also recall that the Conley-Zehnder index *only depends on the germs of the Legendrians* at the endpoints, and is unchanged as long the projections to $\mathbb{R}^2$ remain transverse and $\mathfrak{s}$ remains non-zero along the chord. This gives us a lot of flexibility for proving the validity of the crossing moves.

5.4.1. *Signs of punctures.* Every half-infinite holomorphic strip asymptotic to a Reeb chord has a well-defined sign. This is summarized in the following figure.

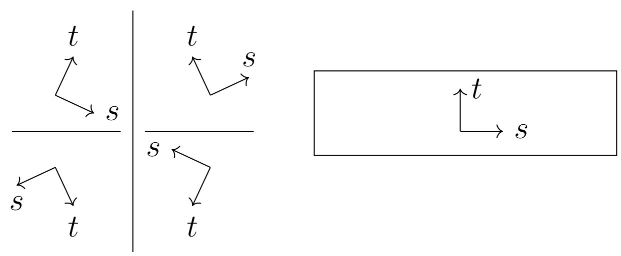


FIGURE 9. The t-direction always points from the lower strand to the upper strand. If the s direction points towards the puncture, the puncture is *positive*. If the s direction points away from the puncture, the puncture is negative.

5.4.2. *The dimension formula for holomorphic curves.* Let $\mu_{\mathrm{CZ}}(c, \mathfrak{s})$ be the Conley-Zehnder index assigned to c by a compatible $\mathfrak{s}$. If u is a holomorphic curve in the symplectization (of Y^3), then the virtual dimension of the space of nearby parametrized maps is given by:

$$d(u) = 2\mathrm{X}(\bar{\Sigma}) - |\Gamma_-| - |\Gamma_{\mathrm{int}}| + M_{\mathfrak{s}} \cdot [u] + \textstyle\sum_{\Gamma_+} \mu_{\mathrm{CZ}}(c, \mathfrak{s}) - \sum_{\Gamma_-} \mu_{\mathrm{CZ}}(c, \mathfrak{s}),$$

using Theorem 1.3 with $n = 1$. Indeed, for any smooth u on a boundary punctured surface which is holomorphic near the punctures (and has finite energy), the quantity $d(u)$ can be defined as the Fredholm index of a certain family of linearized operators associated to u, i.e., we do not require u to be holomorphic everywhere in order to conclude that $d(u)$ is independent of $\mathfrak{s}$. See §4.4.1.

5.4.3. *Lifting holomorphic curves to the symplectization.* Let (σ, z, x, y) be the standard coordinates on the symplectization of the 1-jet space of $\mathbb{R}$. In these coordinates, a complex structure J is *admissible* if and only if it is σ-invariant and satisfies the equation:

$$\mathrm{d}\sigma \circ J = -\mathrm{d}z + y\mathrm{d}x.$$

This implies that $\xi = \ker(\mathrm{d}z - y\mathrm{d}x) \cap \ker \mathrm{d}\sigma$ is a complex subspace.

The space of admissible almost complex structures for $\mathbb{R}^3$ is therefore parametrized by z-dependent complex structures on $\mathbb{R}^2$, using the identification $\xi \simeq \mathrm{pr}^*\mathbb{R}^2$.

Henceforth, let us suppose that we take the standard complex structure on $\mathbb{R}^2$, independent of z. In these coordinates, a holomorphic curve u is holomorphic if and only if $x + iy$ is holomorphic and:

$$\frac{\partial z}{\partial t} - \frac{\partial \sigma}{\partial s} = y\frac{\partial x}{\partial t} \qquad\qquad \frac{\partial \sigma}{\partial t} + \frac{\partial z}{\partial s} = y\frac{\partial x}{\partial s}.$$

For our purposes, it suffices to know how to holomorphically lift ends near the Reeb chords, where we may suppose that lower and upper strands Legendrians are described by $y = y_0$ and $y = y_0 + cx$. It follows that the z-coordinates of the upper and lower strands are described by:

$$z = y_0 x \text{ and } z = T + y_0 x + \frac{1}{2}c^2 x.$$

Let $\tilde{z} = z - (1-t)y_0 x - t(T + y_0 x + \frac{1}{2}c^2 x)$, let $\tilde{\sigma} = \sigma - sT$, and observe that these modified coordinates need to satisfy the equations:

$$\bar{\partial}(\tilde{\sigma} + i\tilde{z}) = E(x, y) \cdot \nabla x.$$

Let us suppose that u is defined on a half infinite strip, and $y = y_0 + cx$, $y = y_0$ holds along the top and bottom boundaries, respectively.

The next step is to appeal to a local solvability result to ensure there exists a solution where $z = 0$ on the boundary (this corresponds to (σ, z, x, y) taking boundary values on L). The usual $\bar{\partial}$ operator is a bit non-optimal on infinite strips, as it as degenerate asymptotics. If we suppose that x, y are in $W^{1,p,\delta}$ for all k and for some rate δ, then we can write

$$\sigma' = e^{\delta|s|}\tilde{\sigma} \text{ and } z' = e^{\delta|s|}\tilde{z},$$

and observe that

$$(\bar{\partial} \pm \delta)(\sigma' + iz') = E(x, y) \cdot (e^{\delta|s|}\nabla x).$$

Since $\bar{\partial} \pm \delta$ is an isomorphism acting on $W^{1,p}(\mathbb{R} \times [0, 1], \mathbb{C}, \mathbb{R})$, as proven in Theorem 6.20. We can solve this equation for $\sigma' + iz'$ on the sub-end $[1, \infty) \times [0, 1]$ (by using a cut-off function on the first part $[0, 1] \times [0, 1]$). It follows easily that $\sigma' + iz'$ is smooth and converges to zero at infinity.

There are obvious candidates for the x, y projections. We can take

$$x + iy = x_0 + iy_0 + ae^{\pm\theta(s+it)},$$

for appropriate a, θ, x_0, y_0. This decays exponentially (in one of the ends).

To summarize, when the Legendrians take the above form, we can find local holomorphic ends converging to the associated Reeb chord. The sign of the puncture is determined by Figure 9.

5.4.3.1. *Proving the chord crossing moves.* We prove the chord crossing moves by reducing to the case of two particular knots, one of which is shown in Figure 10 and the other is its reflection. Each has a single Reeb chord at the origin.

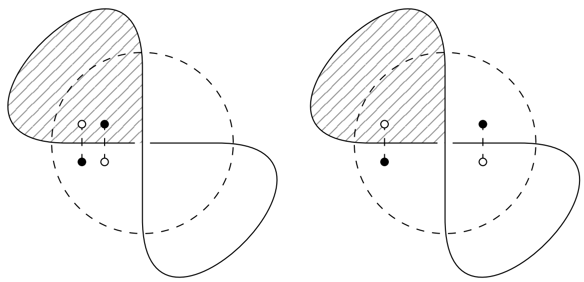


FIGURE 10. Adding a pair of cancelling links to $M_{\mathfrak{s}}$. We only show the part of the Maslov class located in $\{x^2 + y^2 < \delta\}$. The shaded region represents the smooth map u.

As explained previously, there exist smooth maps $u : \bar{\mathbb{H}} \to \mathbb{R}^4$ which are holomorphic near ∞, have boundary on L, and whose projections to $\mathbb{R}^2$ are given by $x + iy = ae^{-\theta(s+it)}$, near ∞, where $\theta = \pi/2$ and $a \in \{i, -i\}$. Both have positive punctures.

Let $\mathfrak{s}$ be a compatible section for this knot, defining a Maslov class $M_{\mathfrak{s}}$. By an isotopy of $M_{\mathfrak{s}}$, we may suppose that $M_{\mathfrak{s}}$ is disjoint from $\{x^2 + y^2 < \delta\}$. By a similar linear surgery argument given above, we can invert the cancellation process, and add a pair of cancelling links to $M_{\mathfrak{s}}$, without changing the Conley-Zehnder index associated to c.

Referring to Figure 10, when we slide the rightmost link over, the virtual dimension $d(u)$ does not change. However, it is easy to see that $M_{\mathfrak{s}} \cdot [u]$ must *decrease* by 1, and hence the Conley-Zehnder index of the chord must *increase* by 1, since u has a *positive* puncture. This proves the first chord crossing rule in Figure 8. The second rule is proved in the same manner. The only difference is that u has a *negative* puncture, which affects the dimension formula. We leave the details to the reader.

5.5. Comparison with other gradings

In [**Etn04**, §4.1], the author defines the *grading* of a Reeb chord c, joining p_- to p_+, to be the degree of the Gauss map restricted to either embedded arc joining p_+ to p_-, post-concatenated with a clockwise rotation from $\mathrm{dpr}(TL_-)$ to $\mathrm{dpr}(TL_+)$. In [**EES02**, §2.3], the authors define the *Conley-Zehnder index* of a Reeb chord c in the same way, except using a *counterclockwise* rotation in the last step. Then the authors define the *grading* to be the Conley-Zehnder index minus 1. The grading, in general, is only well-defined when we consider the "capping path" as part of the data; however, in the case when the rotation number of the knot is zero, the grading is independent of the path.

The goal of this section is to prove that our canonical Conley-Zehnder indices (§5.2) agree with the "gradings" defined above for knots with rotation number zero.

The argument is simple; let us suppose, without loss of generality, that all of the crossings in our knot take the form shown in Figure 13:

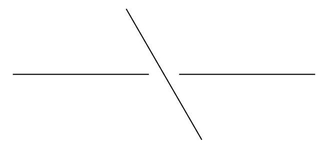


FIGURE 11. Local model for the Reeb chord. The upper strand has negative slope

At the start, we use the $\mathfrak{s}$ arising from surgery at the vertical tangencies, whose Maslov class is a collection of linking circles at the vertical tangencies. As we travel from p_+ to p_-, we "pick up" each Maslov linking circle we pass along the way, thereby obtaining a picture of the form shown in Figure 13.

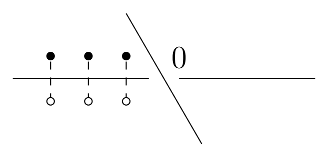


FIGURE 12. While travelling from p_+ to p_- along half of the knot, we gather all the linking circles in the Maslov class.

It is easy to see that the signed count of the linking circles, say d, is equal to the degree of the Gauss map *when we close up by a clockwise rotation*, since no vertical tangengies are introduced when we rotate clockwise from a horizontal line to a line with negative slope. If we instead rotated counter-clockwise to complete the loop, then we cross another vertical tangency and would introduce a +1 to the winding number. In other words, d equals the grading as defined in [**Etn04**] and [**EES02**].

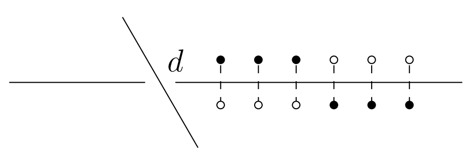


FIGURE 13. Gather the linking circles on the other half of the knot and cancel them with the ones we crossed over.

Now, cross all the links to the other side of the Reeb chord, which (by the chord crossing rules) increases its index from 0 to d. Then, since the knot is presumed to have rotation number 0, we can gather all the linking circles on the *other half* of the knot and cancel them with the ones near the chord, as shown above. At the end of this process, we have killed the Maslov class, and hence the resulting $\mathfrak{s}$ computes the canonical Conley-Zehnder indices. Thus the grading d is equal to the canonical Conley-Zehnder index, as desired. The verification of the signs is left to the reader.

5.6. Examples

5.6.1. *Killing the Maslov class example.* We compute the canonical Conley-Zehnder indices for the knot shown in Figure 14, using the algorithm explained in §5.3.2.

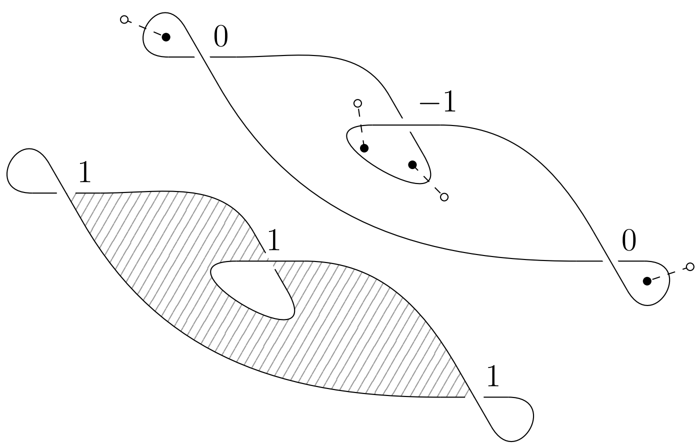


FIGURE 14. Killing the Maslov class for a knot with rotation number zero, and the resulting canonical Conley-Zehnder indices. We apply the dimension formula to the shaded region below.

5.6.1.1. *Applying the dimension formula.* Recall that the dimension formula is given by:

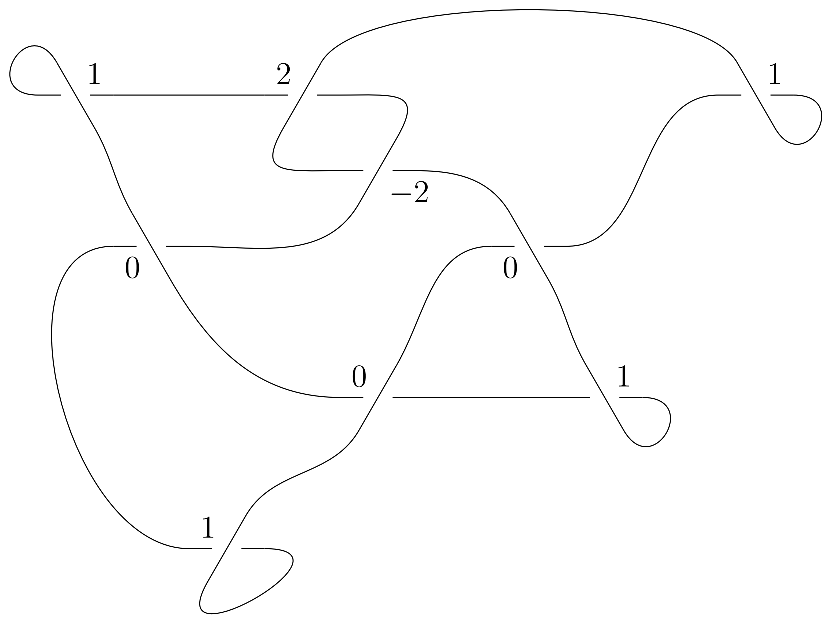
$$d(u) = 2\mathrm{X}(\bar{\Sigma}) - |\Gamma_-| + M_{\mathfrak{s}} \cdot [u] + \sum_{\Gamma_+} \mu_{\mathrm{CZ}}(c, \mathfrak{s}) - \sum_{\Gamma_-} \mu_{\mathrm{CZ}}(c, \mathfrak{s}).$$

When we apply this to the shaded region, using the fact that two of the punctures are positive and two are negative, we obtain $d(u) = 2 - 2 + 0 + 2 - 2 = 0$. This is the expected dimension of the space of parametrized holomorphic maps *with the same underlying domain.* Since the domain is a four-times punctured disk, there is a 1-dimensional space of conformal parameters P. The condition that $d(u) = 0$ implies that the expected dimensions of the *fibers* of the parametric moduli space $\mathcal{M} \to P$ are 0-dimensional.

Assuming that $\mathcal{M}$ is cut transversally in the parametric sense, any regular value of the map $\mathcal{M} \to P$ must have empty fiber, because of the $\mathbb{R}$-action by translation on the domain. After quotienting by this action, we obtain a discrete set of points in $\mathcal{M}/\mathbb{R}$, (and most conformal structures do not contribute to the count).

In this sense, any holomorphic curve which projects to the shaded region contributes to the *parametric* count of holomorphic curves where we allow varying conformal parameters.

5.6.1.2. *One of the Chekanov-Eliashberg knots.* We cancel the Maslov Class for the Chekanov-Eliashberg knot shown in Figure 1, and obtain its canonical Conley-Zehnder indices.



Chapter 6

The index formula

The main goal of this chapter is to prove that *asymptotically non-degenerate Cauchy-Riemann operators* on surfaces with boundary punctures are Fredholm, and give a formula for their Fredholm index in terms of topological data, i.e., provide an *index formula*. Our index formula generalizes the result stated in [**Sch95**, Theorem 3.3.11] (see also [**Ger18**, Theorem 3.1.2] and [**Wen20**, Theorem 5.4]).

To actually prove the index formula we adopt the technique introduced in [**Tau96**, §7] (subsequently generalized by [**Ger18**, Chapter 3]), and deform our Cauchy-Riemann operator D by an anti-linear lower order term σB. As explained in [**Tau96**], [**Ger18**], and [**Wen20**], as $\sigma \to \infty$ the kernel of $D + \sigma B$ concentrates near the positive zeros of B and the cokernel of $D+\sigma B$ is represented by sections supported near the negative zeros of B. With some further analysis, one concludes that the signed count of zeros of B equals the index of $D + \sigma B$.

Our argument is complicated by the boundary $\partial\Sigma$. The most apparent difference is that the anti-linear perturbation B can have zeros on the boundary, and, as we will show, the boundary zeros split into *four* cases, two of which contribute 0 to the index, and the other two contribute $+1$ and -1. See Figure 1 and 3. This phenomenon leads to the "relative Euler characteristic" term in the index formula depending on the signs of punctures – this is a novel phenomenon when compared with the $\partial\Sigma = \emptyset$ case. The details of this part of the argument are presented in §8.

If the asymptotics of D do not match the asymptotics of $D + \sigma B$ for σ large, then the Fredholm index will likely change during the deformation $\sigma \to \infty$. There are two approaches to deal with this: one way is to try to find B so that the asymptotics of D match the asymptotics $D + \sigma B$ for all $\sigma \in [0, \infty)$ – this is the approach taken in [**Ger18**, Chapter 3] and [**Wen20**, §5.8]. The other approach is to pick B without regard to D, and then analyze the change in index as an "index gluing" problem. This is the approach developed in §7.3, and it leads to a natural definition of the Conley-Zehnder indices as Fredholm indices of certain operators. The necessary analytic ingredient to make this work is the *linear kernel gluing operation*. See [**Sch95**, §3.2], [**FH93**], and [**Sei08**, §11c] for similar gluing problems.

Remark 6.1. There is other work which proves index formulas for Cauchy-Riemann operators on surfaces with boundary punctures. See, for instance, [**CEJ10**, Appendix A], and

[**Sei08**, §11e]. Our work differs from theirs in how we present the index formula (e.g., they do not define Conley-Zehnder indices for Reeb chords), and how we prove the result.

6.1. Statement of the index formula

Let D be an asymptotically non-degenerate Cauchy-Riemann operator for the data

$$(\Sigma, \partial\Sigma, \Gamma_\pm, E, F, C, [\tau]).$$

Briefly:

(1) $\Gamma = \Gamma_+ \cup \Gamma_-$ is collection of punctures which may be on the boundary (we denote the punctured surface by $\dot{\Sigma}$),

(2) (E, F) is a complex vector bundle with totally real sub-bundle $F \subset E|_{\partial\dot{\Sigma}}$

(3) for each $z \in \Gamma$, $C_z \subset \dot{\Sigma}$ is a chosen cylindrical/strip-like end with holomorphic coordinate $s + it$ (there are four possibilities for C_z, depending on whether $z \in \Gamma_\pm$ and $z \in \partial\Sigma$),

(4) $[\tau]$ is an equivalence class of trivializations $\tau_z : (E|_{C_z}, F|_{\partial C_z}) \to (\mathbb{C}^n, \mathbb{R}^n)$ called *asymptotic trivializations*. See §6.3.2 for more details.

We recall that Cauchy-Riemann operators are defined by their symbol. The *asymptotically non-degenerate* condition means that for any $\tau \in [\tau]$, the coordinate representation D_τ in the end C_z is asymptotic to $\partial_s - A_z^\tau$ as $s \to \pm\infty$ where $A_z^\tau = -i\partial_t - S(t)$ is a non-degenerate asymptotic operator:

$$C^\infty([0,1], \mathbb{C}^n, \mathbb{R}^n) \to C^\infty([0,1], \mathbb{C}^n) \quad \text{or} \quad C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{C}^n, \mathbb{R}^n) \to C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{C}^n),$$

as explained in §1.

Theorem 6.2. For $p > 1$, $D : W^{1,p}(E, F) \to L^p(\Lambda^{0,1} \otimes E)$ is Fredholm and its index is given by

$$\mathrm{ind}(D) = n\mathrm{X}(\Sigma, \Gamma_\pm) + \mu_{\mathrm{Mas}}^\tau(E, F) + \sum_{z \in \Gamma_+} \mu_{\mathrm{CZ}}(A_z^\tau) - \sum_{z \in \Gamma_-} \mu_{\mathrm{CZ}}(A_z^\tau),$$

where n is the complex rank of E, τ is an asymptotic trivialization of (E, F), and:

(i) The *relative Euler characteristic* $\mathrm{X}(\Sigma, \Gamma_\pm)$ is the count of zeros of a generic vector field on $\dot{\Sigma}$ which is tangent to $\partial\dot{\Sigma}$ and points inwards along Γ_- and outwards along Γ_+ (e.g., equal to ∂_s in the ends C_z). Boundary zeros are counted according to the rules in Figure 1. Interior are zeros are counted as usual. See §3 for examples.

(ii) The Maslov index $\mu_{\mathrm{Mas}}^\tau(E, F)$ is the signed count of zeros of a generic section σ of $(\det E)^{\otimes 2}$ which (a) restricts to the canonical positive generator of $(\det F)^{\otimes 2}$ along the boundary, and (b) is identically 1 in the asymptotic trivializations induced by τ. The zeros are all interior.

(iii) The Conley-Zehnder index is the Fredholm index of any Cauchy-Riemann operator on the trivial bundle $E = \mathbb{C}^n$, $F = \mathbb{R}^n$ over an infinite strip/cylinder which equals

$$\partial_s u + J_0 \partial_t u + \bar{u} = \partial_s u + J_0 \partial_t u + Cu$$

at the negative end and $\partial_s - A_z^\tau$ at the positive end. See Figure 2.

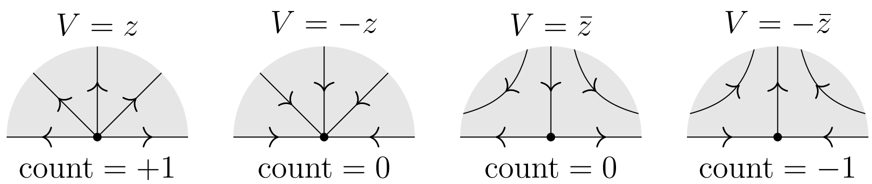


Figure 1. Boundary zeros either contribute ± 1 or 0 to the index.

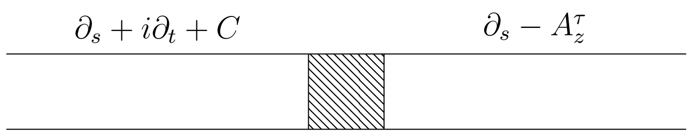


Figure 2. The Conley-Zehnder index is the Fredholm index of any Cauchy-Riemann operator on the infinite strip or cylinder which interpolates between the two asymptotic conditions. The matrix C represents complex conjugation.

Remark 6.3. If $\partial\Sigma = \emptyset$, then this agrees with [**Wen20**, Theorem 5.4]. If $\Gamma = \emptyset$ then this agrees with [**MS12**, Theorem C.1.10].

Remark 6.4. The definition of the Conley-Zehnder index as a Fredholm index suggests a way to define *determinant lines* for asymptotic operators, namely as the Fredholm determinant of the operator in Figure 2. This is similar to [**Abo14**, Definition 1.4.3] or [**Par19**, Definition 2.46].

A kernel gluing theorem analogous to the one in §7.3 should establish a relationship between the Fredholm determinant of D, the determinant lines of the asymptotic operators, and the Fredholm determinant of a different Cauchy-Riemann operator D^1 where all the asymptotic operators are changed to $-i\partial_t - C$.

The method of large anti-linear deformations considers a family $D^\sigma = D_0 + \sigma B$ (which agrees with D^1 when $\sigma = 1$). Moreover, B can be chosen so that D^σ is Fredholm for all $\sigma \geq 1$. See §8.2 for a precise definition of D^σ.

For large σ, we can explicitly describe the kernel and cokernel of D^σ as the $\mathbb{R}$-vector space generated by certain sections concentrated near certain zeros of B (i.e., each zero either contributes ± 1, or 0 to the index). In particular, the problem of orienting the Fredholm determinant of D^σ reduces to the problem of orienting a vector space generated by certain subsets of zeros of B. We do not pursue the question of "coherently orienting" Fredholm determinants any further in this thesis.

6.2. Relative Euler characteristics for Riemann surfaces with boundary punctures

In this section we give a more precise definition of the relative Euler characteristic term appearing in the index formula. Suppose that $(\Sigma, \partial\Sigma, \Gamma_+, \Gamma_-, C)$ is a Riemann surface with punctures $\Gamma = \Gamma_+ \cup \Gamma_-$, some of which may be on the boundary, and cylindrical/strip-like ends C_z for each $z \in \Gamma_\pm$. Each puncture in Γ_+ has a cylindrical end biholomorphic to $[0, \infty) \times [0, 1]$ or $[0, \infty) \times \mathbb{R}/\mathbb{Z}$, and similarly for Γ_- with $[0, \infty)$ replaced by $(-\infty, 0]$. Let $s + it$ denote the holomorphic coordinate in these cylindrical ends.

Let V be a vector field on $\dot{\Sigma} := \Sigma \setminus \Gamma_+ \setminus \Gamma_-$ which agrees with ∂_s in the cylindrical ends, and which is everywhere tangent to $\partial\dot{\Sigma}$. See Figure 3 for an illustration. By choosing V generically, we can assume that the linearizations of V at its zeros are non-degenerate. Let us agree to call such a vector field *admissible* for $(\Sigma, \partial\Sigma, \Gamma_\pm)$.

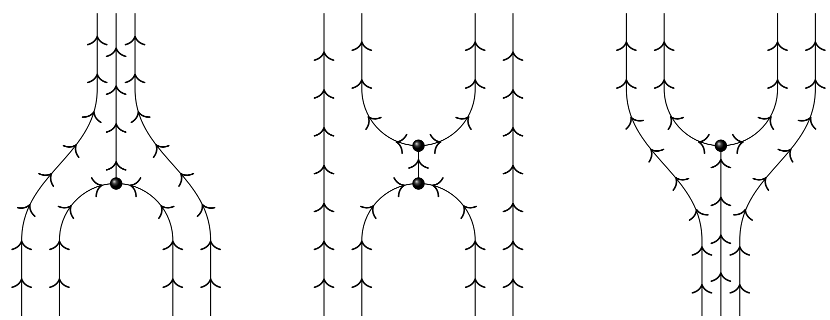


FIGURE 3. Vector fields on surfaces with boundary punctures. Positive punctures (i.e., in Γ_+) are placed at the top of the figure while negative punctures are placed at the bottom. The relative Euler characteristic X is the count of zeros weighted as in Figure 1.

If $p \in \Sigma$ is a zero of V, and $z = s + it$ is a holomorphic coordinate with $z(p) = 0$, we can write V as

$$V = \begin{bmatrix} \partial_s & \partial_t \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} + \text{higher order terms},$$

where the 2×2 matrix is invertible.

If p is an interior zero, then we define the count of p to be the sign of the determinant of the 2×2 matrix. We can deform our vector field near p so that $a = 1$, $d = \pm 1$ and $b = c = 0$ – this uses the fact that $\mathrm{GL}_2(\mathbb{R})$ has two connected components. After this deformation, the local coordinate representation of V is either z or $\bar{z}$, depending on whether the count of the p is ± 1.

Suppose now that p is a boundary zero. Then we can pick z so that it takes values in $\bar{\mathbb{H}}$, in which case we must have $c = 0$ and $a > 0$. We define:

$$\text{count of } p = \begin{cases} +1 \text{ if } a > 0 \text{ and } c > 0, \\ \quad 0 \text{ if } a < 0 \text{ and } c < 0, \\ \quad 0 \text{ if } a > 0 \text{ and } c < 0, \\ -1 \text{ if } a < 0 \text{ and } c < 0. \end{cases}$$

Unlike the case when p was an interior zero, we cannot freely deform the linearization, since the linearization is required to map $T\partial\Sigma$ into $T\partial\Sigma$. The four cases above depend on whether the coordinate representation of V can be deformed to $\pm z$ or $\pm\bar{z}$, as shown in Figure 1. Note that $a \in \mathbb{R}^{1\times 1}$ can be thought of as the linearization of the *restriction* of V to $\partial\Sigma$, considered as a section of $T\partial\Sigma \to \partial\Sigma$.

Proposition 6.5. The sum of the counts of the zeros of V is independent of the choice of V and the coordinate systems used. It does depend on the assignment of signs to the boundary punctures Γ. The resulting integer is denoted $\mathrm{X}(\Sigma, \Gamma_+, \Gamma_-)$.

Proof. We do not actually use this invariance to prove the index formula, and hence the proposition follows from the index formula. Indeed, one can use the large anti-linear perturbation method from §8 to show that our count of zeros of V equals the Fredholm index of the operator:

$$f \mapsto D(f) := \mathrm{d}f + i \cdot \mathrm{d}f \cdot j + \mu(-, V)\bar{f}$$

acting on sections of the trivial line bundle $\mathbb{C}$ which take real values on $\partial\dot{\Sigma}$. Here μ is a Hermitian metric on $T\dot{\Sigma}$ which is cylindrical in the ends. This completes the proof. $\square$

6.2.1. *Formula for the relative Euler characteristic.* This relative Euler characteristic is not too strange; indeed, we have the following simple formula which computes it:

Lemma 6.6. Let $\bar{\Sigma}$ denote the unpunctured surface. Then

$$\mathrm{X}(\Sigma, \Gamma_+, \Gamma_-) = \mathrm{X}(\bar{\Sigma}) - \left|\Gamma^{\mathrm{int}}\right| - \left|\partial\Gamma_-\right|,$$

where Γ^{int} and $\partial\Gamma_-$ are the interior punctures and boundary negative punctures, respectively.

Proof. The proof is summarized in Figure 4. It is clear that by capping off:

(i) each interior puncture by a disk with a single interior zero ($V(z) = \pm z$),

(ii) each negative boundary puncture with a half-plane of the form $V(z) = z$, and

(iii) each positive boundary punctures with a half-plane of the form $V(z) = -z$, we will add 1 to the relative Euler characteristic for each interior puncture and each *negative* boundary puncture. In the end, we obtain a vector field on an *unpunctured surface*. As shown above, the relative Euler characteristic term is an invariant, independent of the choice of vector field, and hence we can use a different vector field which is everywhere non-zero along the boundary. We conclude the desired result. □

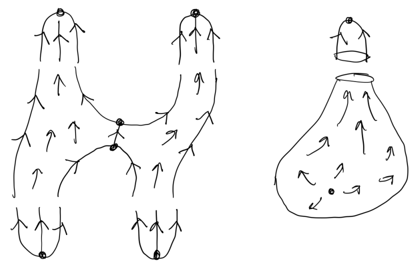


FIGURE 4. Computing the relative Euler characteristic by capping off each puncture. Each addition will change the relative Euler characteristic by some computable amount, and we will obtain a vector field on a surface without any punctures.

6.3. Asymptotically non-degenerate Cauchy-Riemann operators

6.3.1. *Strip-like and cylindrical ends.* Fix a Riemann surface Σ with boundary $\partial\Sigma$ and punctures $\Gamma = \Gamma_+ \cup \Gamma_-$. Fix cylindrical ends around each of the punctures of Γ; this means that we pick holomorphic coordinate disks or half-disks around each $z \in \Gamma_\pm$, and identify the disks with $\mathbb{R}_\pm \times \mathbb{R}/\mathbb{Z}$ via the map $(s,t) \mapsto e^{\mp 2\pi(s+it)}$ and the half-disks with $\mathbb{R}_\pm \times [0,1]$ via the map $(s,t) \mapsto e^{\mp\pi(s+it)}$. Note that in order for this to make sense, we require picking lower half-disks around positive punctures and upper half-disks around negative punctures.

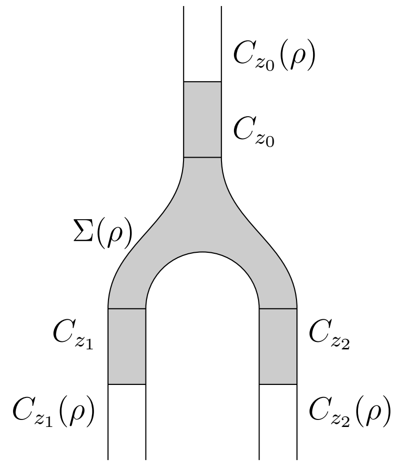


FIGURE 5. A surface Σ with $\Gamma_+ = \{z_0\}$ and $\Gamma_- = \{z_1, z_2\}$ and chosen cylindrical ends. The precompact sub-domain $\Sigma(\rho)$ is shown as the shaded region. The bundle E has an equivalence class of unitary trivializations defined on the ends.

For each $z \in \Gamma$, let C_z denote the cylindrical end corresponding to z, and let $C_z(\rho) \subset C_z$ denote the closed which translated by ρ deeper into the end, i.e., if z is a positive boundary puncture then $C_z(\rho) = [\rho, \infty) \times [0, 1]$, and similarly for the other possibilities for z. See Figure 5. Let $C(\rho) = \bigcup_{z \in \Gamma} C_z(\rho)$, with $C = C(0)$. We let $\Sigma(\rho) = \dot{\Sigma} \setminus C(\rho)$, so that $\Sigma(\rho)$ is a precompact sub-domain of $\dot{\Sigma}$.

6.3.2. *Asymptotically Hermitian structures.* Suppose that (E, J) is a complex vector bundle of rank n over $\dot{\Sigma}$ and $F \subset E|_{\partial\dot{\Sigma}}$ is a totally real sub-bundle. Similarly to [**Wen20**, §4.1], we define an *asymptotically Hermitian structure* on (E, F, J) to be an equivalence class of trivializations τ of $E|_C \simeq \mathbb{R}^{2n}$ which identify J with the standard complex structure J_0 and send F to $\mathbb{R}^n$. Two trivializations are equivalent provided the transition map between them converges to an s-independent unitary transformation (i.e., multiplication by a t-dependent family $\Omega(t) \in U(n)$). The inverse $X = \tau^{-1}$ of a trivialization will be called an *asymptotic unitary frame.*

To be more precise, we require that the transition between τ_1 and τ_2 is multiplication by $\Omega(s, t)$ and $\Omega(s, t) - \Omega_\infty(t) \in W^{k,p}$ for all k for some smooth $\Omega_\infty(t)$.

6.3.3. *Sobolev spaces.* We recall that the space of sections $W^{k,p}_{\mathrm{loc}}(E)$ is well-defined independently of any choice of auxiliary data on $\dot{\Sigma}$ for all $k \geq 0, p \geq 1$. These are sections which are of class $W^{k,p}_{\mathrm{loc}}$ in any coordinate chart equipped with a trivialization of E.

For $k - 2/p > 0$, the Sobolev embedding theorem (see [**MS12**, Theorem B.1.11]) implies that $W^{k,p}_{\mathrm{loc}}(E)$ sections are continuous, and hence we can define $W^{k,p}_{\mathrm{loc}}(E, F) \subset W^{k,p}_{\mathrm{loc}}(E)$ as the sections taking boundary values in F.

For $k = 1$ and $p \in [1, 2]$, we say $\xi \in W^{1,p}_{\mathrm{loc}}(E, F)$ if, for any choice of $\bar{\mathbb{H}}$-valued coordinates equipped with trivializations identifying E with $\mathbb{R}^{2n}$ and F with $\mathbb{R}^n$, the doubling[1] of ξ (i.e., extension by $\bar{\xi}$ on $-\mathbb{H}$) is still of class $W^{1,p}$. It can be shown that this agrees with the other definition of $W^{1,p}_{\mathrm{loc}}(E, F)$ when $p > 2$. See Remark 6.16 for more details.

Using the asymptotic Hermitian structure we can define Sobolev spaces which admit Banach space norms. We say $\xi \in W^{k,p}(E, F)$ if $\xi \in W^{k,p}_{\mathrm{loc}}(E, F)$ and $\tau \circ \xi \in W^{k,p}$ using the standard Euclidean structure on the cylindrical ends (for any asymptotic trivialization τ). To define a Banach space topology on $W^{k,p}(E, F)$, we introduce the norm:

$$\|\xi\|_{\tau,k,p,g,\mu,\nabla} := \sum_{z\in\Gamma} \sum_{\ell=0}^{k} \sum_{a+b=\ell} \left[\int_{C_z} \left| \partial_s^a \partial_t^b (\tau \circ \xi) \right|^p \mathrm{d}s\mathrm{d}t \right]^{1/p} + \|u\|_{W^{k,p}_{g,\mu,\nabla}(\Sigma(1))}.$$

Here we make an arbitrary choice of metric g on $\dot{\Sigma}$, and fiber-wise metric μ and connection ∇ on $E \to \dot{\Sigma}$. It is straightforward to show that for any other choice of g, μ, ∇ we obtain an equivalent norm (since $\Sigma(1)$ is precompact). It is also not hard to show that two different choices of τ give equivalent norms.

The same process defines $W^{k,p}(E)$ for $k \geq 0$. We denote $W^{0,p}(E) =: L^p(E)$.

6.3.3.1. *Exponentially weighted Sobolev spaces.* Let σ be any function which equals $|s|$ on the ends C, and is bounded in $[-1, 1]$ on $\Sigma(1)$. For $\delta \in \mathbb{R}$ define:

$$W^{k,p,\delta}(E, F) = \left\{ u \in W^{k,p}_{\mathrm{loc}} : e^{\delta\sigma} u \in W^{k,p}(E, F) \right\},$$

with the unique equivalence class of norms so that $e^{\delta\sigma} : W^{k,p,\delta} \to W^{k,p}$ is an isomorphism. This equivalence class of norm does not depend on σ.

Note that the L^p size of $\nabla^k(e^{\delta\sigma}u)$ can be bounded by the L^p norms of $e^{\delta\sigma}u, e^{\delta\sigma}\nabla u, \ldots, e^{\delta\sigma}\nabla^k u$ (by constants depending on σ and δ). We can therefore equivalently define $W^{k,p,\delta}$ as those u with $e^{\delta\sigma}\nabla^\ell u \in L^p$ for all $\ell \leq k$.

6.3.4. *Cauchy-Riemann operators with non-degenerate asymptotics.* A first order partial differential operator $D : \Gamma(E) \to \Gamma(\Lambda^{0,1} \otimes E)$ is called a *Cauchy-Riemann operator* if

$$D(f \otimes \xi) = \mathrm{d}f \otimes \xi + (\mathrm{d}f \cdot j) \otimes J\xi + f \cdot D\xi$$

for all real-valued functions f and smooth sections ξ. Here j is the complex structure on Σ and J is the fiber-wise complex structure on E.

We begin with a discussion of the local coordinate representations of Cauchy-Riemann operators. If $z = s + it$ is a holomorphic coordinate, then $\mathrm{d}s - i\mathrm{d}t$ trivializes $\Lambda^{0,1}$. Suppose that

[1]To make this precise, we double ξ in the sense of distributions.

$\tau : E \to \mathbb{C}^n$ is a complex linear trivialization over E. Then $\tau^{-1}(e_k) = X_k$ and $\tau^{-1}(ie_k) = JX_k$ define a local frame for E.

Write $\xi = \tau^{-1}(u) = \sum_k u_k X_k = \sum_k a_k X_k + b_k JX_k$, where $u = a + ib$ is a $\mathbb{C}^n$-valued function. We obtain:

$$D(\sum a_k X_k) = \sum \mathrm{d}a_k \otimes X_k + (\mathrm{d}a_k \cdot j) \otimes JX_k + a_k \cdot DX_k$$
$$D(\sum b_k JX_k) = \sum \mathrm{d}b_k \otimes JX_k - (\mathrm{d}b_k \cdot j) \otimes X_k + b_k \cdot D(JX_k).$$

Hence, using $c \otimes JX_k = ic \otimes X_k$ (for sections of $\Lambda^{0,1} \otimes E$) we obtain:

$$D(\tau^{-1}(u)) = \sum (\mathrm{d}u_k + i \cdot \mathrm{d}u_k \cdot j) \otimes X_k + a_k \cdot DX_k + b_k \cdot D(JX_k).$$

It is straightforward to compute $\mathrm{d}u_k + i \cdot \mathrm{d}u_k \cdot j = (\partial_s u_k + i\partial_t u_k)(\mathrm{d}s - i\mathrm{d}t)$. In particular, we have

$$D(\tau^{-1}(u)) = \sum_{k=1}^{n} (\partial_s u_k + i\partial_t u_k) \cdot (\mathrm{d}s - i\mathrm{d}t) \otimes X_k.$$

We note that $(\mathrm{d}s - i\mathrm{d}t) \otimes X_k$ is a local (complex) frame for $\Lambda^{0,1} \otimes E$. We denote the inverse trivialization by τ_1. If conjugate D by the complex trivializations τ^{-1} and $\tau_1^{-1} : w \mapsto w(\mathrm{d}s - i\mathrm{d}t) \otimes X$, we conclude that

$$\tau_1 \circ D \circ \tau^{-1} =: D_\tau(u) = \partial_s u + i\partial_t u + S(s,t)u, \tag{6.1}$$

where $S(s,t)$ is some smooth family of real linear matrices. Note that D_τ depends both on the holomorphic coordinates used on the base and on the trivialization τ.

Let τ be an asymptotic trivialization for E. Using the holomorphic coordinate in the ends C, we can compute the coordinate representation for D using τ. We say that D has *non-degenerate asymptotics* provided that (6.1) satisfies

$$\sup_t \left|\partial_s^k \partial_t^\ell (S(s,t) - S_\infty(t))\right| \to 0 \text{ as } |s| \to \infty, \tag{6.2}$$

for all $k, \ell \in \mathbb{N}$, for some smooth family of *symmetric* matrices S_∞, and the corresponding *asymptotic operator*

$$A_\tau = -i\partial_t - S_\infty(t) : \begin{cases} C^\infty([0,1], \mathbb{C}^n, \mathbb{R}^n) \to C^\infty([0,1], \mathbb{C}^n) \\ C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{C}^n) \to C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{C}^n) \end{cases} \tag{6.3}$$

is injective. In this case we say that A_τ is non-degenerate. The two cases in (6.3) are whether the cylindrical end corresponds to a boundary or interior puncture.

Since the transition function between two asymptotic trivializations converges to $\Omega(t)$, the condition that D has non-degenerate asymptotics is independent of the chosen τ. Indeed, if $\tau_1\tau_2^{-1} \to \Omega(t)$, then

$$A_{\tau_2} = \Omega(t)^{-1} A_{\tau_1} \Omega(t) = -i\partial_t - i\Omega(t)^{-1}\Omega'(t) - \Omega(t)^{-1} S_\infty(t) \Omega(t).$$

A straightforward computation shows that $i\Omega(t)^{-1}\Omega'(t) - \Omega(t)^{-1}S_\infty(t)\Omega(t)$ is still symmetric.

6.3.5. *Some facts about non-degenerate asymptotic operators.* In this section we fix a non-degenerate asymptotic operator $A = -i\partial_t - S(t)$ on $[0,1]$. The analogous results with $[0,1]$ replaced by $\mathbb{R}/\mathbb{Z}$ are left to the reader.

Proposition 6.7. The map $A : C^\infty([0,1], \mathbb{C}^n, \mathbb{R}^n) \to C^\infty([0,1], \mathbb{C}^n)$ extends to a self-adjoint isomorphism

$$A : W^{1,2}([0,1], \mathbb{C}^n, \mathbb{R}^n) \to L^2([0,1], \mathbb{C}^n).$$

By self-adjoint we mean that $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in W^{1,2}([0,1], \mathbb{C}^n, \mathbb{R}^n)$.

See [**Wen20**, Corollary 3.14] for an alternative approach, yielding a proof in the $\mathbb{R}/\mathbb{Z}$ case.

Proof. It is clear that A extends to a bounded linear operator between the advertised Banach spaces. Since $C^\infty([0,1], \mathbb{C}^n, \mathbb{R}^n)$ is dense in $W^{1,2}([0,1], \mathbb{C}^n, \mathbb{R}^n)$ it suffices to prove the self-adjointness for smooth functions u, v. This follows from a straightforward integration-by-parts computation, using the fact that the matrix $S(t)$ is symmetric, and i is anti-symmetric. We leave this computation to the reader. Note that it is crucial that both u, v take boundary values in $\mathbb{R}^n$, otherwise the integration by parts will fail.

It suffices to prove that A is a bijection, since continuous bijections between Banach spaces are isomorphisms. Observe that any element in the kernel of A must be smooth (by 1-dimensional elliptic regularity). Since we assume that $u \mapsto Au$ is injective for smooth u, we conclude that A is injective on $W^{1,2}$.

We will prove that A is surjective. Fix a smooth η, and we attempt to solve $A(\xi) = \eta$ for a smooth ξ:

$$\begin{aligned} & i\frac{\partial \xi}{\partial t} + S(t)\xi = -\eta(t) \iff \frac{\partial \xi}{\partial t} - iS(t)\xi = i\eta(t). \\ \iff & \frac{\partial}{\partial t}(\mathrm{F}(t)\xi(t)) = \mathrm{F}(t)i\eta(t), \\ \iff & \xi(t) = \mathrm{F}(t)^{-1}\xi(0) + \mathrm{F}(t)^{-1}\int_0^t \mathrm{F}(t')i\eta(t')\mathrm{d}t'. \end{aligned} \tag{6.4}$$

where $\mathrm{F}'(t) = -\mathrm{F}(t)iS(t)$ and $\Sigma(0) = 0$. This shows that we can solve $A(\xi) = \eta$ for many different choices of ξ, namely there is an $\mathbb{R}^{2n}$ dimensional family of solutions corresponding to the choice of $\xi(0)$. We claim that (exactly) one of these solutions will satisfy $\xi(0), \xi(1) \in \mathbb{R}^n$. To see why, consider the affine map:

$$f : \xi(0) \in \mathbb{R}^n \mapsto \mathrm{F}(1)^{-1}\xi(0) + F(1)^{-1}\int_0^1 \mathrm{F}(t')i\eta(t')\mathrm{d}t' \in \mathbb{R}^{2n}.$$

This map parameterizes an n-dimensional affine subspace of $\mathbb{R}^{2n}$. Note that the associated linear subspace $\mathrm{F}(1)^{-1}\mathbb{R}^n$ is transverse to $\mathbb{R}^n$ (otherwise we could find a vector $v \in \mathbb{R}^{n-1}$ so

$\mathrm{F}(1)^{-1}v \in \mathbb{R}^n$, and the above computation with $\eta = 0$ would imply $\xi(t) = \mathrm{F}(t)^{-1}v$ lies in the kernel of A).

Therefore $f(\mathbb{R}^n)$ intersects $\mathbb{R}^n$ in a unique point $f(\xi(0)) = \xi(1)$. Thus (6.4) with this special $\xi(0)$ shows that A is surjective onto the smooth elements η.

To show that A is surjective in general, it suffices to prove that the image of A is closed. This follows from the estimate

$$\|\xi\|_{W^{1,2}} \leq C(\|A(\xi)\|_{L^2} + \|\xi\|_{L^2})$$

and the fact that $W^{1,2} \to L^2$ is a compact inclusion. This completes the proof of the lemma. $\square$

Proposition 6.8. There exists an orthonormal basis of $L^2([0,1],\mathbb{R}^n)$ consisting of (smooth) eigenvectors of A. The union of all the eigenvalues is a discrete set $\Lambda \subset \mathbb{R}$ disjoint from 0.

Proof. The key observation is that the following composition is a compact self-adjoint operator (called the *resolvent* of A):

$$L^2 \xrightarrow{A^{-1}} W^{1,2} \xrightarrow{\subset} L^2.$$

This is because $W^{1,2} \subset L^2$ is a compact inclusion (Proof: if $\partial_t f_n$ is bounded in L^2 then $|f_n(x) - f_n(x+t)| \leq ct^{1/2}$, and hence f_n is equicontinuous). Self-adjoint compact operators have orthonormal eigenbases whose spectrum accumulates only at 0 (see [**Sim15**, Theorem 3.2.3]). The desired result follows. $\square$

6.3.6. *Formal adjoints.* The purpose of this section is to define the *formal adjoint* of a Cauchy-Riemann operator. Formal adjoints will play an important role in establishing the Fredholm property. A good reference in the case when $\partial\Sigma = \emptyset$ is [**Wen20**, §4.7].

Let D be a Cauchy-Riemann operator for the data $(\Sigma, \partial\Sigma, \Gamma_\pm, C, E, F, [\tau])$ as explained above. Fix a j-invariant Riemannian metric g on $T\Sigma$ which is the Euclidean metric in the cylindrical ends. The corresponding volume form is given by $\mathrm{dvol} = g(j-,-)$.

Pick a Hermitian structure on (E, F) which agrees with the asymptotically Hermitian structure in the cylindrical ends C. This means that E is equipped with a fiber-wise metric g which is J-invariant and F is g-orthogonal to JF. In other words, F is Lagrangian for the symplectic form $g(J-,-)$.

We define $\mathbb{C}$-valued Hermitian metrics on E (and $T\dot{\Sigma}$) by the formulas:

$$\mu(X,Y) = g(X,Y) + ig(JX,Y).$$

By our conventions, $\mu(X,JY) = i\mu(X,Y)$ and $\mu(JX,Y) = -i\mu(X,Y)$.

The bundle isomorphism $Y \mapsto \mu(-, Y)$ identifies $T\dot{\Sigma}$ with $\Lambda^{0,1}$. We use this to push forward a Hermitian metric onto $\Lambda^{0,1}$.

Given two complex vector bundles E_1, E_2 with Hermitian metrics μ_1, μ_2 we can endow a Hermitian metric $\mu_1 \otimes \mu_2$ on $E_1 \otimes E_2$ by the formula:

$$\mu_1 \otimes \mu_2(X_1 \otimes X_2, Y_1 \otimes Y_2) = \mu_1(X_1, Y_1)\mu_2(X_2, Y_2).$$

Via this construction, the bundles E and $\Lambda^{0,1} \otimes E$ are both equipped with Hermitian metrics.

With these preliminaries out of the way, we say that $D^* : \Gamma(\Lambda^{0,1} \otimes E) \to \Gamma(E)$ is a *formal adjoint* of D if

$$\operatorname{Re} \int_{\dot{\Sigma}} \mu(D(\xi), \eta) \,\mathrm{dvol} = \operatorname{Re} \int_{\dot{\Sigma}} \mu(\xi, D^*(\eta)) \,\mathrm{dvol}, \tag{6.5}$$

for all $\xi \in \Gamma_0(E, F)$ and $\eta \in \Gamma_0(\Lambda^{0,1} \otimes E, F^*)$. Here $F^* \subset \Lambda^{0,1} \otimes E$ is the totally-real sub-bundle of maps which map $T\partial\Sigma$ into F, and $\Gamma_0(E, F)$ is the set of smooth compactly supported sections of E which take boundary values in F.

Since $\operatorname{Re}(\mu)$ is a Riemannian metric, formal adjoints are necessarily unique. We will derive a formula for the formal adjoint in local trivializations below. By patching together the local descriptions we deduce that formal adjoints always exist.

Let $z = s + it$ be a holomorphic coordinate and $\tau : E \to \mathbb{C}^n$ a local unitary trivialization of E defined on the domain of z. Let $X_1, \cdots, X_n$ be the unitary frame of E induced by τ. Recall that we have an associated trivialization $\tau_1 : \Lambda^{0,1} \otimes E \to \mathbb{C}^n$ which satisfies

$$\tau_1^{-1}(w) = \sum_k w_k(\mathrm{d}s - i\mathrm{d}t) \otimes X_k.$$

The equation (6.1) shows $\tau_1 \circ D \circ \tau^{-1}(u) = \partial_s u + i\partial_t u + S(s,t)u$.

To incorporate the boundary conditions, we require that z takes values in $\mathbb{R} \times [0,1]$, $z(\partial\dot{\Sigma}) \subset \mathbb{R} \times \{0,1\}$, and τ identifies F with $\mathbb{R}^n$. We do not require that z is surjective, e.g., it could take values in $D(1) \cap \bar{\mathbb{H}}$, or $i/2 + D(1/2)$.

Lemma 6.9. If D^* is a formal adjoint for D, then for sections w with compact support in the above coordinate chart we have

$$|\partial_s|^2\, \tau \circ D^* \circ \tau_1^{-1}(w) = -\partial_s w + i\partial_t w + S(s,t)^T w, \tag{6.6}$$

where $|\partial_s|^2 = \mu(\partial_s, \partial_s)$ and $S(s,t)^T$ is the transpose matrix.

Proof. The first thing we do is derive formulas for the Hermitian metrics μ. Because τ is a unitary transformation, we have

$$\mu(\tau^{-1}(u), \tau^{-1}(v)) = \sum_k \bar{u}_k v_k =: \mu_0(u, v).$$

Unfortunately, τ_1 is not a unitary transformation because $\mathrm{d}s - i\mathrm{d}t$ is not a unitary frame of $\Lambda^{0,1}$. We easily compute $\mu(-, \partial s) = |\partial_s|^2 (\mathrm{d}s - i\mathrm{d}t)$ (by inserting ∂_s, ∂_t into both sides). Since the Hermitian metric on $\Lambda^{0,1}$ is pushed forward from $T\Sigma$ we have

$$|\partial_s|^4 \mu(\mathrm{d}s - i\mathrm{d}t, \mathrm{d}s - i\mathrm{d}t) = \mu(\partial_s, \partial_s) = |\partial_s|^2 \implies |\mathrm{d}s - i\mathrm{d}t|^2 = |\partial_s|^{-2}.$$

We can therefore compute

$$\mu(\tau_1^{-1}(u), \tau_1^{-1}(v)) = \sum_{k,\ell} \mu(u_k(\mathrm{d}s - i\mathrm{d}t) \otimes X_k, v_\ell(\mathrm{d}s - i\mathrm{d}t) \otimes X_\ell) = |\partial_s|^{-2} \mu_0(u, v).$$

It is also easy to compute that $\mathrm{dvol} = |\partial_s|^2 \mathrm{d}s\mathrm{d}t$.

Let w and u be $\mathbb{C}^n$ valued functions which takes values in $\mathbb{R}^n$ on $\mathbb{R} \times \{0, 1\}$. We compute

$$\mu(\tau^{-1}(u), D^* \circ \tau_1^{-1}(w)) = \mu_0(u, \tau \circ D^* \circ \tau_1^{-1}(w)).$$

On the other hand, we have

$$\mu(D \circ \tau^{-1}(u), \tau_1^{-1}(w)) = |\partial_s|^{-2} \mu_0(\tau_1 \circ D \circ \tau^{-1}(u), w).$$

Since real linear combinations of X_k lie in F, we are allowed to apply the formal adjoint property with $\xi = \tau^{-1} \circ u$ and $\eta = \tau_1^{-1} \circ w$:

$$\mathrm{Re} \int \mu(D \circ \tau^{-1}(u), \tau_1^{-1}(w))\mathrm{dvol} = \mathrm{Re} \int \mu(\tau^{-1}(u), D^* \circ \tau_1^{-1}(w))\mathrm{dvol}.$$

This implies that:

$$\mathrm{Re} \int \mu_0(\tau_1 \circ D \circ \tau^{-1}(u), w)\mathrm{d}s \wedge \mathrm{d}t = \mathrm{Re} \int \mu_0(u, \tau \circ D^* \circ \tau_1^{-1}(w)) |\partial_s|^2 \mathrm{d}s \wedge \mathrm{d}t.$$

In particular, $|\partial_s|^2 \tau \circ D^* \circ \tau_1^{-1}$ is the formal adjoint of $D_\tau := \tau_1 \circ D \circ \tau^{-1}$ with respect to the standard metric μ_0 and volume form $\mathrm{d}s \wedge \mathrm{d}t$. Equation (6.1) gives a formula for D_τ and so we can explicitly compute its adjoint:

$$\mathrm{Re} \int \mu_0(\partial_s u + i\partial_t u + S(s,t)u, w)\mathrm{d}s\mathrm{d}t = \mathrm{Re} \int \mu_0(u_k, -\partial_s w + i\partial_t u + S(s,t)^T w)\mathrm{d}s\mathrm{d}t.$$

The boundary terms in the integration by parts are given by

$$\mathrm{Re} \int_{\mathbb{R} \times \{0,1\}} \mu_0(iu, w)\mathrm{d}s\mathrm{d}t = 0.$$

It follows that $|\partial_s|^2 \tau \circ D^* \circ \tau_1^{-1} = -\partial_s + i\partial_t + S(s,t)^T$ as desired. □

As a consequence, if $s + it$ is the coordinate system in a cylindrical end C_z and τ is an asymptotic trivialization, then

$$\tau \circ D^* \circ \tau_1^{-1} = -\partial_s + i\partial_t + S(s,t)^T \to -\partial_s - A \text{ as } s \to \infty,$$

where $A_\tau w = -i\partial_t w - S_\infty(t) w$ is the asymptotic operator for D_τ in the end C_z.

6.4. Regularity and the Fredholm property

The references for this section are [**Sal97**, §2.3], [**Wen20**, Chapter 4], [**Sch95**, Chapter 3], and [**MS12**, Appendix B] (for the local L^p elliptic estimates).

6.4.1. *Local elliptic estimates.* Our first result is the following local elliptic estimate for $u \mapsto \partial_s u + i\partial_t u$.

Theorem 6.10. Fix $r < 1$ and $q > 1$. There is a constant $c_{q,r}$ so that for all smooth maps $u : \bar{D}(1) \cap \bar{\mathbb{H}} \to \mathbb{C}^n$ satisfying $u(D(1) \cap \mathbb{R}) \subset \mathbb{R}^n$ we have

$$\int_{D(r)\cap\bar{\mathbb{H}}} |u|^q + |\partial_x u|^q + |\partial_y u|^q \,\mathrm{d}x\mathrm{d}y \leq c_{q,r} \int_{D(1)\cap\bar{\mathbb{H}}} |u|^q + |\partial_x u + i\partial_y u|^q \,\mathrm{d}x\mathrm{d}y.$$

Proof. The theorem follows from [**MS12**, Theorem B.3.2] which concerns weak solutions of the equation

$$\Delta w = f_0 + \partial_x f_1 + \partial_y f_2,$$

with $w, f_0, f_1, f_2 \in L^p(D(1))$. The conclusion is that the $W^{1,q}$ size of w on a smaller disk is bounded by the L^q sizes of w, f_0, f_1, f_2. This uses the Calderon-Zygmund inequality proved in [**MS12**, §B.2].

To apply their result to our setting, we extend u across the boundary by

$$u(x, -y) = \overline{u(x, y)}.$$

The extended function is no longer smooth. Let $\eta = \partial_x u + i\partial_y u$ (which potentially has a jump discontinuity along $\mathbb{R}$, but is still in L^q). We note that

$$\eta(x, -y) = \partial_x \bar{u} - i\partial_y \bar{u} = \overline{\partial_x u + i\partial_y u} = \overline{\eta(x, y)}.$$

In particular, the size $|\eta|$ is invariant under $y \mapsto -y$.

Using the fact that $(\partial_x - i\partial_y)(\partial_x + i\partial_y) = \Delta$ we have

$$\int_{D(1)} g_0(u, \Delta\phi) \,\mathrm{d}x\mathrm{d}y = -\int_{D(1)} g_0(\eta, (\partial_x + i\partial_y)\phi)\mathrm{d}x\mathrm{d}y.$$

To see why, apply Stokes' Theorem separately on the upper and lower half-disks, and then observe that the boundary terms will cancel; this uses $u(D(1)\cap\mathbb{R}) \subset \mathbb{R}^n$. The equality above satisfies the hypothesis of [**MS12**, Theorem B.3.2] and allows us to conclude that $W^{1,q}$ size of u is controlled by the L^q size of η and u. This implies the desired result. □

6.4.2. *Local elliptic regularity.* In this section we wish to prove that weak solutions of $D^*(\eta) = f$ are in fact smooth, provided f is smooth. In order to talk about $D^*(\eta)$, we require the choice of Hermitian metrics μ on $E, T\dot{\Sigma}$, as in §6.3.6.

More precisely, we wish to prove the following:

Proposition 6.11. Let $q > 1$. If $\eta \in L^q_{\mathrm{loc}}(\Lambda^{0,1} \otimes E)$, f is smooth, and $D^*(\eta) = f$ weakly in the sense that

$$\mathrm{Re} \int_{\dot{\Sigma}} \mu(D(\xi), \eta)\mathrm{dvol} = \mathrm{Re} \int_{\dot{\Sigma}} \mu(\xi, f)\mathrm{dvol}$$

for all $\xi \in \Gamma_0(E, F)$, then η is in smooth and lies in $\Gamma(\Lambda^{0,1} \otimes E, F^*)$.

The same holds true with $(\eta, \Lambda^{0,1} \otimes E, F^*, D^*)$ swapped with (ξ, E, F, D) throughout the statement.

Remark 6.12. Note that "weakly" solving the equation implicitly incorporates the boundary conditions, since we allow the test functions to be non-zero along the boundary. We do require, however, that the test functions take values the appropriate sub-bundle F.

Since smoothness is a local property, we can prove Proposition 6.11 by restricting our attention to a coordinate chart $z = s + it$ on which we have a unitary trivialization $\tau : E \to \mathbb{C}^n$. Without loss of generality, let us suppose that z takes values in $D(1) \cap \bar{\mathbb{H}}$. Writing $\tau(\xi) = u$, $\tau_1(\eta) = w$, and $f := \tau(f)$, we compute

$$D^*(\eta) = f \text{ weakly} \implies -\partial_s w + i\partial_t w + S(s,t)^T w = |\partial_s|^{-2} f \text{ weakly}$$
$$D(\xi) = f \text{ weakly} \implies \partial_s u + i\partial_t u + S(s,t)u = f \text{ weakly.}$$

We can simplify this a bit further by observing that

$$-\partial_s w + i\partial_t w + S(s,t)^T w = |\partial_s|^{-2} f$$
$$\iff \partial_s \bar{w} + i\partial_t \bar{w} - CS(s,t)^T C\bar{w} = -|\partial_s|^{-2} \bar{f},$$

where C is the matrix representing complex conjugation. Thus, Proposition 6.11 follows from:

Lemma 6.13. Let $q > 1$. Write $\Omega(r) = D(r) \cap \bar{\mathbb{H}}$, and suppose that

$$u \in L^q(\Omega(1), \mathbb{C}^n),\ f \in C^\infty(\bar{\Omega(1)}, \mathbb{C}^n), \text{ and } S \in C^\infty(\bar{\Omega(1)}, \mathbb{R}^{2n\times 2n})$$

satisfy

$$\partial_s u + i\partial_t u + S(s,t)u = f \text{ weakly,} \tag{6.7}$$

in the sense that

$$\mathrm{Re} \int_{\Omega(1)} \mu_0(u, -\partial_s\varphi + i\partial_t\varphi + S(s,t)^T\varphi) = \mathrm{Re} \int_{\Omega(1)} \mu_0(f, \varphi),$$

for all compactly supported test functions φ which take values in $\mathbb{R}^n$ on $\Omega(1) \cap \mathbb{R}$. Then u is smooth and takes boundary values in $\mathbb{R}^n$. Moreover for $k \in \mathbb{N}$ and $r < 1$, there exists a constant $c = c(k, q, S)$ so that

$$\|u\|_{W^{k,q}(D(r)\cap\mathbb{H})} \leq c(\|u\|_{L^q(D(1)\cap\mathbb{H})} + \|f\|_{W^{k-1,q}(D(1)\cap\mathbb{H})}). \tag{6.8}$$

Proof. Throughout the argument we will need to shrink the domain countably many times. For this purpose, fix a sequence $1 > r_1 > r_2 > \cdots > r_\infty = r$. Each time we need to shrink the domain we will pass from $\Omega(r_j)$ to $\Omega(r_{j+1})$. To obtain the constant in (6.8), we will only need to shrink the domain finitely many times.

Our first goal is to upgrade u to a $W^{1,q}$ distribution. We observe that

$$(\partial_s + i\partial_t)u = -S(s,t)u + f \text{ weakly.}$$

Notice that the right hand side lies in L^q. More generally, let us consider equations of the form

$$(\partial_s + i\partial_t)u = F \text{ weakly,}$$

where $F \in L^q$. Our strategy is to approximate u by a sequence of smooth sections u_n taking real values along the boundary so that:

(i) $u_n \to u \in L^q$ and

(ii) $\|(\partial_s + i\partial_t)u_n\|_{L^q(\Omega(r_1))}$ is bounded by $c_1 \|F\|_{L^q(\Omega(1))}$.

We will explain how to do this approximation at the end of the proof. See [**Wen20**, §2.4] for another approach in the case with $\bar{\mathbb{H}}$ replaced by $\mathbb{C}$. See [**MS12**, §B.4] for a similar bootstrapping argument (in a non-linear context). The estimate from Theorem 6.10 then implies that u_n is bounded in the $W^{1,q}$ topology (on a smaller domain $\Omega(r_2)$). Indeed, we have

$$\|u_n\|_{W^{1,q}(\Omega(r_2))} \le c_2(\|u_n\|_{L^q(\Omega(1))} + \|F\|_{L^q(\Omega(1))}).$$

Since the $W^{1,q}$ spaces are reflexive, the Banach-Alaoglu theorem implies that some subsequence of u_n converges in the weak topology to an element $u' \in W^{1,q}$. Since $(L^q)^* \subset (W^{1,q})^*$ we conclude that $\lim_{n\to\infty}\langle u_n, w\rangle = \langle u', w\rangle$ for all $w \in (L^q)^*$. However, the same holds with u' replaced by u (because u_n converges to u in the L^q norm). Thus $u = u'$, and hence $u \in W^{1,q}$. Moreover, the Banach-Alaoglu theorem implies the $W^{1,q}$ norm of u is bounded above by $\limsup \|u_n\|_{W^{1,q}}$, and hence we conclude that

$$\|u\|_{W^{1,q}(\Omega(r_2))} \le c_2(\|u\|_{L^q(\Omega(1))} + \|F\|_{L^q(\Omega(1))}).$$

Suppose that we have shown that u is of class $W^{k,q}$ on some region $\Omega(r_{2k})$. Moreover, suppose that the $\|u\|_{W^{k,q}(\Omega(r_{2k}))}$ is bounded by $c(\|u\|_{L^q(\Omega(1))} + \|f\|_{W^{k-1,q}(\Omega(1))})$ for some c. Then we can differentiate the equation (6.7) k times in the s-direction to conclude:

$$(\partial_s + i\partial_t)\partial_s^k u = \partial_s^k f - \sum_{\ell=0}^{k} \partial_s^\ell S(s,t)\cdot \partial_s^{k-\ell} u = F_k \text{ weakly.} \tag{6.9}$$

This differentiation is a bit subtle because the "weak" condition incorporates the boundary conditions; we will explain this step in greater detail at the end of the proof.

By our assumption on u, the right hand side is in L^q. The same argument given above implies that $\partial_s^k u$ is in $W^{1,q}$ on a smaller region $\Omega(r_{2k+2})$ and that

$$\left\|\partial_s^k u\right\|_{W^{1,q}(\Omega(r_{2k+2}))} \leq c'(\left\|\partial_s^k u\right\|_{L^q(\Omega(r_{2k}))} + \left\|\partial_s^k f\right\|_{L^q(\Omega(r_{2k}))} + C(S)\left\|u\right\|_{W^{k,q}(\Omega(r_{2k}))}).$$

It is straightforward to use (6.7) to establish that, for $a+b=k$,

$$\partial_s^a\partial_t^b u = i^b\partial_s^k u + \text{lower order terms.}$$

This equality should be interpreted as saying that both sides agree when integrated against a test function which is supported in the interior of the domain (i.e., we do not need to worry about the boundary). Since $\partial_s^k u$ and the "lower order terms" are of class $W^{1,q}$ we conclude that all the kth order derivatives are in $W^{1,q}(\Omega(r_{2k+2}))$, and hence u is in $W^{k+1,q}(\Omega(r_{2k+2}))$. Keeping track of the various estimates implies that

$$\left\|u\right\|_{W^{k+1,q}(\Omega(r_{2k+2}))} \leq c''(\left\|u\right\|_{L^q(\Omega(1))} + \left\|f\right\|_{W^{k,q}(\Omega(1))}).$$

The Sobolev embedding theorem [**MS12**, Theorem B.1.11] implies that u is smooth on $\Omega(r_\infty)$. Part of the conclusion of the Sobolev embedding theorem is that u extends smoothly to the boundary. We claim that u takes $\mathbb{R}^n$ values along the boundary. This follows from (6.7); pick any test function φ taking boundary values in $\mathbb{R}^n$. It is easy to see (by integration by parts) that

$$\mathrm{Re}\int_{D(1)\cap\bar{\mathbb{H}}} \mu_0(\partial_s u + i\partial_t u, \varphi) - \mu_0(u, -\partial_s\varphi + i\partial_t\varphi)\mathrm{d}s\mathrm{d}t = \mathrm{Re}\int_{D(1)\cap\mathbb{R}} \mu_0(u, i\varphi)\mathrm{d}s.$$

If u did not take $\mathbb{R}^n$-values along $D(1)\cap\mathbb{R}$, we could pick φ so that the right hand side was non-zero. This would contradict (6.7).

This completes the proof, modulo our explanation of how to pick the approximations $u_n \to u$ so that (i) and (ii) hold, and also why we can differentiate the weak equation with respect to ∂_s to obtain (6.9).

First we explain how to differentiate the weak equation. Suppose that $(\partial_s + i\partial_t)w = F$ weakly and $w, F \in W^{1,q}$. Then for any test function φ taking real-values along the boundary, $\partial_s\varphi$ still takes real values along the boundary, and hence

$$\mathrm{Re}\int \mu_0(w, (-\partial_s + i\partial_t)\partial_s\varphi)\mathrm{d}s\mathrm{d}t = \mathrm{Re}\int \mu_0(F, \partial_s\varphi)\mathrm{d}s\mathrm{d}t. \tag{6.10}$$

The distributional derivative ∂_s is defined (by duality) by how it integrates against sections ψ supported in the *interior* of $\Omega(r)$:

$$\mathrm{Re}\int \mu_0(F, \partial_s\psi)\mathrm{d}s\mathrm{d}t = \mathrm{Re}\int -\mu_0(\partial_s F, \psi)\mathrm{d}s\mathrm{d}t,$$

However, the above holds even if ψ is non-zero along the boundary $\Omega(r) \cap \mathbb{R}$. To see why, observe that

$$\mathrm{Re} \int_{\Omega(r)} \mu_0(\partial_s F, \psi) \mathrm{d}s\mathrm{d}t = \mathrm{Re} \lim_{\delta \to 0} \int_{\Omega(r)} \mu_0(\partial_s F, \beta(t/\delta)\psi) \mathrm{d}s\mathrm{d}t,$$

where $\beta : [0, \infty) \to [0, 1]$ vanishes near 0 and equals 1 on $[1, \infty)$. Since $\beta(t/\delta)$ is independent of s, we can integrate by parts and conclude

$$\begin{aligned} \mathrm{Re} \int_{\Omega(r)} \mu_0(\partial_s F, \psi) \mathrm{d}s\mathrm{d}t &= \mathrm{Re} \lim_{\delta \to 0} \int_{\Omega(r)} \mu_0(F, \beta(t/\delta)\partial_s \psi) \mathrm{d}s\mathrm{d}t \\ &= \mathrm{Re} \int_{\Omega(r)} \mu_0(F, \partial_s \psi) \mathrm{d}s\mathrm{d}t. \end{aligned}$$

In particular, this observation applied to (6.10) yields

$$\mathrm{Re} \int \mu_0(\partial_s w, (-\partial_s + i\partial_t)\varphi) \mathrm{d}s\mathrm{d}t = \mathrm{Re} \int \mu_0(\partial_s F, \varphi) \mathrm{d}s\mathrm{d}t,$$

which implies that $(\partial_s + i\partial_t)\partial_s w = \partial_s F$ still holds weakly.

Finally, we explain how to choose the approximations $u_n \to u$ so that (i) and (ii) hold. First we extend u as an L^q distribution to $D(1)$ by $E(u)(s, -t) = \bar{u}(s, t)$ for $t \leq 0$. This can be defined in the sense of distributions as

$$\langle E(u), \varphi \rangle = \mathrm{Re} \int_{D(1) \cap \Omega} \mu_0(u(s, t), \varphi(s, t) + \bar{\varphi}(s, -t)) \mathrm{d}s\mathrm{d}t.$$

Let Φ be a radially symmetric bump function of unit mass supported in $D(1)$, and let $\Phi_n(s, t) = \Phi(sn, tn)$. Then define $u_n = \Phi_n * E(u)$. Clearly (i) holds. It can be shown that

$$\langle \Phi_n * E(u), (-\partial_s + i\partial_t)\varphi \rangle = \langle E(u), (-\partial_s + i\partial_t)(\Phi_n * \varphi) \rangle = \langle E(F), (\Phi_n * \varphi) \rangle.$$

This uses the distributional definition of $E(u)$ and the assumption that $(\partial_s + i\partial_t)u = F$ weakly. It also uses the fact that convolution commutes with $\partial_s + i\partial_t$ (as it is a differential operator with constant coefficients).

We therefore conclude that L^q size of $(\partial_s + i\partial_t)(\Phi_n * E)(u)$ is bounded by the L^q size of F. This proves (ii). We observe that since Φ_n is a radially symmetric and $u(s, -t) = \bar{u}(s, t)$, u_n must take real values along the real axis. This completes the proof. $\square$

Note that a consequence of the above proof is the following smooth approximation result:

Proposition 6.14. Let $\Omega(r) = \bar{\mathbb{H}} \cap D(r)$ and $q > 1$. Suppose that $u \in L^q(\Omega(1), \mathbb{C}^n)$ has the property that $\partial_s u + i\partial_t u = F$ holds weakly for some $F \in L^q(\Omega(1), \mathbb{C}^n)$. Then for any $r < 1$, the doubling $E(u)$ lies in $W^{1,q}(D(r))$ and there is a family of smooth functions u_n on $\Omega(r)$ taking real values on $\partial\Omega(r)$ so that $u_n \to u$ in $W^{1,q}(\Omega(r))$.

Proof. Let $E(u)$ be the doubling of u, as in the previous proof, and recall that $\Phi_n * E(u)$ converges to $E(u)$ in $L^q(D(r'))$ and is bounded in $W^{1,q}(D(r'))$. As we argued above, this implies that some subsequence of $\Phi_n * E(u)$ converges to $E(u)$ in the weak topology for $W^{1,q}(\Omega(r'))$. In particular $E(u)$ is in $W^{1,q}(D(r'))$. Basic properties of convolutions ensure that $\Phi_n * E(u)$ converges to $E(u)$ in the $W^{1,q}(D(r))$ norm. Thus we can set $u_n = \Phi_n * E(u)$, as desired. □

Remark 6.15. Let $u \in W^{1,q}(\Omega(1), \mathbb{C}^n, \mathbb{R}^n)$, i.e., $E(u) \in W^{1,q}(D(1), \mathbb{C}^n, \mathbb{R}^n)$. Then $(\partial_s + i\partial_t)\Phi_n * E(u) = F_n$ converges to some element $F \in L^q$ (in the sense of distributions). We claim that

$$\partial_s u + i\partial_t u = F \text{ holds weakly.}$$

This is a sort of converse to the above proposition. Indeed, if φ takes real-values along the boundary, we compute

$$\langle u, -\partial\varphi\rangle = \lim_{n\to\infty}\langle \Phi_n * E(u), -\partial\varphi\rangle = \lim_{n\to\infty}\langle \bar{\partial}\Phi_n * E(u), \varphi\rangle = \langle F, \varphi\rangle,$$

as desired.

Remark 6.16. If $u \in W^{1,p}(\Omega(1), \mathbb{C}^n)$ with $p > 2$, then u has well-defined boundary values. Suppose that u takes real values along the boundary. We will show that $E(u) \in W^{1,p}_{\mathrm{loc}}(D(1), \mathbb{C}^n)$. Let φ be a test function taking real values along the boundary. Let $h : \bar{\mathbb{H}} \to [0,1]$ be a function which (a) vanishes on $\mathbb{R} \times [0,1]$, (b) which equals 1 on $\mathbb{R} \times [2,\infty)$ and (c) which depends only on the t coordinate. Let $F = \bar{\partial}u$ (an L^p distribution). We compute:

$$\langle u, -\partial\varphi\rangle = \lim_{\sigma\to\infty}\langle h(\sigma t)u, -\partial\varphi\rangle = \lim_{\sigma}[\sigma\langle \bar{\partial}(h)(\sigma t)u, \varphi\rangle + \langle h(\sigma t)F, \varphi\rangle].$$

Note that $\bar{\partial}(h)(\sigma t)$ is concentrated on a region R_σ of area at most σ^{-1}. Since $\bar{\partial}(h)$ is purely imaginary, we can write

$$\langle \bar{\partial}(h)(\sigma t)u, \varphi\rangle = \langle \bar{\partial}(h)(\sigma t)\mathrm{Im}(u), \mathrm{Re}(\varphi)\rangle + \langle \bar{\partial}(h)(\sigma t)\mathrm{Re}(u), \mathrm{Im}(\varphi)\rangle.$$

Our discussion of R_σ and $\bar{\partial}(h)$ implies that

$$\left|\sigma\langle \bar{\partial}(h)(\sigma t)u, \varphi\rangle\right| \leq C(\sup_{R_\sigma}|\mathrm{Re}(\varphi)|\,|\mathrm{Im}(u)| + |\mathrm{Re}(u)|\,|\mathrm{Im}(\varphi)|).$$

Because $\mathrm{Im}(u)$ and $\mathrm{Im}(\varphi)$ are both continuous and vanish on the boundary, we can take the limit as $\sigma \to \infty$ and conclude that $\lim_\sigma \sigma\langle \bar{\partial}(h)(\sigma t)u, \varphi\rangle$ vanishes. We are left with

$$\langle u, -\partial\varphi\rangle = \langle F, \varphi\rangle$$

This says that $\bar{\partial}u = F$ weakly. As a consequence of Proposition 6.14 we conclude that $E(u) \in W^{1,p}(D(r))$ and u can be approximated in $W^{1,p}(D(r))$ by smooth functions taking real values along the boundary. This completes the proof.

6.4.3. *Injectivity estimates for translation invariant operators.* The next result concerns various estimates for operators of the form

$$u \mapsto \partial_s u + i\partial_t u + S(t)u = \partial_s u - Au$$

on the infinite strip $\mathbb{R} \times [0,1]$ with A a non-degenerate asymptotic operator. See [**Sal97**, Lemma 2.4], [**Wen20**, §4.4], and [**Sch95**, §3.1.2] for similar results for the infinite cylinder.

Proposition 6.17. Let $D(u) = \partial_s u - Au$ on the infinite strip $\mathbb{R} \times [0,1]$, where A is a non-degenerate asymptotic operator. Let $\|-\|$ denote the L^2 norm over $[0,1]$.

There exist constants $c_1, c_2, c_{3,p}$ so that, for all $u \in C_0^\infty(\mathbb{R} \times [0,1], \mathbb{C}^n, \mathbb{R}^n)$, we have

(i) $\displaystyle\int_{\mathbb{R}} \|u\|^2 + \|\partial_s u\|^2 + \|\partial_t u\|^2 \, \mathrm{d}s \leq c_1 \int_{\mathbb{R}} \|D(u)\|^2 \, \mathrm{d}s$,

(ii) $\displaystyle\int_{\mathbb{R}} \|u\|^p \, \mathrm{d}s \leq c_2^p \int_{\mathbb{R}} \|D(u)\|^p \, \mathrm{d}s$,

(iii) $\|u\|_{W^{1,p}(\mathbb{R}\times[0,1])} \leq c_{3,p} \|D(u)\|_{L^p(\mathbb{R}\times[0,1])}$, for $p \geq 2$.

The same result holds with $[0,1]$ replaced by $\mathbb{R}/\mathbb{Z}$.

Remark 6.18. Before we prove the theorem, we wish to make a few remarks.

(1) All of these estimates roughly measure the injectivity of D.

(2) The results are proved for smooth functions with compact support, although they imply estimates on various Banach space completions of C_0^∞ by taking smooth approximations. For instance, the reflection plus convolution technique used in the proof of Lemma 6.13 can be used to approximate $u \in W^{1,p}(\mathbb{R} \times [0,1], \mathbb{C}^n, \mathbb{R}^n)$ by smooth u_n taking real values along the boundary.

(3) Note that (i) is (iii) in the case $p = 2$. After we prove the proposition, we will be able to upgrade (iii) to include the case $q < 2$. See Theorem 6.20.

(4) Note that (ii) can be considered as an estimate on a mixed $(2,p)$ norm.

(5) We will give an elementary proof of (i), which is similar to the one given in [**Sch95**]. See [**Wen20**] for an alternate proof of (i) which considers the Fourier transformation in the s-variable.

(6) Our proofs of (ii) and (iii) are directly inspired by [**Sal97**]. The proof of (ii) will use the spectral properties of A proved in Proposition 6.8. See [**Sch95**] for an alternative proof of (iii).

Proof (of Proposition 6.17)**.** Suppose that $D(u) = \eta$, i.e., $\partial_s u = Au + \eta$. To prove (i), the idea is to consider the quantity $\gamma(s) = \|u(s,t)\|^2 = \langle u, u\rangle$, where $\langle -, -\rangle$ denotes the real inner product on $L^2([0,1],\mathbb{C}^n)$. We differentiate $\gamma(s)$ twice:

$$\begin{aligned}\gamma''(s) &= \langle \partial_s u, \partial_s u\rangle + \langle u, \partial_s\partial_s u\rangle = \|\partial_s u\|^2 + \langle u, \partial_s(Au+\eta)\rangle \\ &= \|\partial_s u\|^2 + \langle Au, \partial_s u\rangle + \partial_s\langle u,\eta\rangle - \langle \partial_s u, \eta\rangle \\ &= \|\partial_s u\|^2 + \|Au\|^2 + \langle Au,\eta\rangle + \partial_s\langle u,\eta\rangle - \langle \partial_s u,\eta\rangle.\end{aligned}$$

Here we have used the fact that $\partial_s Au = A\partial_s u$ and $\langle f, Ag\rangle = \langle Af, g\rangle$. We integrate this equality over $\mathbb{R}$. Since u is smooth and compactly supported the integrals of $\gamma''(s)$ and $\partial_s\langle u,\eta\rangle$ both vanish. We are left with:

$$\int \|\partial_s u\|^2 + \|Au\|^2 \,\mathrm{d}s = \int \langle \partial_s u - Au, \eta\rangle \mathrm{d}s.$$

Using Cauchy-Schwarz and $2ab \le a^2 + b^2$ we have

$$\begin{aligned}\int \langle \partial_s u - Au, \eta\rangle \mathrm{d}s &\le \int \|\partial_s u\|\,\|\eta\| + \|Au\|\,\|\eta\|\,\mathrm{d}s \\ &\le \frac{1}{2}\int \|\partial_s u\|^2 + \|Au\|^2 \,\mathrm{d}s + \int \|\eta\|^2 \,\mathrm{d}s.\end{aligned}$$

Recalling that $D(u) = \eta$, it follows that

$$\int \|\partial_s u\|^2 + \|Au\|^2 \,\mathrm{d}s \le 2\int \|D(u)\|^2 \,\mathrm{d}s.$$

Finally, using the fact that $A : W^{1,2}([0,1],\mathbb{C}^n,\mathbb{R}^n) \to L^2(\mathbb{C}^n)$ is an isomorphism, we conclude a constant $c \ge 1$ so that $\|u\|^2 + \|\partial_t u\|^2 \le c\,\|Au\|^2$, and hence

$$\int \|u\|^2 + \|\partial_s u\|^2 + \|\partial_t u\|^2 \,\mathrm{d}s \le 2c\int \|D(u)\|^2 \,\mathrm{d}s,$$

as desired. This completes the proof of (i).

For (ii) we use the spectral properties of A. Let $E_\pm$ denote the splitting of $L^2([0,1],\mathbb{C}^n)$ into positive and negative eigenspaces of A. The operator $\exp(-sA)$ converges on E_+ for $s \ge 0$ while the operator $\exp(-sA)$ converges on E_- for $s \le 0$.

We can decompose $u = u_+ + u_-$ where $u_+(s,-) \in E_+$ and $u_-(s,-) \in E_-$ for all s. It is straightforward to show that

$$\begin{aligned}&\text{for } s \ge 0\text{:} \quad \partial_s(\exp(-sA)u_+(s+s_0,t)) = \exp(-sA)\eta_+(s_0+s,t), \\ &\text{for } s \le 0\text{:} \quad \partial_s(\exp(-sA)u_-(s+s_0,t)) = \exp(-sA)\eta_-(s_0+s,t),\end{aligned}$$

where $\eta_\pm = \partial_s u_\pm - Au_\pm$. Integrate the first ODE over $[0,\infty)$ and integrate the second ODE over $(-\infty, 0]$, concluding that

$$u_+(s_0,t) = -\int_0^\infty \exp(-sA)\eta_+(s_0+s,t)\mathrm{d}s, \tag{6.11}$$
$$u_-(s_0,t) = \int_{-\infty}^0 \exp(-sA)\eta_-(s_0+s,t)\mathrm{d}s.$$

Following [**Sal97**, Lemma 2.4], the idea is now to interpret this as a convolution $u_\pm = K_\pm * \eta_\pm$, and then apply Young's convolution inequality to conclude (ii).

Here are the details of the argument. First, we show the mixed $(2,p)$ norm satisfies a variational definition:

$$\left[\int_{\mathbb{R}} \|u\|^p \,\mathrm{d}s\right]^{1/p} = \sup\left\{\int_{\mathbb{R}} \langle u, g\rangle \mathrm{d}s : \int_{\mathbb{R}} \|g\|^q \,\mathrm{d}s = 1,\ \text{where } p^{-1}+q^{-1}=1\right\}.$$

It is easy to show that $\geq$ holds, and to show $\leq$ it suffices to prove it when the left hand side is 1. In this case we can simply take $g = \|u\|^{p-2} u$ (if $p>2$ this is fine, if $p<2$ then we can take a sequence g approximating $\|u\|^{p-2} u$).

We fix g and compute

$$\int_{\mathbb{R}} \langle u_+(s_0), g(s_0)\rangle \mathrm{d}s_0 = -\int_{\mathbb{R}}\int_0^\infty \langle \exp(-sA)\eta_+(s_0+s), g(s_0)\rangle \mathrm{d}s\mathrm{d}s_0.$$

It is straightforward to check that

$$\langle \exp(-sA)\eta_+(s_0+s), g(s_0)\rangle \leq e^{-s\lambda^+_{\min}} \|\eta_+(s_0+s)\| \, \|g(s_0)\|,$$

where $\lambda^+_{\min}$ is the smallest positive eigenvalue.

Switching the order of integration and using Hölder's inequality yields:

$$\left|\int_{\mathbb{R}}\int_0^\infty \langle \exp(-sA)\eta_+(s_0+s), g(s_0)\rangle \mathrm{d}s\mathrm{d}s_0\right| \leq \int_0^\infty e^{-s\lambda^+_{\min}}\mathrm{d}s \left[\int_{\mathbb{R}} \|\eta_+\|^p \,\mathrm{d}s_0\right]^{1/p}.$$

As a consequence the variational definition of the mixed $2,p$ norm implies that

$$\left[\int_{\mathbb{R}} \|u_+(s)\|^p \,\mathrm{d}s\right]^{1/p} \leq \frac{1}{\lambda^+_{\min}} \left[\int_{\mathbb{R}} \|\eta_+\|^p \,\mathrm{d}s\right]^{1/p}.$$

A similar argument shows that

$$\left[\int_{\mathbb{R}} \|u_-(s)\|^p \,\mathrm{d}s\right]^{1/p} \leq \frac{-1}{\lambda^-_{\max}} \left[\int_{\mathbb{R}} \|\eta_-\|^p \,\mathrm{d}s\right]^{1/p},$$

where $\lambda^-_{\max}$ is the largest negative eigenvalue.

Using the fact that $\|\eta_\pm\| \leq \|\eta\|$ we conclude that

$$\int_{\mathbb{R}} \|u\|^p \,\mathrm{d}s \leq (\frac{1}{\lambda^+_{\min}} - \frac{1}{\lambda^-_{\max}})^p \int_{\mathbb{R}} \|D(u)\|^p \,\mathrm{d}s.$$

This proves (ii).

To prove (iii) we again follow [**Sal97**, Lemma 2.4]. Fix $p > 2$. Let $\Omega(r) = [-r, r] \times [0, 1]$. It is straightforward to apply Theorem 6.10 and Sobolev embedding [**MS12**, Theorem B.1.11] to conclude constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ so that

$$\begin{aligned}
\|u\|_{W^{1,p}(\Omega(1))} &\leq \kappa_1(\|D(u)\|_{L^p(\Omega(1.5))} + \|u\|_{L^p(\Omega(1.5))}),\\
\|u\|_{L^p(\Omega(1.5))} &\leq \kappa_2 \|u\|_{W^{1,2}(\Omega(1.5))},\\
\|u\|_{W^{1,2}(\Omega(1.5))} &\leq \kappa_3(\|D(u)\|_{L^2(\Omega(2))} + \|u\|_{L^2(\Omega(2))}),\\
\|D(u)\|_{L^2(\Omega(2))} &\leq \kappa_4 \|D(u)\|_{L^p(\Omega(2))}.
\end{aligned}$$

The constant κ_4 can be explicitly computed as $4^{1-1/p}$. Combining these yields

$$\|u\|_{W^{1,p}(\Omega(1))} \leq \kappa_1(1 + \kappa_2\kappa_3\kappa_4) \|D(u)\|_{L^p(\Omega(1.5))} + \kappa_1\kappa_2\kappa_3 \|u\|_{L^2(\Omega(2))}.$$

Using $(a+b)^p \leq 2^p(a^p + b^p)$ we conclude

$$\|u\|^p_{W^{1,p}(\Omega(1))} \leq 2^p\kappa_1^p(1 + \kappa_2\kappa_3\kappa_4)^p \|D(u)\|^p_{L^p(\Omega(2))} + (\kappa_1\kappa_2\kappa_3)^p \|u\|^p_{L^2(\Omega(2))}.$$

It is straightforward to compute that

$$\|u\|^p_{L^2(\Omega(2))} = [\int_{-2}^{2} \|u\|^2 \,\mathrm{d}s]^{p/2} \leq 4^{p/2-1} \int_{-2}^{2} \|u\|^p \,\mathrm{d}s.$$

The above holds with $\Omega(r)$ replaced by $2k + \Omega(r)$ for the same constants since D is translation invariant. Hence we conclude that there is a constant C so that

$$\|u\|^p_{W^{1,p}(2k+\Omega(1))} \leq C(\|D(u)\|^p_{L^p(2k+\Omega(2))} + \int_{2k-2}^{2k+2} \|u\|^p \,\mathrm{d}s).$$

Summing over all $k \in \mathbb{Z}$ yields

$$\|u\|^p_{W^{1,p}(\mathbb{R}\times[0,1])} \leq 2C(\|D(u)\|^p_{L^p(\mathbb{R}\times[0,1])} + \int_{\mathbb{R}} \|u\|^p \,\mathrm{d}s).$$

The factor of 2 is because the domains $2k + \Omega(2)$ cover $\mathbb{R} \times [0, 1]$ "twice over." Finally, using part (ii), we conclude that

$$\|u\|^p_{W^{1,p}(\mathbb{R}\times[0,1])} \leq 2C(1 + c_2^p) \|D(u)\|^p_{L^p(\mathbb{R}\times[0,1])}.$$

Setting $c_{3,p} = (2C(1 + c_2^p))^{1/p}$ completes the proof. □

The formula (6.11) can be used to prove the following regularity result. We still assume that $D = \partial_s - A$ is a translation invariant operator on the infinite strip or cylinder. As in the proof of Proposition 6.17 we assume that $\lambda^-_{\max} < 0 < \lambda^+_{\min}$ are the maximal negative and minimal positive eigenvalues of A.

Lemma 6.19. Let $q > 1$. If $u \in L^q$ and $D(u) = \eta$ for smooth η with compact support, then u is smooth. Moreover, for all $k, \ell \in \mathbb{N}$ there are constants $C^+_{k,\ell}$ and $C^-_{k,\ell}$ depending on η so that

$$\begin{aligned} \left|\partial^k \partial^\ell u(s,t)\right| &\leq C^+_{k,\ell} e^{\lambda^-_{\max} s} \text{ as } s \to +\infty, \\ \left|\partial^k \partial^\ell u(s,t)\right| &\leq C^-_{k,\ell} e^{\lambda^+_{\min} s} \text{ as } s \to -\infty. \end{aligned} \tag{6.12}$$

Both estimates are of exponential decay type. In particular, $u \in W^{k,q}$ for all k and all q.

Proof. Let $\Omega(r) = [-r, r] \times [0, 1]$. By the local elliptic regularity result (Lemma 6.13) we know that u is smooth, and that it satisfies elliptic estimates of the form

$$\|u\|_{W^{k+1,q}(s+\Omega(1))} \leq c(\|\eta\|_{W^{k,q}(s+\Omega(2))} + \|u\|_{L^q(s+\Omega(2))}).$$

We can take c to be independent of s since D is translation invariant. In particular, it is clear that the $L^2([0,1])$ size $\|u(s,-)\|$ decays as $s \to \infty$.

Decompose $u = u_+ + u_-$. It is straightforward to show that $u_\pm$ are still elements of $C^\infty(\mathbb{R}, W^{1,2}([0,1], \mathbb{C}^n, \mathbb{R}^n))$. As in the proof of Proposition 6.17, we think of the equation $\partial_s u_+ - Au_+ = \eta_+$ as an ordinary differential equation which we can explicitly solve:

$$\begin{aligned} &\partial_s(\exp(-sA)u_+(s_0 + s)) = \exp(-sA)\eta(s_0 + s) \\ \implies &\exp(-NA)u_+(s_0 + N) - u_+(s_0) = \int_0^N \exp(-sA)\eta(s_0 + s)\, \mathrm{d}s. \end{aligned}$$

Taking the limit as $N \to \infty$ and using the fact that $\lim_{N\to\infty} \|u_+(s_0 + N)\| = 0$ we conclude that

$$u_+(s_0, t) = -\int_0^\infty \exp(-sA)\eta_+(s_0 + s, t)\, \mathrm{d}s.$$

A similar argument shows that the other equation in (6.11) also holds, and hence we have:

$$u(s_0, t) = \int_{-\infty}^0 \exp(-sA)\eta_-(s_0 + s, t)\, \mathrm{d}s - \int_0^\infty \exp(-sA)\eta_+(s_0 + s, t)\, \mathrm{d}s.$$

Suppose that η is supported in $[-R, R]$. Then for $s_0 < -R$, the first integral always vanishes, and the second integrand is supported on the region where $s > -s_0 - R$ and so we have:

$$\|u(s_0, -)\| \leq e^{(s_0+R)\lambda^+_{\min}} \int_{-\infty}^\infty \|\eta_+\|\, \mathrm{d}s = C(\eta_-, R)e^{s_0\lambda^+_{\min}} \qquad (\text{as } s_0 \to -\infty).$$

A similar deduction proves that

$$\|u(s_0, -)\| \leq C(\eta_+, R)e^{s_0\lambda^-_{\max}} \qquad (\text{as } s_0 \to +\infty).$$

Now, by simply integrating the norm $\|u(s,-)\|$ over $s \in [s_0-2, s_0+2]$, we conclude that

$$\begin{aligned} \|u\|_{L^2(s+\Omega(2))} &\leq C_2 e^{s\lambda^-_{\max}} \text{ as } s \to +\infty \\ \|u\|_{L^2(s+\Omega(2))} &\leq C_2 e^{s\lambda^+_{\max}} \text{ as } s \to -\infty. \end{aligned} \tag{6.13}$$

Using the elliptic estimates for $q = 2$, we conclude that the $W^{k,2}$ size of u on $s + \Omega(1)$ also decays exponentially like (6.13). Since the C^ℓ size is controlled by the $W^{k+2,2}$ size, we ultimately conclude the desired result (6.12). $\square$

We can upgrade the injectivity estimates to the following important result:

Theorem 6.20. Let $D = \partial_s u - Au$ with A a non-degenerate asymptotic operator. Let $q > 1$. The induced map $D : W^{1,q}(\mathbb{R}\times[0,1], \mathbb{C}^n, \mathbb{R}^n) \to L^q(\mathbb{R}\times[0,1], \mathbb{C}^n)$ is an isomorphism.

Proof. First we prove the case when $q \geq 2$. Part (iii) of Proposition 6.17 implies that D is injective and has closed image. Thus it suffices to prove that the image of D is dense. If η is a smooth function with compact support, then (6.11) gives an explicit formula for some u satisfying $D(u) = \eta$. As in the proof of Lemma 6.19, u is smooth and the formula (6.11) implies that u and its derivatives decay exponentially as $s \to \pm\infty$, hence $u \in W^{1,p}$.

Next we prove the case when $q < 2$. We follow the argument outlined in [**Sal97**, Exercise 2.5]. The idea is to prove an injectivity estimate for $D : L^q \to W^{-1,q}$, and then upgrade this to a $D : W^{1,q} \to L^q$ injectivity estimate.

By definition, we set $W^{-1,q} = (W^{1,p})^*$ where p is Hölder dual to q, and

$$\|u\|_{-1,q} = \sup_{\|\varphi\|_{1,p}=1} \langle u, \varphi\rangle.$$

Let $D^* = -\partial_s - A$. By the above results (applied to $-D^*$) we conclude D^* is an isomorphism $W^{1,p} \to L^p$. Thus

$$c^{-1}\|u\|_{L^q} \leq \sup_{\|\varphi\|_{1,p}=1} \langle u, D^*(\varphi)\rangle \leq c\,\|u\|_{L^q}.$$

Observe that $D^* = -\partial_s - A$ is the formal adjoint to D, and hence (using distributional definitions) we have:

$$\|u\|_{L^q} \leq c\,\|D(u)\|_{-1,q}.$$

In particular, if $v \in W^{1,q}$, then we can apply the above to $u = \partial_s v$ and conclude that

$$\|\partial_s u\|_{L^q} \leq c\,\|D(\partial_s u)\|_{-1,q}.$$

Now it is clear that, in the sense of distributions, we have $D(\partial_s u) = \partial_s D(u)$. We claim that

$$\|\partial_s D(u)\|_{-1,q} \leq c_2\,\|D(u)\|_{L^q}.$$

This is easy to see using the above variational definition of the $W^{-1,q}$ norm. Thus we conclude that

$$\|\partial_s u\|_{L^q} \leq c_3 \|D(u)\|_{L^q}.$$

It is clear that $W^{-1,q}$ norm is less than the L^q norm, hence $\|u\|_{L^q} \leq c\|D(u)\|_{L^q}$. It then follows easily that $\|\partial_t u\|_{L^q} \leq c_4 \|D(u)\|_{L^q}$, and so we conclude the desired injectivity estimate

$$\|u\|_{W^{1,q}} \leq c \|D(u)\|_{L^q}.$$

It follows easily that $D(u)$ has closed range and hence it suffices to prove that the image of D is dense. The arguments given in (6.11) and Lemma 6.19 show that we can (explicitly) solve for compactly supported smooth functions and the solutions are certainly of class $W^{1,q}$. Thus D is surjective. This completes the proof that D is an isomorphism. □

6.5. Proof of the Fredholm property

The main result of this section is the following:

Proposition 6.21. Let $p > 1$ and let D be an asymptotically non-degenerate Cauchy-Riemann operator for the data $(\Sigma, \Gamma_\pm, E, F, C, [\tau])$. Then the induced maps

$$D : W^{1,p}(E, F) \to L^p(\Lambda^{0,1} \otimes E) \text{ and } D^* : W^{1,p}(E, F) \to L^p(\Lambda^{0,1} \otimes E)$$

are Fredholm.

Similar arguments can be found in [**Sal97**, §2.3], [**Sch95**, Proposition 3.1.30], and [**Wen20**, §4.5].

Proof. Let φ_ρ be a cutoff function supported in the ends which equals 0 on $\Sigma(\rho-1)$ and equals 1 on $C(\rho)$. We can choose φ_ρ so that its derivatives are bounded as $\rho \to \infty$.

We observe that

$$D(\varphi_\rho u) = \partial_s(\varphi_\rho u) - A(\varphi_\rho u) + \Delta(s)\varphi_\rho u,$$

where $\Delta(s)$ is a lower order term which converges to 0 as $s \to \pm\infty$. We know that $\partial - A : W^{1,p} \to L^p$ is an isomorphism and so $\|\varphi_\rho u\|_{W^{1,p}} \leq C\|(\partial_s - A)(\varphi_\rho u)\|$ for some C. We estimate

$$\begin{aligned} \|\varphi_\rho u\|_{W^{1,p}} &\leq C(\|D(\varphi_\rho u)\|_{L^p} + \|\Delta(s)\varphi_\rho u\|_{L^p}) \\ &\leq C(\|D(u)\|_{L^p} + \|\Delta(s)\varphi_\rho u\|_{L^p} + \left\|\bar\partial\varphi_\rho \cdot u\right\|_{L^p}) \\ \implies \|\varphi_\rho u\|_{L^p} &\leq C'(\|D(u)\|_{L^p} + \|u\|_{L^p(\Sigma(\rho))}), \end{aligned}$$

where we pick ρ large enough so that $C|\Delta(s)| < 0.5$ on the support of φ_ρ. We also use that $\bar\partial\varphi_\rho$ is supported in $\Sigma(\rho)$.

Next we combine the local elliptic estimates from 6.13 (to finitely many disks covering $\Sigma(\rho)$) and conclude some constant $C(\rho)$ so that

$$\|(1-\varphi_\rho)u\|_{W^{1,p}} \leq C(\rho)(\|Du\|_{L^p} + \|u\|_{L^p(\Sigma(\rho+1))}).$$

Combining our two estimates (and updating the constant) yields

$$\|u\|_{W^{1,p}} \leq C(\|Du\|_{L^p} + \|u\|_{L^p(\Sigma(\rho+1))}). \tag{6.14}$$

Crucially, ρ does not depend on u. Since $W^{1,p} \to L^p(\Sigma(\rho+1))$ (inclusion followed by restriction) is a compact operator, we conclude from (6.14) that D is semi-Fredholm; i.e., has closed image and finite dimensional kernel. See [**MS12**, Appendix A] for the argument. The same argument shows that D^* is semi-Fredholm.

Now suppose that D were not surjective. Since the image of D is closed, we can apply the Hahn-Banach theorem to find $w \in L^q(\Lambda^{0,1} \otimes E)$ so that $\langle D(u), w\rangle = 0$ for all $u \in W^{1,p}(E,F)$ and $w \neq 0$. But then w is smooth and takes boundary values in F^* by the local regularity results. Moreover, in the ends we have $(\partial_s + A)w = \Delta^* w$ which implies that $w \in W^{1,q}(\Lambda^{0,1} \otimes E, F^*)$ (using the injectivity estimates for $\partial_s + A$). We conclude that $w \in \ker D^*$. Since D^* is semi-Fredholm, its kernel is finite dimensional. This implies that $\operatorname{coker} D$ is finite dimensional, and this completes the proof that D is Fredholm. The same argument works for D^*. $\square$

Chapter 7

Conley-Zehnder indices and kernel gluing

In this section our goal is to prove that the index behaves additively under a gluing operation. See [**Sch95**, §3.2] for a similar argument.

Throughout this section we fix an asymptotic trivialization τ (i.e., fix τ_z for each $z \in \Gamma$). Suppose that D is an asymptotic operator on $(\Sigma, \Gamma_\pm, E, F)$ whose restriction to the cylindrical ends C_z equals $D = \partial_s - A_z$ with respect to τ_z and where each A_z is a non-degenerate asymptotic operator. We have shown that D is Fredholm. Moreover, it is clear that if D' has the same asymptotic operators A_z (in the same trivialization), then we can homotope D to D' while remaining in the space of Fredholm operators. Then the index of D will equal the index of D'. Therefore, the index depends only on the choice of non-degenerate asymptotic operators $z \mapsto A_z$ (and (Σ, Γ, E, F) of course).

Introduce the reference operator D^{al} whose restrictions to the cylindrical ends equals $\partial_s + i\partial_t + C$ with respect to the same trivialization τ. Here C is the matrix of complex conjugation, i.e., in each end we have $D^{\mathrm{al}}(u) = \partial_s u + i\partial_t u + \bar{u}$. The "al" stands for "anti-linear." The associated asymptotic operator is $A^{\mathrm{al}} = -i\partial_t - C$. In other words, D^{al} has all of its asymptotics equal to A^{al}.

In this section we will prove the following formula for index difference

$$\operatorname{ind}(D) - \operatorname{ind}(D^{\mathrm{al}}) = \sum_{z \in \Gamma_+} \mu_{\mathrm{CZ}}(A_z) - \sum_{z \in \Gamma_+} \mu_{\mathrm{CZ}}(A_z),$$

where $\mu_{\mathrm{CZ}}(A_z)$ is the *Conley-Zehnder* index of A_z, defined in §7.1 below. This formula determines how the index depends on changing asymptotic operators (i.e., we can compute $\operatorname{ind}(D_1) - \operatorname{ind}(D_2)$ for any pair D_1, D_2). In §8 we will prove that $\operatorname{ind}(D^{\mathrm{al}}) = n\mathrm{X} + \mu^\tau_{\mathrm{Mas}}(E, F)$, which will complete the proof of the index formula.

7.1. Conley-Zehnder indices as Fredholm indices

First we need to show that D^{al} is actually Fredholm. This follows from:

Lemma 7.1. For $\sigma > 0$, the reference operator $A^{\mathrm{al},\sigma} = -i\partial_t - \sigma C$ is non-degenerate.

Proof. We prove the strip case, leaving the (very similar) $\mathbb{R}/\mathbb{Z}$ case to the reader. Suppose $u : [0, 1] \to \mathbb{C}^n$ takes real values when $t = 0$ and $A^{\mathrm{al},\sigma}(u) = 0$. A straightforward computation shows that

$$\partial_t u_j = i\sigma\bar{u}_j \iff \partial_t(x_j + iy_j) = \sigma(y_j + ix_j) \implies u_j = x_j(0)(\cosh(\sigma t) + i\sinh(\sigma t)).$$

In particular, since $\sinh(\sigma t) > 0$ for $t > 0$, we cannot also have $u_j(1) \in \mathbb{R}^n$. This proves that $A^{\mathrm{al},\sigma}$ is non-degenerate. □

Fix a non-degenerate asymptotic operator A. We will define a special Cauchy-Riemann operator on the infinite strip/cylinder which will interpolate between $\partial_s - A^{\mathrm{al}}$ and $\partial_s - A$. Let $s \mapsto \beta(s)$ be a $[0,1]$-valued bump function which equals 0 on $(-\infty, 0]$ and 1 on $[1, \infty)$, and define:

$$D_A^{\mathrm{CZ}} := \partial_s - (1 - \beta(s))A^{\mathrm{al}} - \beta(s)A.$$

As a corollary to Lemma 7.1, the operator D_A^{CZ} is Fredholm. We define the *Conley-Zehnder index* of A as the Fredholm index of D_A^{CZ}:

$$\mu_{\mathrm{CZ}}(A) := \mathrm{ind}(D_A^{\mathrm{CZ}}).$$

It is clear that $\mu_{\mathrm{CZ}}(A)$ is independent of the choice of β used to define D_A^{CZ}, since any deformation of bump functions will keep D_A^{CZ} in the space of Fredholm operators.

Remark 7.2. See [**Flo89a**, page 595] for an argument which explains why $\mathrm{ind}(D_A^{\mathrm{CZ}})$ is the *spectral flow* of the path of self-adjoint operators $A(s) = (1 - \beta(s))A^{\mathrm{al}} + \beta(s)A$.

Note that, since $\partial_s - A^{\mathrm{al}}$ is an isomorphism $W^{1,p} \to L^p$ (Theorem 6.20) we conclude that $\mu_{\mathrm{CZ}}(A^{\mathrm{al}}) = 0$.

The main result of this section is:

Proposition 7.3. Let D, D^{al} be as above, i.e., for a fixed choice of trivialization τ and for each $z \in \Gamma$ the restrictions of D, D^{al} are

$$D = \partial_s - A_z, \qquad D^{\mathrm{al}} = \partial_s - A^{\mathrm{al}}.$$

Note that the operators D^{al} and A_z depend on the choice of triviliazation. We have

$$\mathrm{ind}(D) = \mathrm{ind}(D^{\mathrm{al}}) + \sum_{z \in \Gamma_+} \mu_{\mathrm{CZ}}(A_z) - \sum_{z \in \Gamma_-} \mu_{\mathrm{CZ}}(A_z).$$

Proof. The proposition follows from a kernel gluing lemma for stabilized operators (Lemma 7.4) proved in §7.3.1. □

The kernel gluing argument we will use is similar to the one used in [**Sch95**, §3.2]. See also [**FH93**, Proposition 9]. The rough idea is to deform D by a parameter ρ so that it equals a "glued" operator D^ρ obtained from D^{al} by gluing on the asymptotic operator $D_{A_z}^{\mathrm{CZ}}$ for each $z \in \Gamma$, as suggested by in Figure 1.

Note that at negative ends we actually need to glue $D^{\mathrm{CZ}}_{A_z}$ on "backwards." For this reason, we define:

$$D^{\mathrm{ZC}}_A := \partial_s - (1-\beta(s))A - \beta(s)A^{\mathrm{al}},$$

which interpolates from $\partial_s - A$ on the negative end to $\partial_s - A^{\mathrm{al}}$ at the positive end.

Our kernel gluing argument will imply two things:

(i) $\mathrm{ind}(D) = \mathrm{ind}(D^{\mathrm{al}}) + \sum_{z\in\Gamma_+} \mathrm{ind}(D^{\mathrm{CZ}}_A) + \sum_{z\in\Gamma_-} \mathrm{ind}(D^{\mathrm{ZC}}_A)$,

(ii) $\mathrm{ind}(D^{\mathrm{CZ}}_A) + \mathrm{ind}(D^{\mathrm{ZC}}_A) = 0 \implies \mathrm{ind}(D^{\mathrm{ZC}}_A) = -\mu_{\mathrm{CZ}}(A)$.

These results together imply Proposition 7.3.

Before we perform the gluing argument we will explain how to *stabilize* the relevant operators in order to make them surjective. This is the topic of the next subsection.

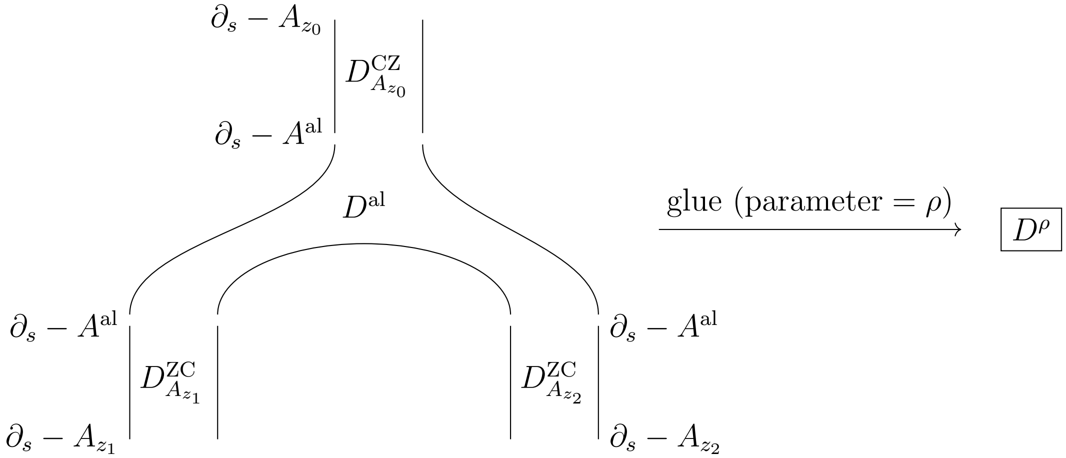


FIGURE 1. Gluing together operators D^{al}, D^{CZ}_A, and backwards versions D^{ZC}_A to form D^ρ, which can be deformed back to D through Fredholm operators. For large gluing parameter ρ, we will be able to relate the kernel of D^ρ to the kernels of $D^{\mathrm{CZ}}_{A_z}$, $D^{\mathrm{ZC}}_{A_z}$, and D^{al}.

7.2. Stabilizing Cauchy-Riemann operators

Let D be a Cauchy-Riemann operator on $(E, F, \Sigma, \Gamma, C, [t])$ as usual.

As we have seen in §6.5, the operator $D : W^{1,p}(E,F) \to L^p(\Lambda^{0,1}\otimes E)$ has a finite dimensional cokernel which can be identified with $\ker D^* \subset W^{1,p}(\Lambda^{0,1}\otimes E, F^*)$.

Pick a basis $c : \mathbb{R}^d \to \ker(D^*)$, considered as a map $c : \mathbb{R}^d \to L^p(\Lambda^{0,1}\otimes E)$.

$$\begin{bmatrix} D & c \end{bmatrix} : W^{1,p}(E,F)\oplus\mathbb{R}^d \to L^p(\Lambda^{0,1}\otimes E). \tag{7.1}$$

For our choice of c, this operator is surjective and its kernel is $\ker(D)\oplus 0$. Since $\ker(D)$ is finite dimensional, the above operator has a right inverse. Since having a right inverse is open in the norm topology, we can smoothly "cut-off" the cokernel elements $c_1, \cdots, c_d$ so that they vanish outside of $\Sigma(\rho_0)$ for ρ_0 sufficiently large (i.e., they vanish on the ends $C(\rho_0)$).

This leads us to the following definition: a *stabilized* operator for D is any surjective operator D_{st} of the form (7.1) where $d = \dim \operatorname{coker}(D)$ and the cokernel elements $c_1, \cdots, c_d$ are smooth and supported in $\Sigma(\rho_0)$ for some ρ_0. The preceding discussion shows that stabilized operators always exist.

By computing the Fredholm index of (7.1) when $c = 0$, we easily see that

$$\operatorname{ind}(D^{\text{st}}) = \operatorname{ind}(D) + d = \operatorname{ind}(D) + \dim \operatorname{coker}(D) = \dim \ker(D).$$

Since D^{st} is surjective, $\dim \ker(D^{\text{st}}) = \dim \ker(D)$, and hence

$$\ker D^{\text{st}} = \ker D \oplus 0. \tag{7.2}$$

7.3. The kernel gluing argument

Let D be a Cauchy-Riemann operator as above. Fix a single positive puncture z with asymptotic trivialization τ, and suppose that D is asymptotic to $\partial_s - A$ in the end C_z.

By perturbing D through the space of Fredholm operators, we may suppose that on C_z we have

$$D = \partial_s - (1 - \beta(s))A^{\text{al}} - \beta(s)A.$$

Here β is the bump function from before (i.e., 0 on $(-\infty, 0]$ and 1 on $[1, \infty)$). This local model is nice because it is the beginning of a family of Fredholm operators, namely

$$D^\rho = \partial_s - (1 - \beta(s - 3\rho))A^{\text{al}} - \beta(s - 3\rho)A.$$

We suppose that D^ρ is fixed away from C_z. Consequently, the index of D^ρ is constant since it is always Fredholm (its asymptotics are fixed).

Introduce the operator $D^- = \lim_{\rho\to\infty} D^\rho$ (pointwise limit). In other words, D^- agrees with D on the complement of C_z and equals $\partial_s - A^{\text{al}}$ on C_z.

Observe that the restriction of D^ρ to C_z is a translated copy of

$$D_+ := D_A^{\text{CZ}} = \partial_s - (1 - \beta(s))A^{\text{al}} - \beta(s)A.$$

We can therefore think of D^ρ as obtained by *gluing* D^+ to the positive end of D^-. See Figure 2.

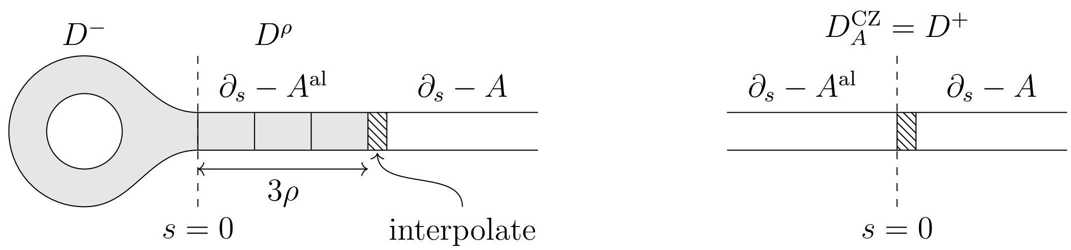


FIGURE 2. The relationship between D_-, D^ρ and D^+. We can think of D_- as the pointwise limit of D^ρ. However, D^ρ is always a translated (and truncated) version of D^+ on C_z.

To perform the actual gluing argument, we will need to stabilize the operators. Let $c = (c_1, \cdots, c_d)$ be cokernel elements for D^- and let $\gamma = (\gamma_1, \cdots, \gamma_\delta)$ be cokernel elements for D^+. We suppose that the c_j are supported in $\Sigma(\rho_0)$ and similarly the γ_k are supported where $|s| < \rho_0$. These choices define stabilized operators:

$$D^-_{\mathrm{st}} : W^{1,p}(E, F) \oplus \mathbb{R}^d \to L^p(\Lambda^{1,0} \otimes E) \qquad (\xi_1, a) \mapsto D^-(\xi_1) + \sum a_j c_j,$$

$$D^+_{\mathrm{st}} : W^{1,p}(\mathbb{C}^n, \mathbb{R}^n) \oplus \mathbb{R}^\delta \to L^p(\mathbb{C}^n) \qquad (\xi_2, b) \mapsto D^+(\xi_2) + \sum b_k \gamma_k.$$

Then, for $\rho > \rho_0$, we define:

$$\begin{aligned} D^\rho_{\mathrm{st}} &: W^{1,p}(E, F) \oplus \mathbb{R}^d \oplus \mathbb{R}^\delta \to L^p(\Lambda^{0,1} \otimes E) \\ &\text{by } (\xi, a, b) \mapsto D^\rho(\xi) + \sum a_j c_j + \sum b_k \gamma_k(s - 3\rho). \end{aligned} \tag{7.3}$$

Notice that D^ρ_{st} is well-defined since $\gamma_k(s - 3\rho)$ is supported in $C_z(2\rho)$ for $\rho > \rho_0$.

The following lemma establishes a relationship between $D^-_{\mathrm{st}}, D^\rho_{\mathrm{st}}$ and D^+_{st}.

Lemma 7.4 (Kernel gluing lemma)**.** For ρ sufficiently large,

(i) D^ρ_{st} is surjective,

(ii) $\dim\ker D^\rho_{\mathrm{st}} = \dim\ker D^-_{\mathrm{st}} + \dim\ker D^+_{\mathrm{st}}$.

7.3.1. *Consequences of the Kernel gluing lemma.* Before we give the proof we explain why Lemma 7.4 implies Proposition 7.3. First we observe that (i) and (ii) above imply

$$\mathrm{ind}(D^\rho_{\mathrm{st}}) = \mathrm{ind}(D^-_{\mathrm{st}}) + \mathrm{ind}(D^+_{\mathrm{st}}),$$

since all the operators are surjective. Using $\mathrm{ind}(D^\rho_{\mathrm{st}}) = \mathrm{ind}(D^\rho) + d + \delta$ and similar formulas for $\mathrm{ind}(D^\pm_{\mathrm{st}})$, we conclude

$$\mathrm{ind}(D) = \mathrm{ind}(D^\rho) = \mathrm{ind}(D^-) + \mathrm{ind}(D^+). \tag{7.4}$$

Once we recall the definitions of D^+, D^- and how they compare with D^{al} and D^{CZ}_A, we conclude Proposition 7.3 in the case when $\Gamma_+ = \{z\}$ and $\Gamma_- = \emptyset$.

More generally, we can apply Lemma 7.4 one time for each positive puncture and conclude that Proposition 7.3 holds when $\Gamma^- = \emptyset$.

There is an obvious variant of Lemma 7.4 in the case of a negative puncture z, where we consider the deformation

$$D^\rho = \partial_s - (1 - \beta(s + 3\rho))A - \beta(s + 3\rho)A^{\mathrm{al}},$$

defined for $s \le 0$. As above, we suppose D^ρ is fixed on the complement of C_z. The same gluing argument shows that $\mathrm{ind}(D^\rho)$ agrees with the sum of the indices of the operators

$$D^+ = \partial_s - A^{\mathrm{al}} \qquad D^- = \partial_s - (1 - \beta(s))A - \beta(s)A^{\mathrm{al}} =: D_A^{\mathrm{ZC}}.$$

Here D^+ extends to $\dot{\Sigma}$ (i.e., $D^\rho = D^+$ is fixed on the complement of C_z) while D^- is defined on an infinite strip or cylinder.

By performing these deformations at all punctures (one at a time), we ultimately conclude that

$$\mathrm{ind}(D) = \mathrm{ind}(D^{\mathrm{al}}) + \sum_{z \in \Gamma_+} \mathrm{ind}(D_A^{\mathrm{CZ}}) + \sum_{z \in \Gamma_-} \mathrm{ind}(D_A^{\mathrm{ZC}}). \tag{7.5}$$

Finally, consider the following family of operators on the infinite cylinder or strip:

$$D^\rho = \partial_s - (1 - \beta(s))A - \beta(s)(1 - \beta(s - 3\rho))A^{\mathrm{al}} - \beta(s)(\beta(s - 3\rho))A.$$

We can think of this as gluing D_A^{CZ} to the positive end of D_A^{ZC}. Indeed, this fits into the framework considered in Lemma 7.4, and so we conclude that

$$\mathrm{ind}(D^\rho) = \mathrm{ind}(D_A^{\mathrm{ZC}}) + \mathrm{ind}(D_A^{\mathrm{CZ}}).$$

It is clear that if we let ρ become very negative, then D^ρ agrees with $\partial_s - A$, which has Fredholm index 0 (by Theorem 6.20). Since the Fredholm index of D^ρ is constant as a function of ρ we must have

$$\mathrm{ind}(D_A^{\mathrm{ZC}}) = -\mathrm{ind}(D_A^{\mathrm{CZ}}) = -\mu_{\mathrm{CZ}}(A).$$

This combined with (7.5) completes the proof of Proposition 7.3.

7.3.1.1. *The proof of Lemma 7.4.* The argument has two parts: first we prove D^ρ_{st} is uniformly surjective, and second, compute the dimension of its kernel. The first part is used in the second part.

Proof. To prove that D^ρ_{st} is surjective, we will attempt to solve the equation $D^\rho_{\mathrm{st}}(\xi) = \eta$ for some $\eta \in L^p$. Fix three bump functions $b_1^\rho, b_2^\rho, b_3^\rho$, all supported in C_z by the formulas

$$b_1^\rho(s) = \beta(s/\rho) \qquad b_2^\rho(s) = \beta(1 - s/\rho) \qquad b_3^\rho(s) = \beta(2 - s/\rho),$$

as shown in Figure 3. By picking ρ large enough, we may suppose that $(c_1, \cdots, c_d)$ are supported where $b_2^\rho = 1$ and $(\gamma_1(s-3\rho), \cdots, \gamma_\delta(s-3\rho))$ are supported where $b_2^\rho = 0$. This assumption will simplify some calculations later on.

Let $\eta \in L^p(E)$ be some section. Since D^-_{st} has a bounded right inverse, we can find ξ_1 and $\mathfrak{c}_1 = \sum a_j c_j$ so that

$$D^-(\xi_1) + \mathfrak{c}_1 = b_2^\rho \eta.$$

Moreover we can achieve this so that $\|(\xi_1, \mathfrak{c}_1)\| = \|\mathfrak{c}_1\| + \|\xi_1\|_{W^{1,p}}$ is bounded by $C^- \|\eta\|_{L^p}$ for a fixed constant C^- (by fixing a bounded right inverse for D^-_{st}). Here $\|\mathfrak{c}_1\|$ is any norm on $\mathbb{R}^d$ (which we fix throughout the proof).

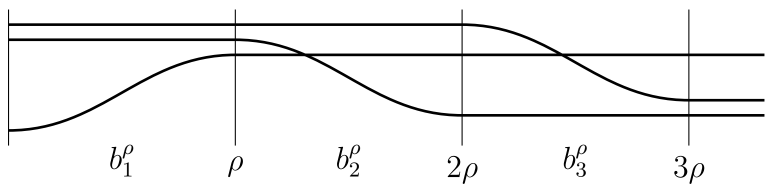


FIGURE 3. Three bump functions drawn with slight vertical offsets to better show their behavior.

Because of $b_3^\rho \mathfrak{c}_1 = \mathfrak{c}_1$ and $b_3^\rho b_2^\rho = b_2^\rho$, we have

$$D^-(b_3^\rho \xi_1) + \mathfrak{c}_1 - b_2^\rho \eta = D^-(b_3^\rho \xi_1) + b_3^\rho \mathfrak{c}_1 - b_3^\rho b_2^\rho \eta = \bar{\partial}(b_3^\rho) \otimes \xi_1.$$

Since $b_3^\rho \xi_1$ is supported in the region where $D^- = D^\rho$, we can rewrite the above as

$$D^\rho(b_3^\rho \xi_1) + \mathfrak{c}_1 = b_2^\rho \eta + (\bar{\partial} b_3^\rho) \otimes \xi_1.$$

Observe that $\Delta = \eta - b_2^\rho \eta$ is supported in the region where $s \geq \rho$. Since D^+_{st} is surjective, we can find ξ_2' and $\mathfrak{c}_2' = \sum b_k \gamma_k$ so that

$$D^+(\xi_2') + \mathfrak{c}_2' = \Delta(s + 3\rho, t).$$

We can achieve this with $\|\mathfrak{c}_2'\| + \|\xi_2'\|_{W^{1,p}} \leq C^+ \|\eta\|_{L^p}$ for a fixed constant C^+.

Let $\xi_2(s,t) = \xi_2(s - 3\rho, t)$ and $\mathfrak{c}_2(s,t) = \mathfrak{c}_2'(s - 3\rho, t)$. Since $b_1^\rho \Delta = \Delta$ and $b_1^\rho \mathfrak{c}_2 = \mathfrak{c}_2$, we conclude that

$$D^\rho(b_1^\rho \xi_2) + \mathfrak{c}_2 = \Delta + (\bar{\partial} b_1^\rho) \otimes \xi_2.$$

Consequently, we have

$$D^\rho(b_3^\rho \xi_1 + b_1^\rho \xi_2) + \mathfrak{c}_1 + \mathfrak{c}_2 = \eta + (\bar{\partial} b_3^\rho) \otimes \xi_1 + (\bar{\partial} b_1^\rho) \otimes \xi_2.$$

Observe that the derivatives of b_i^ρ are of order ρ^{-1}. We think of this as *approximately* solving $D^\rho_{\mathrm{st}}(b_3^\rho \xi_1 + b_1^\rho \xi_2, \mathfrak{c}_1, \mathfrak{c}_2) = \eta$. Indeed, we have just shown that for any η we can find $\xi, \mathfrak{c}_1, \mathfrak{c}_2$ so that

$$\|\xi\|_{W^{1,p}} + \|\mathfrak{c}_1\| + \|\mathfrak{c}_2\| \leq C \|\eta\|_{L^p} \text{ and } \|D^\rho_{\text{st}}(\xi, \mathfrak{c}_1, \mathfrak{c}_2) - \eta\|_{L^p} \leq C\rho^{-1} \|\eta\|_{L^p}, \tag{7.6}$$

for constants C independent of ρ.

The equation (7.6) implies that D^ρ_{st} is surjective for ρ large enough, as follows: pick ρ so $C\rho^{-1} < 1/2$. By (7.6) with $\eta := \eta - D^\rho_{\text{st}}(\xi, \mathfrak{c}_1, \mathfrak{c}_2)$ we obtain $\xi^1, \mathfrak{c}_1^1, \mathfrak{c}_2^1$ so that

$$\left\| D^\rho_{\text{st}}(\xi^1, \mathfrak{c}_1^1, \mathfrak{c}_2^1) - (\eta - D^\rho_{\text{st}}(\xi, \mathfrak{c}_1, \mathfrak{c}_2)) \right\|_{L^p} \leq \frac{1}{4} \|\eta\|_{L^p},$$

and $\|(\xi^1, \mathfrak{c}_1^1, \mathfrak{c}_2^1)\| \leq C2^{-1} \|\eta\|_{L^p}$. In other words, if we try to solve for the *error* arising from our first attempt to solve for η, then $D^\rho_{\text{st}}(\xi, \mathfrak{c}_1, \mathfrak{c}_2) + D^\rho_{\text{st}}(\xi^1, \mathfrak{c}_1^1, \mathfrak{c}_2^1)$ is a better approximation by a factor of $1/2$ than our initial attempt.

By repeating this process, we can find a sequence $\xi^n, \mathfrak{c}_1^n, \mathfrak{c}_2^n$ so that

$$\|(\xi^n, \mathfrak{c}_1^n, \mathfrak{c}_2^n)\| \leq C2^{-n} \|\eta\|_{L^p} \text{ and } \left\| \textstyle\sum_{j=0}^n D^\rho_{\text{st}}(\xi^j, \mathfrak{c}_1^j, \mathfrak{c}_2^j) - \eta \right\|_{L^p} \leq 2^{-n-1} \|\eta\|_{L^p}.$$

The above series then converges to an element in the preimage of η, as desired. This completes the proof that D^ρ_{st} is surjective for ρ sufficiently large. Moreover, we see that D^ρ_{st} actually has a right inverse which is bounded in norm by $2C$. This uniformly bounded right inverse will play a role later on.

Next we need to prove that $\dim\ker D^\rho_{\text{st}} = \dim\ker D^-_{\text{st}} + \dim\ker D^+_{\text{st}}$. First we will prove that:

$$\dim\ker D^\rho_{\text{st}} \leq \dim\ker D^-_{\text{st}} + \dim\ker D^+_{\text{st}}. \tag{7.7}$$

Suppose that $(\xi, \mathfrak{c}_1, \mathfrak{c}_2)$ lies in the kernel of D^ρ_{st}. Using the same bump functions from before, we compute:

$$D^-(b_2^\rho \xi) + \mathfrak{c}_1 = D^\rho(b_2^\rho \xi) + \mathfrak{c}_1 = b_2^\rho(D^\rho(\xi) + \mathfrak{c}_1 + \mathfrak{c}_2) + \bar{\partial}(b_2^\rho) \otimes \xi = \bar{\partial}(b_2^\rho) \otimes \xi.$$

In particular, $(b_2^\rho \xi, \mathfrak{c}_1)$ is *close* to lying in the kernel of D^-_{st} (up to an error of size $\rho^{-1} \|\xi\|$). Indeed, using the bounded right inverse for D^-_{st} we can estimate:

$$\left\| (b_2^\rho \xi, \mathfrak{c}_1) - \ker(D^-_{\text{st}}) \right\| \leq C \|\xi\| \rho^{-1}.$$

On the other hand, we have:

$$0 = (1 - b_2^\rho)(D^\rho(\xi) + \mathfrak{c}_1 + \mathfrak{c}_2) = D^\rho((1 - b_2^\rho)\xi) + \mathfrak{c}_2 - \bar{\partial} b_2^\rho \otimes \xi,$$

so the translated element $((1 - b_2^\rho)\xi, \mathfrak{c}_2)(s + 3\rho, t)$ is close to the kernel of D^+_{st}.

We can encode these as a linear map $\Phi : \ker D^\rho_{\text{st}} \to (L^p \times \mathbb{R}^d) \oplus (L^p \times \mathbb{R}^\delta)$:

$$\Phi(\xi, \mathfrak{c}_1, \mathfrak{c}_2) = \begin{bmatrix} (b_2^\rho \xi, \mathfrak{c}_1) \\ ((1 - b_2^\rho)\xi, \mathfrak{c}_2)(s + 3\rho, t) \end{bmatrix}.$$

It is clear that Φ is uniformly injective (we simply add together its components to recover $\xi, \mathfrak{c}_1, \mathfrak{c}_2$ – this defines a fixed left inverse). We will now estimate the rank of Φ. By the preceding remarks, we have:

$$\left\|\Phi(\xi, \mathfrak{c}_1, \mathfrak{c}_2) - \ker(D^-_{\mathrm{st}}) \oplus \ker(D^+_{\mathrm{st}})\right\| \le C \|\xi\| \rho^{-1}.$$

Let Π be a projection onto $\ker(D^-_{\mathrm{st}}) \oplus \ker(D^+_{\mathrm{st}})$, so that:

$$\|(1-\Pi)\circ\Phi(\xi, \mathfrak{c}_1, \mathfrak{c}_2)\| \le C' \|\xi\| \rho^{-1}.$$

Thus $\Phi = \Pi \circ \Phi +$ (error of size ρ^{-1}), where the error is measured in the operator norm. Since Φ is uniformly injective, we conclude that $\Pi \circ \Phi$ must also be injective for ρ large enough. Hence $\Pi \circ \Phi$ is an injection from $\ker(D^\rho_{\mathrm{st}})$ into $\ker(D^-_{\mathrm{st}}) \oplus \ker(D^+_{\mathrm{st}})$, proving (7.7).

Finally we prove the reverse inequality:

$$\dim\ker D^\rho_{\mathrm{st}} \ge \dim\ker(D^-_{\mathrm{st}}) + \dim\ker(D^+_{\mathrm{st}}). \tag{7.8}$$

The strategy will be "glue" together elements in the kernels of $D^\pm_{\mathrm{st}}$ and obtain elements approximately in the kernel of D^ρ_{st}, and then use the fact that D^ρ_{st} has a (uniformly) bounded right inverse (which we proved above) to show that we can deform these approximate kernel elements into actual kernel elements.

So, let $(\xi_1, \mathfrak{c}_1) \in \ker(D^-_{\mathrm{st}})$ and let $(\xi_2', \mathfrak{c}_2') \in \ker(D^+_{\mathrm{st}})$. Let $(\xi_2, \mathfrak{c}_2) = (\xi_2', \mathfrak{c}_2')(s - 3\rho, t)$. Recall from §7.2 that we must have $\mathfrak{c}_1 = \mathfrak{c}_2 = 0$.

Then it is straightforward to check that:

$$D^\rho(\beta^\rho_2\xi_1 + (1-\beta^\rho_2)\xi_2) = (\bar\partial\beta^\rho_2)(\xi_1 - \xi_2). \tag{7.9}$$

Let $\Phi(\xi_1, \xi_2') = \beta^\rho_2\xi_1 + (1-\beta^\rho_2)\xi_2$. First we show that Φ is uniformly injective. Indeed, the injectivity estimates for $\partial_s u - Au = 0$ from §6.4.3 imply that:

$$\begin{aligned} \|\xi_1\| &\le C\|\xi_1\|_{\Sigma(\rho)} \le C\|\Phi(\xi_1,\xi_2)\|, \\ \|\xi_2'\| &\le C\|\xi_2'\|_{(-\rho,\infty)\times I} \le C\|\Phi(\xi_1,\xi_2')\|, \end{aligned} \tag{7.10}$$

for a uniform constant C.[1] In particular

$$\|\xi_1\| + \|\xi_2'\| \le 2C\|\Phi(\xi_1,\xi_2)\|. \tag{7.11}$$

[1]The idea is to write

$$\xi_1 = (1-\beta^\rho_1)\xi_1 + \beta^\rho_1\xi_1.$$

Then $D^-(\beta^\rho_1\xi_1) = \bar\partial\beta^\rho_1 \otimes \xi_1$. Observe that $\beta^\rho_1\xi_1$ is supported in the region where D^- is translation invariant. Thus we can apply the injectivity estimates and conclude that the $W^{1,p}$ size of $\beta^\rho_1\xi_1$ is controlled by the L^p size of ξ_1 on $[0,\rho]\times I$. The constant C gets better (closer to 1) as ρ increases. Similar considerations establish the second part of (7.10).

Let B be a bounded right inverse for D^ρ_{st}, and consider

$$\Phi' = \Phi - B \circ D^\rho_{\mathrm{st}} \circ \Phi.$$

Because of (7.9), $B \circ D^\rho_{\mathrm{st}} \circ \Phi$ has operator norm bounded by ρ^{-1} (here we assume that the operator norm of B is bounded as $\rho \to \infty$; the first part of our proof shows that this can be achieved).

Then for ρ large enough, Φ' is also injective as it is a small perturbation of an injective operator (i.e., the estimate (7.11) will still hold, modulo increasing C slightly).

Thus Φ' injects $\ker(D^-) \oplus \ker(D^+)$ into $\ker(D^\rho_{\mathrm{st}})$, establishing (7.8). This completes the proof of the Lemma. $\square$

Chapter 8

The index formula for large anti-linear deformations

Fix a trivialization τ of $(\Sigma, \Gamma, E, F, C, [\tau])$, as above, and let D^{al} be a Cauchy-Riemann operator whose restriction to each end C_z is equal to $\partial_s - A^{\mathrm{al}}$ (in the trivialization τ).

Our goal in this section is to compute the Fredholm index $\mathrm{ind}(D^{\mathrm{al}})$. The formula will be in terms of the following invariants:

(i) The *relative Euler characteristic* $\mathrm{X} := \mathrm{X}(\Sigma, \Gamma_+, \Gamma_-)$ is the weighted count of zeros of a transverse vector field which equals ∂_s in each end C_z and is everywhere tangent to $\partial\Sigma$ (the zeros are counted as explained in §6.2, see also Figure 1).

(ii) The *Maslov index* $\mu^{\tau}_{\mathrm{Mas}} := \mu^{\tau}_{\mathrm{Mas}}(E, F)$ is the signed count of zeros of a transverse section of $(\det E)^{\otimes 2}$ which (a) restricts to the canonical generator of $(\det F)^{\otimes 2}$ along the boundary and (b) equals 1 in each end C_z (this last part uses τ). Notice that all the zeros will necessarily be interior.

The main result in this section is:

Proposition 8.1. The Fredholm index of $D^{\mathrm{al}} : W^{1,p}(E, F) \to L^p(\Lambda^{1,0} \otimes E)$ is

$$\mathrm{ind}(D^{\mathrm{al}}) = n\mathrm{X} + \mu^{\tau}_{\mathrm{Mas}},$$

where n is the complex rank of E.

The proof of Proposition 8.1 breaks into two parts. In §8.1 we reduce to the case when E is a line bundle (so $E = \det(E)$ and $F = \det(F)$). In §8.2 we prove Proposition 8.1 in the case when E is a line bundle by considering the $\sigma \to \infty$ limiting behavior of $D^{\mathrm{al}} + \sigma B^{\mathrm{al}}$ where B^{al} is a special anti-linear deformation (we only deform the lower order terms). This is the strategy introduced in [**Tau96**, §7] and generalized in [**Ger18**, Chapter 3].

8.1. Reduction to the case of line bundles

In this section we assume that Proposition 8.1 is true for line bundles, and we deduce it holds for all bundles. We will split (E, F) into a direct sum

$$(E, F) = \underbrace{(\mathbb{C}, \mathbb{R}) \oplus \cdots \oplus (\mathbb{C}, \mathbb{R})}_{n-1 \text{ copies}} \oplus (\det(E), \det(F))$$

in a way compatible with the trivialization τ. In order to do the splitting, we fix a Hermitian metric μ on (E, F) extending the Hermitian metric in the ends C_z.

Consider the trivialization τ. This defines a unitary frame $X_1, \cdots, X_n$ in the ends. If $n > 1$, we can extend X_1 over $\partial\dot{\Sigma}$ as a non-zero section of F, which we may normalize so $|X_1| = 1$. Let E_1 denote the μ-orthogonal complement of X_1, and let $F_1 = E_1 \cap F$. Note that F_1 is $n-1$ dimensional and is totally real for E_1.

Notice that $X_2, \cdots, X_n$ are all sections of $(E_1, F_1) \subset (E, F)$ in the ends. If $n > 2$ then we can extend X_2 as a nonzero section of (E_1, F_1). We continue in this fashion until we conclude that $X_1, \cdots, X_{n-1}$ extend as a global unitary frame in (E, F) (in the sense that they are mutually μ-orthogonal and all unit vectors).

Let E_n be the μ orthogonal complement to $X_1, \cdots, X_{n-1}$ and $F_n = E_n \cap F$, and notice that X_n trivializes (E_n, F_n) in the ends.

By construction, D^{al} is given by

$$D^{\mathrm{al}}(\sum u_k X_k) = (\partial_s u_k + i\partial_k u_k + \bar{u_k})(\mathrm{d}s - i\mathrm{d}t \otimes X_k)$$

in the ends. In particular D^{al} *splits* in the ends. By perturbing D^{al} away from the ends, we may suppose it splits everywhere. This means that if u takes values in the line $\mathbb{C}X_k$ (resp., E_n), then $D^{\mathrm{al}}(u)$ takes values in $\Lambda^{0,1} \otimes \mathbb{C}X_k$ (resp., $\Lambda^{0,1} \otimes E_n$). It follows that the induced operator splits as a diagonal matrix of Cauchy-Riemann operators asymptotic to the one-dimensional version of $\partial_s - A^{\mathrm{al}}$:

$$D^{\mathrm{al}} : [\bigoplus_{k=1}^{n-1} W^{1,p}(\mathbb{C}X_k, \mathbb{R}X_k)] \oplus (E_n, F_n) \to [\bigoplus_{k=1}^{n-1} L^p(\Lambda^{0,1} \otimes \mathbb{C}X_k)] \oplus L^p(\Lambda^{0,1} \otimes E_n).$$

Let D^{al}_k be the kth factor in the above decomposition. The Fredholm index is additive under diagonal decompositions. Since Proposition 8.1 applies to D^{al}_k we conclude that:

$$\mathrm{ind}(D^{\mathrm{al}}) = [\sum_{k=1}^{n-1} \mathrm{ind}(D^{\mathrm{al}}_k)] + \mathrm{ind}(D^{\mathrm{al}}_n) = n\mathrm{X} + \mu^{\tau}_{\mathrm{Mas}}(E_n, F_n).$$

Finally, fix $\mathfrak{s}$ a transverse section of $E_n^{\otimes 2}$ which restricts to the canonical generator of $F_n^{\otimes 2}$ and which equals $1 \simeq X_n^{\otimes 2}$ in the end. Locally write $\mathfrak{s} = \mathfrak{s}_1 \otimes \mathfrak{s}_2$, and define

$$\mathfrak{s}' = (X_1 \wedge \cdots \wedge X_{n-1} \wedge \mathfrak{s}_1) \otimes (X_1 \wedge \cdots \wedge X_{n-1} \wedge \mathfrak{s}_2). \tag{8.1}$$

This does not depend on the decomposition $\mathfrak{s} = \mathfrak{s}_1 \otimes \mathfrak{s}_2$ since E_n is a complex line bundle.

Then $\mathfrak{s}'$ is a transverse section of $\det(E)^{\otimes 2}$ which restricts to the canonical generator of $\det(F)^{\otimes 2}$. The signed count of zeros of $\mathfrak{s}'$ agrees with the count of zeros of $\mathfrak{s}$ as they locally differ by application of a fiber-wise complex linear isomorphism (namely, the map induced by (8.1)). Thus we conclude $\mu^\tau_{\mathrm{Mas}}(E,F) = \mu^\tau_{\mathrm{Mas}}(E_n,F_n)$. This completes the proof of the reduction to the line bundle case.

8.2. Large anti-linear deformations

Let (E,F) be a line bundle with asymptotic trivialization τ. As in the previous section, we can consider τ as defining a non-vanishing section X in the ends which takes boundary values in F. In other words $(E,F) = (\mathbb{C}X, \mathbb{R}X)$ in each end.

Our strategy will be to define a particular family of operators D^σ, $\sigma > 0$, whose asymptotic form with respect to the trivialization τ is equal to $\partial_s + i\partial_t + \sigma C$. Since we have shown $A^{\mathrm{al},\sigma} = -i\partial_t - \sigma C$ is non-degenerate for all $\sigma > 0$ (Lemma 7.1), we conclude that D^σ is always Fredholm. Moreover, when $\sigma = 1$, $D^\sigma = D^{\mathrm{al}}$. Therefore

$$\mathrm{ind}(D^{\mathrm{al}}) = \lim_{\sigma\to\infty} \mathrm{ind}(D^\sigma).$$

Via another index gluing argument, we will be able to relate $\mathrm{ind}(D^\sigma)$ for large σ to the weighted count of zeros of a certain section used to define D^σ, and ultimately conclude

$$\mathrm{ind}(D^\sigma) = \mathrm{X} + \mu^\tau_{\mathrm{Mas}} \text{ for } \sigma \gg 0.$$

This will complete the proof of Proposition 8.1.

8.2.1. *Defining the family D^σ.* We will now carefully define the family D^σ in such a way which will facilitate the later analysis. Pick a Hermitian metric μ on all of E so that $|X| = 1$. Consider the section $M = X \otimes X$ of $F^{\otimes 2} \to \partial C_z$.

We can extend this section as a *non-vanishing* section of $F^{\otimes 2} \to \partial\dot{\Sigma}$ as follows: on any contractible open subset of $\partial\dot{\Sigma}$, let $M = Y \otimes Y$ where $Y \in \Gamma(F)$ satisfies $|Y| = 1$ using the metric μ. Since there is a unique unit vector lying in F up to ± 1, we conclude that these local descriptions of M agree on their overlaps. Clearly, in each end, $M = X \otimes X$. We should note that X may *not* extend over the boundary $\partial\dot{\Sigma}$, (but, as we have seen, M always does).

We extend M to the interior of $\dot{\Sigma}$ as a section of $E \otimes E$ so that all of its zeros are transverse. By the same considerations of the linearization of a vector field given in §6.2, we can deform M near each zero ζ so that, for some $D(1)$-valued holomorphic coordinate z centered at ζ, and some unitary frame Y for E, we have $M = -zY \otimes Y$ or $M = \bar{z} Y \otimes Y$, depending on the sign of the determinant of the linearization of M at ζ. By definition, $\mu^\tau_{\mathrm{Mas}}(E,F)$ is the signed count of zeros of M. See Figure 1.

It will be useful to recall that $E \otimes E$ is complex linearly isomorphic to $\mathrm{Hom}^{0,1}(E,E)$ via the map $Y \otimes X \mapsto \mu(-,Y)X$, where μ is our chosen Hermitian metric. Let M_* denote the image of M under this isomorphism (so M_* is a section of $\mathrm{Hom}^{0,1}(E,E)$).

Next, we extend the vector field ∂_s (defined in the ends) to all of Σ. We let V be a vector field which (a) is everywhere tangent to $\partial\dot{\Sigma}$, (b) equals ∂_s in the ends, (c) has non-degenerate zeros, and (d) its zeros are disjoint from the zeros of M. Unlike the section $M = X \otimes X$, we expect V to have boundary zeros.

As explained in §6.2 we can slightly deform V (away from the ends), so that near each interior zero p there is a holomorphic coordinate $z = s + it$ so that $V = -z\partial_s$ or $V = \bar{z}\partial_s$, and near each boundary zero we have one of four possibilities $V = \pm z\partial_s$, $V = \pm\bar{z}\partial_s$.

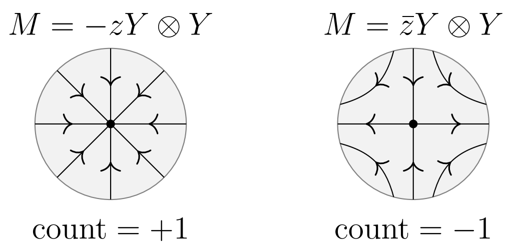


FIGURE 1. After a slight deformation in a neighborhood of each zero, we may assume the zeros of M have coordinate representations as either $-z$ or $\bar{z}$.

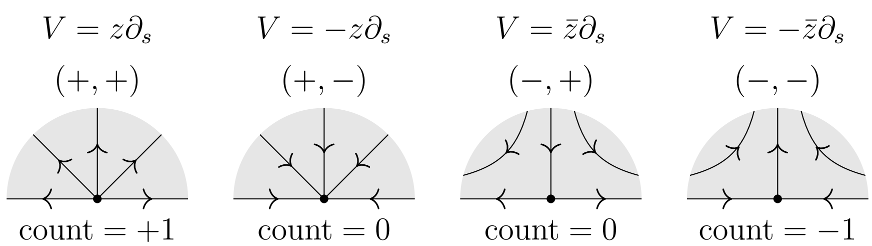


FIGURE 2. The four models for a boundary zero of V. The first sign is from the linearization of $V : \Sigma \to T\Sigma$ and the second sign is from the linearization of the restriction $V : \partial\Sigma \to T\partial\Sigma$.

By definition, the weighted count of the zeros of V is the relative Euler characteristic X.

It will be important to fix a Hermitian metric μ on $T\dot{\Sigma}$. We can do this so that $|\partial_s| = 1$ on all the coordinate charts introduced above (including the coordinate charts on the ends C_z, of course).

We are almost ready to define the operator D^σ. Two further simplifications we can do are the following:

(i) Via a small deformation of V away from the ends and its zeros, we may suppose that in the coordinate charts $z = s + it$ centered on the zeros of M, V takes the form ∂_s, and

(ii) via a small deformation of M away from the ends and its zeros, we may suppose that $M = Y \otimes Y$ for a unitary frame (with $Y|_{\partial\Sigma} \in F$) on the coordinate charts near the zeros of V.

8.2.1.1. *Summary of the setup.* we have the following:

(a) Holomorphic coordinate charts $z = s+it$ centered on each zero of M and V. Boundary holomorphic coordinate charts are valued in $D(1) \cap \bar{\mathbb{H}}$.

(b) Unitary metrics on E and $T\dot{\Sigma}$ extending the metrics in the ends. Moreover we fix unitary sections Y for (E,F) defined on the domains of the coordinate charts from (a), and also suppose that $|\partial_s| = 1$ in each chart.

(c) Near each zero of M, $V = \partial_s$ and M equals $-zY \otimes Y$ or $\bar{z}Y \otimes Y$,

(d) Near each interior zero of V, $M = Y \otimes Y$ and V equals $-z\partial_s$ or $\bar{z}\partial_s$,

(e) Near each boundary zero of V, $M = Y \otimes Y$ and V equals $\pm z\partial_s$ or $\pm\bar{z}\partial_s$, (the $\pm$ signs are independent).

Fix D_0 to be a Cauchy-Riemann operator on (E,F) which equals $\partial_s + i\partial_t$ in C (with respect to τ) and equals $\partial_s + i\partial_t$ with on the local trivializations induced by (a) and (b) above. This operator D_0 is not Fredholm, since its asymptotics are degenerate. We will perturb D_0 by the following lower order term

$$\xi \in \Gamma(E) \mapsto B(\xi) = \mu(-,V) \otimes M_*(\xi) \in \Gamma(\Lambda^{0,1} \otimes E).$$

Note that since M_* is a section of $\mathrm{Hom}^{0,1}(E,E)$, $\xi \mapsto B(\xi)$ is anti-linear. We define

$$D^\sigma = D_0 + \sigma B.$$

Before we proceed, let us verify that D^σ has the correct "al" asymptotics for $\sigma > 0$.

In any of the asymptotic coordinate charts, we have $M_* = \mu(-,X)X$ and $V = \partial_s$ hence $M_*(uX) = \bar{u}X$ and

$$\begin{aligned} D^\sigma(uX) &= (\partial_s u + i\partial_t u)(\mathrm{d}s + i\mathrm{d}t) \otimes X + \sigma\bar{u}\mu(-,\partial_s) \otimes X. \\ &= (\partial_s u + i\partial_t u + \sigma\bar{u})(\mathrm{d}s + i\mathrm{d}t) \otimes X, \end{aligned}$$

where we have used the fact that $\mu(-,\partial_s) = \mathrm{d}s - i\mathrm{d}t$ in the ends (indeed, this holds in all of our coordinate charts by the assumption that $|\partial_s| = 1$). Thus the local representation of D^σ indeed equals $\partial_s + i\partial_t + \sigma C$, as desired.

As explained at the start of this section, this implies that the Fredholm index of D^σ is constant for $\sigma > 0$. Our task therefore reduces to the following lemma, which we will prove by deforming $\sigma \to +\infty$:

Lemma 8.2. The Fredholm index of D^σ is equal

$$\mathrm{ind}(D^\sigma) = \mathrm{X} + \mu^\tau_{\mathrm{Mas}}.$$

This lemma will complete the proof of Proposition 8.1.

8.2.2. *Computing the local coordinate representations of D^σ.* In this section we will derive various formulas for D^σ in coordinate charts. We have just shown that

$$\text{in the ends } C_z \text{ we have: } D^\sigma = \partial_s + i\partial_t + \sigma C. \tag{8.2}$$

Near the zeros of V and M, we compute the coordinate representation of D^σ using the $s+it$ coordinate and the frame Y.

On a chart centered on a zero of M, we have $M_* = \alpha\mu(-,Y)Y$, where $\alpha = -z$ or $\alpha = \bar{z}$, and $V = \partial_s$. Similarly, near an interior zero of V, we have $M_* = \mu(-,Y)Y$ and $V = \alpha\partial_s$. In either case, we conclude:

$$\begin{aligned} &\text{at interior positive zeros: } D^\sigma(u) = \partial_s u + i\partial_t u - \sigma z\bar{u},\\ &\text{at interior negative zeros: } D^\sigma(u) = \partial_s u + i\partial_t u + \sigma \bar{z}\bar{u}. \end{aligned} \tag{8.3}$$

Next we compute the coordinate representation of D^σ near the boundary zeros, which we partition by their pair of signs $(\pm,\pm)$ as in Figure 2:

$$\begin{aligned} &\text{at } (+,\pm) \text{ type zeros: } D^\sigma(u) = \partial_s u + i\partial_t u \pm \sigma z\bar{u},\\ &\text{at } (-,\pm) \text{ type zeros: } D^\sigma(u) = \partial_s u + i\partial_t u \pm \sigma \bar{z}\bar{u}. \end{aligned} \tag{8.4}$$

8.3. Bochner-Weitzenböck estimates and a linear compactness result

Following [**Tau96**, §7] and [**Wen20**, Chapter 5], we show that $D^\sigma = D_0 + \sigma B$ satisfies a "Bochner-Weitzenböck" type estimate which will imply that kernel elements $\xi \in \ker D^\sigma$ and cokernel elements $\eta \in \ker D^{\sigma,*}$ concentrate near zeros of B. The key step is the following L^2 estimate:

Lemma 8.3 (Bochner-Weitzenböck estimates)**.** Let $\xi \in W^{1,2}(E,F)$, then

$$\|D_0\xi\|_{L^2}^2 + \sigma^2 \|B(\xi)\|_{L^2}^2 \leq \|D^\sigma\xi\|_{L^2}^2 + \sigma \|\xi\|_{L^2} \|D_0^*(B(\xi)) + B^*(D_0(\xi))\|_{L^2}.$$

Moreover, $\xi \mapsto D_0^*(B(\xi)) + B^*(D_0(\xi))$ is a *zeroth* order operator (which is translation invariant in the ends). Similarly, if $\eta \in W^{1,2}(\Lambda^{1,0}\otimes E, F^*)$ then

$$\|D_0^*\eta\|_{L^2}^2 + \sigma^2 \|B^*\eta\|_{L^2}^2 \leq \|D^{\sigma,*}\eta\|_{L^2}^2 + \sigma \|\xi\|_{L^2} \|D_0(B^*(\eta)) + B(D_0^*(\eta))\|_{L^2}.$$

and $\eta \mapsto D_0(B^*(\eta)) + B(D_0^*(\eta))$ is also a zeroth order operator (also translation invariant in the ends). We therefore conclude a constant $C = C(D_0, B)$ so that for all ξ, η as above we have

$$\begin{aligned} \|B\xi\|_{L^2}^2 &\leq \sigma^{-2}\|D^\sigma\xi\|_{L^2}^2 + C\sigma^{-1}\|\xi\|_{L^2}^2,\\ \|B^*(\eta)\|_{L^2}^2 &\leq \sigma^{-2}\|D^{\sigma,*}\eta\|_{L^2}^2 + C\sigma^{-1}\|\eta\|_{L^2}^2. \end{aligned} \tag{8.5}$$

In particular, if $D^{\sigma_n}\xi_n$ and ξ_n remain bounded in L^2 and $\sigma_n \to \infty$, then $B\xi_n$ must converge to zero in L^2. This forces the mass of ξ_n to concentrate near the zeros of B.

Proof. Thanks to Proposition 6.14 and the subsequent remarks, it suffices to consider the case when ξ is smooth and takes values in F along the boundary.

Let $\langle -, - \rangle$ denote the L^2 inner product. Naively, the estimate is proved by the following computation

$$\begin{aligned} \|D^\sigma \xi\|^2 &= \langle \xi, D^{\sigma,*} D^\sigma \xi \rangle \\ &= \langle \xi, D_0^* D_0 \xi \rangle + \sigma \langle \xi, D_0^*(B(\xi)) + B^*(D_0(\xi)) \rangle + \sigma^2 \|B(\xi)\|^2 \\ &= \|D_0 \xi\|^2 + \sigma \langle \xi, D_0^*(B(\xi)) + B^*(D_0(\xi)) \rangle + \sigma^2 \|B(\xi)\|^2 . \end{aligned} \tag{8.6}$$

Rearranging easily yields the desired result. Unfortunately, we cannot expect to be able to apply the formal adjoint property in the first and third equality unless $D^\sigma \xi$ and $D_0 \xi$ take boundary values in F^*. One way to circumvent this issue would to be assume that $D_0 \xi$ takes boundary values in F^*. The lower order term B has been constructed so that $D^\sigma \xi$ would automatically also take boundary values in F^*. It seems plausible that smooth sections ξ which take boundary values in F *and* for which $D^0 \xi$ takes boundary values in F^* are dense in $W^{1,2}(E, F)$.[1] However, we will not pursue this density approach here. Rather, we prefer to make the observation that we have applied the formal adjoint property *twice*, once for D^σ and once for D_0, and the failures of formal adjointness will cancel each other out.

Indeed, $D^\sigma - D_0$ is a zeroth order operator whose formal adjoint is $D^{\sigma,*} - D_0^*$. Formal adjoints for zeroth order operators do not require any integration by parts, hence

$$\langle D^\sigma \xi - D_0 \xi, D^\sigma \xi + D_0 \xi \rangle = \langle \xi, (D^{\sigma,*} - D_0^*)(D^\sigma + D_0) \xi \rangle.$$

This simplifies to

$$\|D^\sigma \xi\|^2 - \|D_0 \xi\|^2 = \langle \xi, D^{\sigma,*} D^\sigma \xi \rangle - \langle \xi, D^{0,*} D^0 \xi \rangle + \langle \xi, (D_0^* D^\sigma - D^{\sigma,*} D_0) \xi \rangle.$$

Clearly $D_0^* D^\sigma - D^{\sigma,*} D_0 = \sigma(D_0^* B - B^* D_0)$, and hence

$$\begin{aligned} \langle \xi, (D_0^* D^\sigma - D^{\sigma,*} D_0) \xi \rangle &= \langle \xi, D_0^* \sigma B \xi \rangle - \langle \xi, \sigma B^* D_0 \xi \rangle \\ &= \langle D_0 \xi, \sigma B \xi \rangle - \langle \sigma B \xi, D_0 \xi \rangle = 0, \end{aligned}$$

where we have used the fact that $B\xi$ takes values in F^* (which follows easily from our construction of B and the fact ξ takes values in F). Thus

$$\|D^\sigma \xi\|^2 - \langle \xi, D^{\sigma,*} D^\sigma \xi \rangle = \|D_0 \xi\|^2 - \langle \xi, D^{0,*} D^0 \xi \rangle.$$

[1] This is suggested by the following observation: Locally write $\xi = uY$. The approximation result Proposition 6.14 shows that u can be approximated in $W^{1,2}$ by $u_n = \Phi_n * E(u)$. By picking Φ_n appropriately, these approximations satisfy $(\partial_s + i\partial_t) u_n \in \mathbb{R}$. In general we would require that we can approximate u by smooth functions u_n taking real values on the boundary and also satisfying $\bar{\partial} u_n + \alpha u_n + \beta \bar{u_n} \in \mathbb{R}$, where α, β are arbitrary complex valued functions.

This implies that the conclusion of (8.6) holds, (even if the individual steps do not hold). The first estimate from the statement of the Lemma then follows easily. The second estimate is proved in the same way.

To show that $L(\xi) = B^*D_0(\xi) + D_0^*B(\xi)$ is a zeroth order operator, we will show that $L(f\xi) = fL(\xi)$ for all real-valued functions f and all sections ξ (this implies that L is described as a tensor). It suffices to prove this in the case when f is supported in a coordinate chart $z = s + it$ with frame Y.

We digress for a moment to derive formulas for D_0^* and B^* on this coordinate chart. Write $\xi = uY$ and $B = \varphi(\mathrm{d}s - i\mathrm{d}t)\mu(-, Y)Y$. We can assume that Y is a unitary frame, i.e., $|Y| = 1$, but we do not assume that $|\partial_s| = 1$.

Let η be an arbitrary smooth section of $\Lambda^{1,0} \otimes E$ taking values in F^* along the boundary. Write $\eta = w(\mathrm{d}s - i\mathrm{d}t) \otimes Y$. Then we easily compute (similarly to how we argued in §6.3.6):

$$B(\xi) = \bar{u}\varphi(\mathrm{d}s - i\mathrm{d}t) \otimes Y \implies \mathrm{Re}\,\mu(B(\xi), \eta) = \mathrm{Re}\,u\bar{\varphi}w\,|\mathrm{d}s - i\mathrm{d}t|^2\,. \tag{8.7}$$

Therefore we must have $B^*(\eta) = \varphi\,|\mathrm{d}s - i\mathrm{d}t|^2\,\bar{w}Y$, since this choice yields the desired pointwise relationship:

$$\mathrm{Re}\,\mu(\xi, B^*(\eta)) = \mathrm{Re}\,\bar{u}\bar{w}\varphi\,|\mathrm{d}s - i\mathrm{d}t|^2 = \mathrm{Re}\,\mu(B(\xi), \eta).$$

Recall that in (6.6) we have computed a formula for D_0^*:

$$|\mathrm{d}s - i\mathrm{d}t|^{-2}\,D_0^*(w(\mathrm{d}s - i\mathrm{d}t) \otimes Y) = (-\partial_s w + i\partial_t w + Sw)Y,$$

for some matrix valued function S. The important part is that the leading order part is $-\partial = -\partial_s + i\partial_t$. We then combine (8.7) with the above formula for D_0^* to obtain

$$D_0^*(B(f\xi)) = -\partial f \cdot \varphi\,|\mathrm{d}s - i\mathrm{d}t|^2\,\bar{u}Y + fD_0^*(B(\xi)). \tag{8.8}$$

This computes half of $L(f\xi)$. For the other half, we use the defining property of Cauchy-Riemann operators to conclude

$$B^*(D_0(f\xi)) = B^*(\bar\partial f \cdot (\mathrm{d}s - i\mathrm{d}t) \otimes \xi) + fB^*(D_0(\xi)), \tag{8.9}$$

where $\bar\partial = \partial_s + i\partial_t$. Recall that we assume f is real-valued. Then our formula for $B^*(\eta)$ with $\eta = \bar\partial f \cdot (\mathrm{d}s - i\mathrm{d}t) \otimes \xi = \bar\partial f \cdot u \cdot (\mathrm{d}s - i\mathrm{d}t) \otimes Y$ implies

$$B^*(\bar\partial f \cdot (\mathrm{d}s - \mathrm{d}t) \otimes \xi) = \overline{\bar\partial f}\varphi\,|\mathrm{d}s - i\mathrm{d}t|^2\,\bar{u}Y = \partial f \cdot \varphi\,|\mathrm{d}s - i\mathrm{d}t|^2\,\bar{u}Y.$$

Adding together (8.8) and (8.9), the $\pm\partial f \cdot \varphi\,|\mathrm{d}s - i\mathrm{d}t|^2\,\bar{u}Y$ terms cancel and we obtain

$$L(f\xi) = D_0^*(B(f\xi)) + B^*(D_0(f\xi)) = f[D_0^*(B(\xi)) + B^*(D_0(\xi))] = fL(\xi),$$

as desired. A similar argument shows that $BD_0^* + D_0B^*$ is also a zeroth order operator. This completes the proof. $\square$

8.3.1. *Local Bochner-Weitzenböck estimates for sections supported near the zeros.* In this section we will do a case-by-case analysis of the operator D^σ near the zeros. See [**Wen20**, §5.6] for similar results. To simplify the calculations ahead, let's write $\bar\partial = \partial_s + i\partial_t$. In the next section we will explain how to rescale $D^\sigma = \bar\partial \pm \sigma\alpha(z)C$ to $D^1 = \bar\partial \pm \alpha(z)C$. In this section we will focus only on the rescaled operator D^1.

There are four possibilities for D^1, namely $\bar\partial \pm zC$ and $\bar\partial \pm \bar z C$. We have the following estimates for these operators:

Lemma 8.4 (Local Bochner-Weitzenböck)**.** Let $v \in W^{1,2}(\mathbb{C},\mathbb{C})$ or $v \in W^{1,2}(\bar{\mathbb{H}},\mathbb{C},\mathbb{R})$. Then we have the following estimates:

$$\left\|\bar\partial v\right\|_{L^2}^2 + \|zv\|_{L^2}^2 \le \left\|\bar\partial v \pm z\bar v\right\|_{L^2}^2 + 2\|v\|_{L^2}^2$$
$$\left\|\bar\partial v\right\|_{L^2}^2 + \|zv\|_{L^2}^2 \le \left\|\bar\partial v \pm \bar z\bar v\right\|_{L^2}^2 .$$

Proof. Using the smooth approximation result Proposition 6.14, we may suppose that v is smooth, compactly supported, and takes real values along the boundary.

To prove the inequalities, we will need to integrate by parts two times. Let us focus on the first estimate. We start by computing:

$$(-\partial \pm zC)(\bar\partial \pm zC)v = -\partial\bar\partial v \pm z\overline{\bar\partial v} - \pm z\partial\bar v - \pm 2\bar v + |z|^2 v.$$

Using the fact that $\overline{\bar\partial v} = \partial\bar v$ we conclude that two terms cancel and we are left with

$$(-\partial \pm zC)(\bar\partial \pm zC)v = -\partial\bar\partial v \mp 2\bar v + |z|^2 v. \tag{8.10}$$

The naive idea is to multiply both sides by $\operatorname{Re}\mu_0(v,-)$, integrate, and use the formal adjoint property for $-\partial \pm zC = (\bar\partial \pm zC)^*$ and $-\partial = \bar\partial^*$. This naive argument would require that $\bar\partial v$ and $\bar\partial v \pm z\bar v$ take real values along the boundary, which we do not assume. However, as in the previous section, the fact that we integrate by parts twice will imply that the failures of formal adjointness will cancel out.

Indeed, we compute

$$\operatorname{Re}\int \mu_0(v,-\partial\bar\partial v)\mathrm{d}s\mathrm{d}t = \operatorname{Re}\int \mu_0(v,-\partial_s\bar\partial v)\mathrm{d}s\mathrm{d}t + \operatorname{Re}\int \mu_0(v,i\partial_t\bar\partial v)\mathrm{d}s\mathrm{d}t.$$

It is clear that the can integrate by parts with respect to ∂_s, and conclude

$$\operatorname{Re}\int \mu_0(v,-\partial\bar\partial v)\mathrm{d}s\mathrm{d}t = \operatorname{Re}\int \mu_0(\partial_s v,\bar\partial v)\mathrm{d}s\mathrm{d}t + \operatorname{Re}\int \mu_0(v,i\partial_t\bar\partial v)\mathrm{d}s\mathrm{d}t.$$

We can also integrate by parts with respect to ∂_t, at the expense of a boundary integral term, and (after some simplification) end up with:

$$\operatorname{Re}\int \mu_0(v,-\partial\bar\partial v)\mathrm{d}s\mathrm{d}t = \operatorname{Re}\int \mu_0(\bar\partial v,\bar\partial v)\mathrm{d}s\mathrm{d}t - \operatorname{Re}\int_{\mathbb{R}} \mu_0(v,i\bar\partial v)\mathrm{d}s\mathrm{d}t.$$

We do the same computation with $\bar\partial$ replaced by $D = \bar\partial \pm zC$, and conclude that

$$\operatorname{Re}\int \mu_0(v, D^*Dv)\mathrm{d}s\mathrm{d}t = \operatorname{Re}\int \mu_0(Dv, Dv)\mathrm{d}s\mathrm{d}t - \operatorname{Re}\int_{\mathbb{R}} \mu_0(v, iDv)\mathrm{d}s\mathrm{d}t.$$

Finally, we observe that

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{R}} \mu_0(v, iDv)\mathrm{d}s\mathrm{d}t &= \operatorname{Re}\int_{\mathbb{R}} \mu_0(v, i\bar\partial v)\mathrm{d}s\mathrm{d}t \pm \operatorname{Re}\int_{\mathbb{R}} \mu_0(v, iCz\bar v)\mathrm{d}s\mathrm{d}t \\ &= \operatorname{Re}\int_{\mathbb{R}} \mu_0(v, i\bar\partial v)\mathrm{d}s\mathrm{d}t,\end{aligned}$$

where we have used the fact that $Cz\bar v$ takes real values along the boundary. Therefore

$$\operatorname{Re}\int \mu_0(v, D^*Dv)\mathrm{d}s\mathrm{d}t - \|Dv\|_{L^2}^2 = \operatorname{Re}\int \mu_0(v, \bar\partial^*\bar\partial v)\mathrm{d}s\mathrm{d}t - \left\|\bar\partial v\right\|_{L^2}^2.$$

Applying $\operatorname{Re}\mu_0(v, -)$ to (8.10) and integrating implies

$$\left\|\bar\partial v \pm z\bar v\right\|_{L^2}^2 = \left\|\bar\partial v\right\|_{L^2}^2 \mp 2\operatorname{Re}\int \mu_0(v, \bar v) + \|zv\|_{L^2}^2.$$

We rearrange and estimate to conclude that

$$\left\|\bar\partial v\right\|_{L^2}^2 + \|zv\|_{L^2}^2 \le \left\|\bar\partial v \pm z\bar v\right\|_{L^2}^2 + 2\|v\|_{L^2}^2,$$

as desired. The second estimate in the statement of the lemma (with $D = \bar\partial \pm \bar z C$) is proved in the same manner. □

8.3.2. *Classifying the kernels of D^1 (six cases).* The second estimate in Lemma 8.4 implies that $\bar\partial v \pm \bar z\bar v = 0$ has no non-zero solutions – this takes care of three of the six kinds of operators.

Our next lemma shows that $\bar\partial v \pm z\bar v = 0$ has either a one-dimensional space of solutions or a zero-dimensional space of solutions.

Lemma 8.5. Suppose that $v : \mathbb{C} \to \mathbb{C}$ is in L^2, then

$$\begin{aligned}\bar\partial v - z\bar v = 0 &\iff v = ci\exp(-\tfrac{1}{2}|z|^2) \text{ for } c \in \mathbb{R}, \\ \bar\partial v + z\bar v = 0 &\iff v = c\exp(-\tfrac{1}{2}|z|^2) \text{ for } c \in \mathbb{R}.\end{aligned}$$

On the other hand if $v : \bar{\mathbb{H}} \to \mathbb{C}$ is in L^2 and takes real values along the boundary, then

$$\begin{aligned}\bar\partial v - z\bar v = 0 &\iff v = 0 \\ \bar\partial v + z\bar v = 0 &\iff v = c\exp(-\tfrac{1}{2}|z|^2) \text{ for } c \in \mathbb{R}.\end{aligned}$$

Morally, this says that positive interior zeros and $(+,+)$ zeros contribute one dimension to the kernel, but $(+,-)$ zeros do not contribute to the kernel.

Proof. Observe that if we set $v' = iv$, then

$$\partial_s v' + i\partial_t v' - z\bar{v}' = i(\partial_s v + i\partial_t v + z\bar{v}),$$

and hence it suffices to study the equation $\bar{\partial} v - z\bar{v} = 0$. Following [**Wen20**, Proposition 5.22], we prove that the real part of v must vanish identically.

The second estimate from Lemma 8.4 implies that $\bar{\partial} v \in L^2$ and $zv \in L^2$ (proof: both $\left\|\bar{\partial}(\rho(z\delta)v)\right\|_{L^2}$ and $\|z\rho(z\delta)v\|_{L^2}$ remain bounded as $\delta \to 0$). The L^2 elliptic estimates then imply that $v \in W^{1,2}$.

Let $y = \mathrm{Re}(v)$. Since $-\Delta v + 2\bar{v} + |z|^2 v = 0$, we have

$$0 = -\Delta y + (2 + |z|^2)y.$$

Apply $\mathrm{Re}\,\mu_0(\rho(z\delta)y, -)$ to both sides, and integrate by parts to conclude

$$0 = \int \rho(z\delta)\left|\bar{\partial} y\right|^2 \mathrm{d}s\mathrm{d}t + \mathrm{Re}\int \mu_0(\bar{\partial}(\rho(z\delta))\cdot y, \bar{\partial} y)\mathrm{d}s\mathrm{d}t + \int \rho(z\delta)(2 + |z|^2)\,|y|^2\,\mathrm{d}s\mathrm{d}t.$$

When we integrate by parts, we use $\partial_t y = 0$ (which holds in our case). We can now take the limit $\delta \to 0$, since we have verified that $\bar{\partial} y, zy \in L^2$, and conclude that

$$0 = \left\|\bar{\partial} y\right\|_{L^2}^2 + \left\|(2 + |z|^2)^{1/2} y\right\|_{L^2}^2 \implies y = 0,$$

using $\bar{\partial}(\rho(z\delta)) = O(\delta)$. It follows that any L^2 solution of $\bar{\partial} v - z\bar{v} = 0$ as in the statement of the lemma is purely imaginary (and hence vanishes along the boundary, if the boundary exists).

We now observe that $v = i\exp(-\frac{1}{2}|z|^2)$ is certainly in L^2 and solves $\bar{\partial} v - z\bar{v} = 0$. Clearly any other solution v' will be $v' = gv$ for some *holomorphic* g; moreover, by what we have shown above, g must be real. There are no non-constant holomorphic functions defined on $\mathbb{C}$ or $\mathbb{H}$ which take only real values (the rank of the derivative matrix would be always 0). Thus $g = c$ must be a real number.

In the case when v is defined on $\bar{\mathbb{H}}$, the only possibility is $c = 0$, since otherwise v would take non-zero imaginary values along the boundary.

Finally, we return to the second equation from the statement, $\partial_s v + i\partial_t v + z\bar{v} = 0$. We have shown that this solution is conjugate to the first equation under multiplication by i. Therefore all solutions on the disk or half-plane are given by $v = c\exp(-\frac{1}{2}|z|^2)$ for some real c. In this case we *can* have non-zero c when v is defined on $\bar{\mathbb{H}}$.

This completes the proof. $\square$

8.3.3. *The formal adjoint near the zeros.* Since we chose our metric so that $|\partial_s| = 1$ in all of the special coordinate charts centered at the zeros of B, we can easily compute the coordinate representations of $D^{\sigma,*}$:

$$D^\sigma = \bar\partial \pm \sigma z C \implies D^{\sigma,*}(u) = -\partial \pm \sigma z C$$
$$D^\sigma = \bar\partial \pm \sigma \bar z C \implies D^{\sigma,*}(u) = -\partial \pm \sigma \bar z C.$$

Let $D^{\sigma,\dagger} = -C \circ D^{\sigma,*} \circ C$. The above yields:

$$D^\sigma = \bar\partial \pm \sigma z C \implies D^{\sigma,\dagger}(u) = \bar\partial \mp \sigma \bar z C$$
$$D^\sigma = \bar\partial \pm \sigma \bar z C \implies D^{\sigma,\dagger}(u) = \bar\partial \mp \sigma z C.$$

Thus we can think of $D^\sigma \mapsto D^{\sigma,\dagger}$ as defining a "duality involution" on the set of six local model equations. This is illustrated in Figure 3.

To explain the labeling scheme used in the figure, we partition the zero set of B, denoted Z, into six kinds of zeros:

$$\mathrm{Z} = \mathrm{Z}^+ \cup \mathrm{Z}^- \cup \mathrm{Z}^{++} \cup \mathrm{Z}^{+-} \cup \mathrm{Z}^{-+} \cup \mathrm{Z}^{--},$$

where $\mathrm{Z}^\pm$ are interior positive/negative zeros, and $\mathrm{Z}^{\pm\pm}$ are boundary zeros (let's agree for this notation that the two $\pm$ signs are independent). The convention for assigning labels is via the linearization: the first sign is the linearization of B allowing arbitrary deformations, and the second sign is for the linearization only allowing deformations along the boundary. The local form of D^σ near a zero ζ and the corresponding count is summarized in Figure 3.

It follows from the construction in §8.2.1 that the sum of the counts of all the zeros in Z is equal to $\mathrm{X} + \mu^\tau_{\mathrm{Mas}}$.

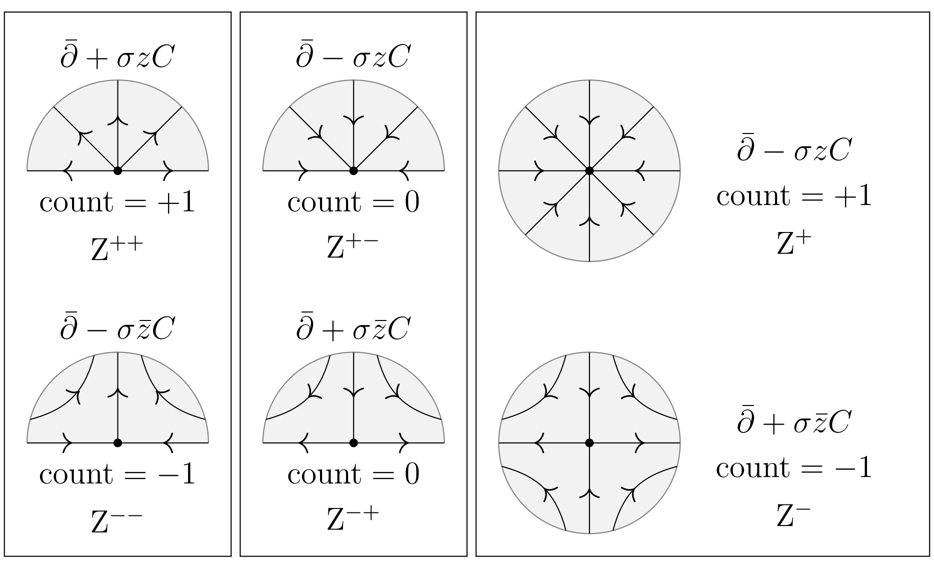


FIGURE 3. The six kinds of zeros and the coordinate representation of D^σ in each chart. Two zeros are in the same box if the operators are dual in the sense defined above.

Applying Lemmas 8.4 and 8.5 to $D^{1,\dagger}$ yields the following result for $D^{1,*}$.

Corollary 8.6. Suppose $v : \mathbb{C} \to \mathbb{C}$ is in L^2. Then

$$-\partial v - z\bar{v} = 0 \iff v = 0$$

$$-\partial v + \bar{z}\bar{v} = 0 \iff v = ic\exp(-\frac{1}{2}|z|^2) \text{ for some } c \in \mathbb{R}.$$

Now suppose that $v : \bar{\mathbb{H}} \to \mathbb{C}$ is in L^2 and takes real values along the boundary. Then

$$-\partial v \pm z\bar{v} = 0 \iff v = 0$$

$$-\partial v + \bar{z}\bar{v} = 0 \iff v = 0$$

$$-\partial v - \bar{z}\bar{v} = 0 \iff v = c\exp(-\frac{1}{2}|z|^2) \text{ for some } c \in \mathbb{R}.$$

Heuristically, this says that the zeros with count -1 in Figure 3 contribute a one-dimensional subspace to the kernel of the formal adjoint $D^{\sigma,*}$ (and all other zeros contribute nothing).

8.4. Linear compactness and a stabilization of D^ρ

In this section we will relate the kernel and cokernel of D^ρ to the kernels and cokernels of the local models D^1. We begin with an explanation of the rescaling scheme we use.

8.4.1. *Modified rescaling maps.* Suppose that ζ is a zero and let z be the special coordinate chart centered at ζ. By convention, z is either $\bar{\mathbb{H}} \cap D(1)$ or $D(1)$ valued. Let ρ be a bump function supported in $D(1)$ which is 1 on $D(1/2)$.

Let $\Phi_\sigma : L^2(\mathbb{C},\mathbb{C}) \to L^2(\dot{\Sigma},\mathbb{C})$ be the modified rescaling map:

$$\Phi_\sigma(v) = \rho \cdot \sigma^{1/2} v(\sigma^{1/2} z).$$

Observe that $\|\Phi_\sigma(v)\|_{L^2} \leq \|v\|_{L^2} = \lim_{\sigma\to\infty} \|\Phi_\sigma(v)\|_{L^2}$. Dually, we let $\Pi_\sigma = \Phi_\sigma^*$ be the adjoint. It is easy to obtain the following explicit formula for Π_σ:

$$\Pi_\sigma(u)(z) = \sigma^{-1/2}\rho(\sigma^{-1/2}z)u(\sigma^{-1/2}z).$$

The relevance of Π_σ, Φ_σ is how they interact with D^σ. Suppose that $D^\sigma = \bar{\partial} + \sigma\alpha(z)C$ and let $D^1 = \bar{\partial} + \alpha(z)C$ (where $\alpha = \pm z, \pm\bar{z}$). Then we easily compute

$$D^\sigma \circ \Phi_\sigma(v) = \sigma^{1/2}\Phi_\sigma(D^1(v)) + (\bar{\partial}\rho)\sigma^{1/2}v(\sigma^{1/2}z).$$

Recall that the L^2 norm of $\lambda v(\lambda z)$ is constant as function of λ. A similar computation can be done using Π_σ, and we conclude:

$$\begin{aligned} &\left\|D^\sigma(\Phi_\sigma(v)) - \sigma^{1/2}\Phi_\sigma(D^1(v))\right\|_{L^2} \leq c(\rho)\|v\|_{L^2(D(\sigma)\smallsetminus D(\sigma/2))}, \\ &\left\|\sigma^{1/2}D^1(\Pi_\sigma(u)) - \Pi_\sigma(D^\sigma(u))\right\|_{L^2} \leq c(\rho)\|u\|_{L^2(D(1)\smallsetminus D(1/2))}. \end{aligned} \tag{8.11}$$

We similarly note the behavior of $D^{\sigma,*}$ under Φ_σ and Π_σ:

$$\begin{aligned} &\left\|D^{\sigma,*}(\Phi_\sigma(v)) - \sigma^{1/2}\Phi_\sigma(D^{1,*}(v))\right\|_{L^2} \leq c(\rho)\left\|v\right\|_{L^2(D(\sigma)\smallsetminus D(\sigma/2))}, \\ &\left\|\sigma^{1/2}D^{1,*}(\Pi_\sigma(u)) - \Pi_\sigma(D^{\sigma,*}(u))\right\|_{L^2} \leq c(\rho)\left\|u\right\|_{L^2(D(1)\smallsetminus D(1/2))}. \end{aligned} \tag{8.12}$$

These estimates will be important later on. They essentially say that a uniform bound on $\|D^\sigma(u)\|_{L^2}$ and $\|u\|_{L^2}$ implies that $\|D^1(v)\|_{L^2} = O(\sigma^{-1/2})$ where $v = \Pi_\sigma(u)$.

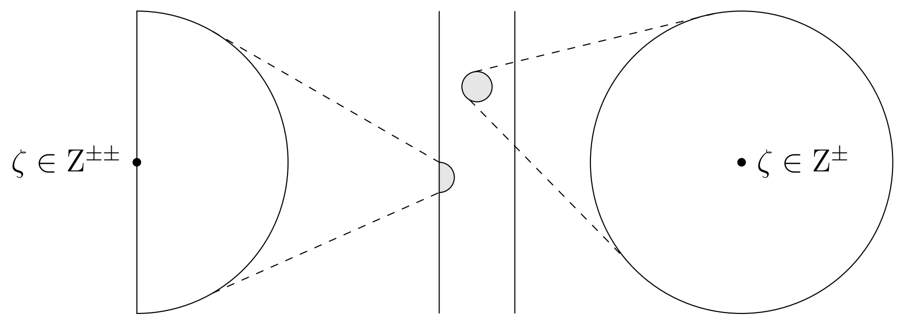


FIGURE 4. Rescaling sections near the zeros of B. The map Φ_σ takes a section on the large domain and compresses it to fit inside the small domain (and then cuts it off by ρ). The map Π_σ does the opposite, it first cuts off by ρ and then expands the domain of the section. The factors have been chosen so that $\|\Phi_\sigma(v)\|_{L^2} = \left\|\rho(\sigma^{-1/2}z)v\right\|_{L^2}$.

8.4.2. *A linear compactness result.* In this section we will prove a compactness theorem which concerns sequences ξ_n with $\|D^{\sigma_n}(\xi_n)\| < C$ and $\sigma_n \to \infty$. To set the stage, let z_ζ be the chosen holomorphic coordinate centered on the zero ζ (as above), and recall the modified rescaling maps:

$$\Phi_{\sigma,\zeta}(v) = \rho \cdot \sigma^{1/2}v(\sigma^{1/2}z_\zeta), \text{ and } \Pi_{\sigma,\zeta}(u) = \sigma^{-1/2}\rho(\sigma^{-1/2}z_\zeta)u(\sigma^{-1/2}z_\zeta).$$

Let $\Pi_\sigma = \oplus_{\zeta\in\mathrm{Z}}\Pi_{\sigma,\zeta}$ be considered as a map

$$\Pi_\sigma : L^2(\dot\Sigma, E) \to \bigoplus_{\zeta\in\mathrm{Z}^\pm} L^2(\mathbb{C},\mathbb{C}) \oplus \bigoplus_{\zeta\in\mathrm{Z}^{\pm\pm}} L^2(\bar{\mathbb{H}},\mathbb{C}) = H.$$

The same formula also defines Π_σ on $L^2(\dot\Sigma, \Lambda^{1,0}\otimes E)$. We can think of H as the Hilbert space of L^2 sections on a disjoint union of finitely many copies of $\mathbb{C}$ and $\bar{\mathbb{H}}$.

We define an operator $D^1 : H \to H$ (with dense domain) whose restriction to each factor equals the choice of $\bar\partial \pm \alpha(z)C$ for $\alpha(z) = z, \bar z$ given by Figure 3. We similarly define $D^{1,*} : H \to H$ where the local form is $-\partial \pm \alpha(z)C$, as appropriate.

The results of Lemmas 8.4, 8.5 and Corollary 8.6 give a complete classification of the elements in $\ker D^1$ and $\ker D^{1,*}$. See (8.15) in the next section for a summary of the kernel of D^1 and $D^{1,*}$.

We let $\mathrm{R}_\sigma(\xi) = \xi - \sum_{\zeta\in \mathrm{Z}} \rho(z_\zeta)\xi$ which we think of as the "remainder" after cutting off. It follows easily from the definitions that

$$
\begin{aligned}
&D^\sigma(\mathrm{R}_\sigma(\xi)) = \bar\partial\rho \otimes \xi + \mathrm{R}_\sigma(D^\sigma(\xi)). \\
&\qquad \|\xi\|_{L^2} \le \|\mathrm{R}_\sigma(\xi)\|_{L^2} + \|\Pi_\sigma(\xi)\|_{L^2} \le 2\,\|\xi\|_{L^2}\,.
\end{aligned} \tag{8.13}
$$

Proposition 8.7 (Linear compactness)**.** Let $\xi_n \in W^{1,2}(E,F)$ be a sequence so that $\|\xi_n\|_{L^2} + \|D^{\sigma_n}(\xi_n)\|_{L^2}$ remains bounded for some sequence $\sigma_n \to \infty$. Then

(a) $\|R_{\sigma_n}(\xi_n)\|_{L^2} \to 0$.

(b) After passing to a subsequence, $\Pi_{\sigma_n}(\xi_n) \to \mathbf{k}$ in L^2 for some element $\mathbf{k} \in \ker D^1$.

The same holds with $(\xi_n, E, F, D^{\sigma_n}, D^1, \mathbf{k})$ replaced by $(\eta_n, \Lambda^{0,1}\otimes E, F^*, D^{\sigma_n,*}, D^{1,*}, \mathbf{c})$.

Proof. We will only prove the ξ_n case, leaving the η_n case to the reader. To avoid too much clutter, we suppress some notation and write $\sigma := \sigma_n$, $\xi_n := \xi$. Keep in mind that ρ is a fixed bump function.

Let's begin the proof. Using (8.13) together with the Bochner-Weitzenböck estimate (8.5) implies that

$$
\|B\mathrm{R}_\sigma(\xi)\|^2_{L^2} \le \sigma^{-2}\left\|\bar\partial\rho\right\|^2_{C^0}\|\xi\|^2_{L^2} + \sigma^{-2}\,\|D^\sigma(\xi)\|^2_{L^2} + C\sigma^{-1}\,\|\xi\|^2_{L^2}\,.
$$

However, $\mathrm{R}_\sigma(\xi)$ is supported on $\dot\Sigma \smallsetminus D(\zeta_1,1/2) \smallsetminus D(\zeta_2,1/2)\smallsetminus\cdots$ and it follows that $|B| > b > 0$ for some fixed constant b on the support of $\mathrm{R}_\sigma(\xi)$. Therefore we conclude that

$$
\|\mathrm{R}_\sigma(\xi)\|^2_{L^2} \le b^{-1}((C\sigma^{-1} + c_\rho\sigma^{-2})\,\|\xi\|^2_{L^2} + \sigma^{-2}\,\|D^\sigma(\xi)\|^2_{L^2}) = O(\sigma^{-1}).
$$

This proves part (a).

For part (b), we use (8.11) to conclude

$$
\left\|D^1(\Pi_\sigma(\xi))\right\| \le \sigma^{-1/2}(\|D^\sigma(\xi))\|_{L^2} + c_\rho\,\|\xi\|_{L^2}) = O(\sigma^{-1/2}).
$$

Let $v_n = \Pi_\sigma(\xi)$. Then $\|v_n\|_{L^2}$ is bounded and $\|D^1(v_n)\| = O(\sigma_n^{-1/2})$. We will now use the local Bochner Weitzenböck estimates (Lemma 8.4) to conclude that we have

$$
\begin{aligned}
&\|v_n\|_{L^2} + \left\|\bar\partial v_n\right\|_{L^2} + \|zv_n\|_{L^2} = O(1) \\
&\qquad\qquad \left\|D^1(v_n)\right\|_{L^2} = O(\sigma_n^{-1/2}).
\end{aligned} \tag{8.14}
$$

The first estimate above is actually enough to imply that a subsequence of v_n converges to some limit v_∞ in L^2; we will explain this step momentarily. The second estimate will imply that $D^1(v_\infty) = 0$. This will complete the proof.

Before we move on, note that the L^2 elliptic estimates for $\bar\partial$ and the first estimate above implies that v_n is uniformly bounded in $W^{1,2}$.

We can phrase the next part of our argument rather generally. If we let

$$W = \{v \in H \text{ and } \|v\|_{W^{1,2}} + \|zv\|_{L^2} \leq C\},$$

(with the obvious induced norm) then the inclusion $W \to H$ is compact; we will prove this below. To see how it applies to our problem, observe that the L^2 estimates for $\bar{\partial}$ and the first part of (8.14) imply that $\|v_n\|_{W^{1,2}} + \|zv_n\|_{L^2}$ is bounded, and hence v_n is bounded in W. Therefore, after passing to a subsequence, v_n converges to some limit v_∞ in L^2. If φ is any test function (taking real values along the boundary) then we have

$$\langle D^{1,*}\varphi, v_\infty\rangle = \lim\langle D^{1,*}\varphi, v_n\rangle \to 0,$$

and hence $D^1 v = 0$ weakly. By our elliptic regularity results v is smooth, takes real values along the boundary, and $D^1 v = 0$ holds pointwise, as desired. We can then set $\mathbf{k} = v_\infty$ to complete the proof.

It remains to show why $W \to H$ is a compact inclusion. It is well-known that $W^{1,2}(\Omega(r)) \subset L^2(\Omega(r))$ is a compact inclusion for $\Omega(r) = D(r)$ or $\Omega(r) = D(r) \cap \bar{\mathbb{H}}$. Thus, by a diagonal argument, we can pass to a subsequence v_n and that $v_n \to v_\infty$ for some limit $v \in L^2_{\text{loc}}$ (in the L^2_{loc} topology).

We easily estimate

$$\|v_n\|^2_{L^2(\Omega(2^k)\smallsetminus\Omega(2^{k-1}))} \leq \frac{1}{4^{k-1}}\|zv_n\|^2_{L^2(\Omega(2^k)\smallsetminus\Omega(2^{k-1}))}.$$

Since $\Omega(2r)\smallsetminus\Omega(r)$ is precompact, we must have

$$\|v_\infty\|^2_{L^2(\Omega(2^k)\smallsetminus\Omega(2^{k-1}))} = \lim\|v_n\|^2_{L^2(\Omega(2^k)\smallsetminus\Omega(2^{k-1}))} \leq \frac{C^2}{4^{k-1}}.$$

Since the right hand side is summable, we conclude that v_∞ is actually in L^2. It follows that, for all k, we have

$$\begin{aligned}\|v_\infty - v_n\|^2_{L^2} &\leq \|v - v_n\|^2_{L^2(\Omega(2^k))} + \sum_{\ell>k}\|v_\infty\|^2_{L^2(\Omega(2^\ell)\smallsetminus\Omega(2^{\ell-1}))} + \|v_n\|^2_{L^2(\Omega(2^\ell)\smallsetminus\Omega(2^{\ell-1}))}.\\ &\leq \|v_\infty - v_n\|^2_{L^2(\Omega(2^k))} + 2C^2 4^{-k}\end{aligned}$$

Pick k large enough that the last term is less than ϵ, and then take the limit $n \to \infty$, yielding

$$\limsup\|v_\infty - v_n\|^2_{L^2} \leq \epsilon.$$

This implies that $v_n \to v_\infty$ in L^2, completing the proof. □

8.4.3. *Stabilizing D^σ and computing its index.* In this section we will stabilize D^σ by adding a cokernel element $\mathbf{c}_\zeta$ for each zero ζ with count -1 (Figure 3). We will also "co"-stabilize it by adding a kernel element $\mathbf{k}_\zeta$ for each ζ with count $+1$.

We define the following elements of $L^2(\mathbb{C},\mathbb{C})$ and $L^2(\bar{\mathbb{H}},\mathbb{C})$:

$$\begin{array}{rl} \text{at } \zeta \in \mathrm{Z}^+ & \mathbf{k}_\zeta = i\exp(-\frac{1}{2}|z|^2) \text{ and } \mathbf{c}_\zeta = 0, \\ \text{at } \zeta \in \mathrm{Z}^- & \mathbf{k}_\zeta = 0 \text{ and } \mathbf{c}_\zeta = i\exp(-\frac{1}{2}|z|^2), \\ \text{at } \zeta \in \mathrm{Z}^{++} & \mathbf{k}_\zeta = \exp(-\frac{1}{2}|z|^2) \text{ and } \mathbf{c}_\zeta = 0, \\ \text{at } \zeta \in \mathrm{Z}^{--} & \mathbf{k}_\zeta = 0 \text{ and } \mathbf{c}_\zeta = \exp(-\frac{1}{2}|z|^2), \\ \text{at } \zeta \in \mathrm{Z}^{+-}\cup\mathrm{Z}^{-+} & \mathbf{k}_\zeta = 0 \text{ and } \mathbf{c}_\zeta = 0, \end{array} \tag{8.15}$$

The results of Lemmas 8.4, 8.5 and Corollary 8.6 show that $\mathrm{span}_{\zeta\in\mathrm{Z}}(\mathbf{k}_\zeta) = \ker D^1 \subset H$, and $\mathrm{span}_{\zeta\in\mathrm{Z}}(\mathbf{c}_\zeta) = \ker D^{1,*} \subset H$.

Keeping track of the counts of the various kinds of zeros, we see that

$$\dim\ker D^1 - \dim\ker D^{1,*} = \mathrm{X} + \mu^\tau_{\mathrm{Mas}}. \tag{8.16}$$

Throughout the subsequent arguments, we will use $\mathbf{k}$ and $\mathbf{c}$ to denote linear combinations of the above basic kernel and cokernel elements.

We consider $\Phi_\sigma(\mathbf{k})$ and $\Phi_\sigma(\mathbf{c})$ as elements of $W^{1,2}(E,F)$ and $W^{1,2}(\Lambda^{0,1}\otimes E,F^*)$, using the special coordinate charts z_ζ and frames Y defined above.

We define the stabilized operator by the formula:

$$\begin{aligned} &D^\sigma_{\mathrm{st}} : W^{1,2}(E,F)\oplus\ker D^{1,*} \to L^2(\Lambda^{0,1}\otimes E)\oplus\ker D^1 \\ &D^\sigma_{\mathrm{st}}(\xi,\mathbf{c}) = (D^\sigma(\xi)+\Phi_\sigma(\mathbf{c}), \textstyle\sum_\zeta \|\mathbf{k}_\zeta\|^{-2}\langle\Pi_\sigma(\xi),\mathbf{k}_\zeta\rangle\mathbf{k}_\zeta). \end{aligned}$$

Note that the second factor is simply an orthogonal projection. The following result will complete the proof of the index formula.

Proposition 8.8. The operator D^σ_{st} is an isomorphism for σ sufficiently large.

See [**Wen20**, §5.7] for a similar result.

Proof. We summarize the strategy. First prove that D^σ_{st} is eventually uniformly injective, in the sense that there are constants C,σ_0 so that

$$\sigma > \sigma_0 \implies \|(\xi,\mathbf{c})\|_{L^2} \leq C\,\|D^\sigma_{\mathrm{st}}(\xi,\mathbf{c})\|_{L^2}. \tag{8.17}$$

Second, we show that D^σ_{st} is eventually surjective. Then D^σ_{st} is eventually an isomorphism, as desired.

We prove (8.17) by contradiction; suppose not and then conclude a sequence $\sigma_n\to\infty$ and elements $(\xi_n,\mathbf{c}_n)$ so that $\|(\xi_n,\mathbf{c}_n)\|_{L^2}=1$ but $\|D^{\sigma_n}_{\mathrm{st}}(\xi_n,\mathbf{c}_n)\|_{L^2}\to 0$. Let's agree to abbreviate $\sigma=\sigma_n$ to avoid excessive subscripts during the course of this argument.

It is clear that

$$\|D^\sigma(\xi_n)\|_{L^2} \leq \|D^\sigma_{\mathrm{st}}(\xi_n, \mathbf{c}_n)\|_{L^2} + C\,\|\mathbf{c}_n\|_{L^2}\,,$$

for a fixed constant C. In particular, we can apply our compactness result to ξ_n and conclude that, after passing to a subsequence $\Pi_\sigma(\xi_n)$ converges to $\mathbf{k}$ and $\mathrm{R}_\sigma(\xi_n)$ converges to 0. However, since $\mathbf{k}_\zeta$ form an orthogonal basis for $\ker D^1$ we have

$$\mathbf{k} = \lim_{n\to\infty} \sum_\zeta \|\mathbf{k}_\zeta\|^{-2} \langle \Pi_\sigma(\xi_n), \mathbf{k}_\zeta\rangle \mathbf{k}_\zeta.$$

Therefore $D^\sigma_{\mathrm{st}}(\xi, \mathbf{c}) \to 0$ implies that $\mathbf{k} = 0$. Therefore $\Pi_\sigma(\xi_n)$ converges to zero in L^2, and since we know $\mathrm{R}_\sigma(\xi_n) \to 0$, we conclude ξ_n converges to zero in L^2.

In order to contradict our initial assumption, it suffices to show that the inner product $\langle \Phi_\sigma(\mathbf{c}), D^\sigma(\xi)\rangle$ converges to zero (because then $\|\mathbf{c}_n\|^2 \leq \|D^\sigma_{\mathrm{st}}(\xi_n, \mathbf{c}_n)\|^2 + \epsilon$ must hold eventually, by Pythagoras' theorem, for arbitrary ϵ). Using the adjointness property and (8.11), we have

$$\langle \Phi_\sigma(\mathbf{c}_n), D^\sigma(\xi_n)\rangle = \langle \mathbf{c}_n, \Pi_\sigma(D^{\sigma(\xi_n)})\rangle = \sigma^{1/2}\langle \mathbf{c}_n, D^1(\Pi_\sigma(\xi_n))\rangle + o(1) = o(1),$$

where we use the fact that $\mathbf{c}_n \in \ker D^{1,*}$. This completes the proof by contradiction, and hence we have (8.17).

To prove that D^σ_{st} is eventually surjective, we also argue by contradiction. Suppose that it were not. Then by standard properties of Hilbert spaces, we could find a unit norm sequence $\eta_n, \mathbf{k}_n$ (with $\sigma_n \to \infty$) so that

$$\langle D^\sigma(\xi) + \Phi_\sigma(\mathbf{c}), \eta_n\rangle + \langle \Pi_\sigma(\xi), \mathbf{k}_n\rangle = 0 \text{ for all } n, \xi, \mathbf{c},$$

Using $\Pi^*_\sigma = \Phi_\sigma$ and $\mathbf{c} = 0$, we conclude that $D^{\sigma,*}(\eta_n) = -\Phi_\sigma(\mathbf{k}_n)$. Since this is bounded in L^2, we can apply the compactness result to conclude that $\Pi_\sigma(\eta_n)$ converges to a solution of $\ker D^{1,*}$. However the assumption that

$$\langle \Phi_\sigma(\mathbf{c}), \eta_n\rangle = 0,$$

for *all* $\mathbf{c} \in \ker D^{1,*}$, allows us to conclude that $\Pi_\sigma(\eta_n)$ converges to 0. It follows that η_n converges to zero (since we already know $\mathrm{R}_\sigma(\eta_n)$ converges to zero). Set $\xi_n = \Phi_\sigma(\mathbf{k}_n)$ and $\mathbf{c} = 0$ to conclude that

$$\begin{aligned}
0 =& \langle D^\sigma(\Phi_\sigma(\mathbf{k}_n)), \eta_n\rangle + \langle \Phi_\sigma(\mathbf{k}_n), \Phi_\sigma(\mathbf{k}_n)\rangle.\\
=& \langle \Phi_\sigma(D^1(\mathbf{k}_n)) + o(1), \eta_n\rangle + \langle \Phi_\sigma(\mathbf{k}_n), \Phi_\sigma(\mathbf{k}_n)\rangle.\\
=& \langle o(1), \eta_n\rangle + \langle \Phi_\sigma(\mathbf{k}_n), \Phi_\sigma(\mathbf{k}_n)\rangle.\\
\implies & \|\Phi_\sigma(\mathbf{k}_n)\| = o(1) \implies \|\mathbf{k}_n\| = o(1).
\end{aligned}$$

We have shown that both $\eta_n, \mathbf{k}_n$ converge to zero, which contradicts our assumption that they were unit norm. This completes the proof. $\square$

Remark 8.9. It follows easily from Proposition 8.8 that

$$\mathrm{ind}(D^\sigma) = \dim\ker D^1 - \dim\ker D^{1,*}.$$

To see why, write D^σ_{st} in matrix form. Deform the operator by keeping the $1,1$ entry fixed and setting all the other entries to zero. This deformation does not change the Fredholm index. It is easy to compute the Fredholm index after the deformation.

Equation (8.16) then implies that $\mathrm{ind}(D^\sigma) = \mathrm{X} + \mu^\tau_{\mathrm{Mas}}$, which completes the proof of Lemma 8.2. This in turn completes the proof of Proposition 8.1 (the index formula for $\mathrm{ind}(D^{\mathrm{al}})$). Applying our earlier result Proposition 7.3 (relating $\mathrm{ind}(D)$ and $\mathrm{ind}(D^{\mathrm{al}})$) completes the proof of our main result, Theorem 6.2.

Chapter 9

Uniform estimates for holomorphic curves in symplectizations

Consider the data (Y, Λ, α, J) of:

(i) a compact manifold Y^{2n+1},

(ii) a contact form α,

(iii) a compatible complex structure J_ξ on $\xi = \ker \alpha$, which induces an admissible complex structure J on $\mathbb{R} \times Y$, and

(iv) a closed Legendrian submanifold Λ.

The goal of this chapter is to prove that any holomorphic cylinder/strip

$$u : \mathbb{R} \times S \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda),$$

where $S = [0,1]$ or $S = \mathbb{R}/\mathbb{Z}$, with finite *Hofer energy* is asymptotic to a trivial cylinder over a Reeb orbit or chord, in the sense explained in §1.3.2. We will prove this result assuming the Reeb orbit/chord is non-degenerate.

We will also prove that $\mathrm{d}u$ satisfies uniform C^k bounds for each $k \geq 0$. The main techniques we will use to establish the C^0 convergence of $\mathrm{pr} \circ u$ and the C^k bounds on $\mathrm{d}u$ are (i) *a bubbling argument* and (ii) *elliptic bootstrapping.*

9.0.1. *Preliminaries.* To state the precise result we will prove, we need to first give a few definitions:

Definition 9.1. We can associate to (α, J_ξ) the almost Kähler triple (g, J, ω) on $\mathbb{R} \times Y$ where $\omega = \mathrm{d}(e^\sigma \mathrm{pr}^*\alpha)$, and

$$g = e^{-\sigma}\omega(-, J-) = \mathrm{d}\sigma^2 + \mathrm{pr}^*\alpha^2 + \mathrm{pr}^*\mathrm{d}\alpha(-, J-). \tag{9.1}$$

This metric defines a translation invariant distance function on $\mathbb{R} \times Y$, which we will use below.

Definition 9.2. Suppose that $u : \dot{\Sigma} \to \mathbb{R} \times Y$ is a smooth map. We define

$$\text{Hofer Energy of } u = \sup_{f \in \mathcal{P}} \int_{\dot{\Sigma}} u^*\mathrm{d}(e^{f(\sigma)}\mathrm{pr}^*\alpha),$$

where $\mathcal{P}$ is the class of increasing diffeomorphisms $f : \mathbb{R} \to (0,1)$.

Another energy quantity which will play a role in our proof is the dα-*energy*, defined by

$$\mathrm{d}\alpha\text{-energy of } u = \int_{\dot{\Sigma}} u^*\mathrm{d}\alpha.$$

It is straightforward to show that the Hofer energy of u is positive for every non-constant holomorphic curve u. Similarly, the dα-energy is non-negative for all holomorphic curves. It is also not hard to show that the dα-energy of a holomorphic curve u is bounded from above by the Hofer energy of u.

Definition 9.3. Let u, v be two smooth maps defined on $[0,\infty) \times S$. We define

$$\mathrm{dist}_{C^\infty(s)}(u; v) = \mathrm{dist}_{C^\infty}(u|_{[s-1,s+1]\times S}, v|_{[s-1,s+1]\times S}).$$

Theorem 9.4. Let $u : [0,\infty) \times S \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$, where $S = \mathbb{R}/\mathbb{Z}$ or $S = [0,1]$, be a J-holomorphic map with finite Hofer energy. We have the following:

(a) The derivative du satisfies uniform C^k bounds in the sense that

$$\sup_{s,t} \left|\nabla^k \mathrm{d}u(s,t)\right| < \infty \text{ for each } k.$$

(b) If $s_n \to \infty$ is any sequence, there is a subsequence (still denoted s_n) so that

$$\begin{aligned} &\lim_{n\to\infty} \mathrm{dist}_{C^\infty(s_n)}(u; Ts + \sigma_0, c(t)) = 0, \text{ or} \\ &\lim_{n\to\infty} \mathrm{dist}_{C^\infty(s_n)}(u; -Ts + \sigma_0, c(1-t)) = 0 \end{aligned} \tag{9.2}$$

for some Reeb chord $c : [0,1] \to (Y, \Lambda)$, or orbit $c : \mathbb{R}/\mathbb{Z} \to Y$, which we parametrize to have constant speed equal to its action T.

(c) If c is a non-degenerate Reeb chord and (9.2) holds then we have

$$\lim_{s\to\infty} \mathrm{dist}_{C^\infty(s)}(\mathrm{pr} \circ u(s,t), c(t)) = 0, \text{ (or } c(1-t)).$$

(d) If c is a non-degenerate Reeb orbit and (9.2) holds then

$$\lim_{s\to\infty} \inf_{t_0\in\mathbb{R}/\mathbb{Z}} \mathrm{dist}_{C^\infty(s)}(\mathrm{pr} \circ u(s,t), c(t+t_0)) = 0, \text{ (or } c(-t+t_0)).$$

In other words, $\mathrm{pr} \circ u(s,t)$ is asymptotically always close to one of the orbits in the $\mathbb{R}/\mathbb{Z}$ family of orbits $\{\gamma(t+t_0) : t_0 \in \mathbb{R}/\mathbb{Z}\}$.

Remark 9.5. In the next Chapter, §10, we will upgrade the above to conclude that $\sigma \circ u(s,t)$ converges, and, in the orbit case, that $\mathrm{pr} \circ u(s,t)$ converge to a fixed parameterization of the limit orbit $\gamma(t)$.

Remark 9.6. After reparametrizing those with negative sign via the map $(s,t) \mapsto (-s, 1-t)$, so the domain changes to $(-\infty, 0] \times S$, we conclude

$$\lim_{n\to\infty} \mathrm{dist}_{C^\infty(\pm s_n)}(u, (Ts + \sigma_0, c(t))) = 0$$

holds in all cases. As explained in 1.3.2, the two types of convergence depend on whether $\sigma \circ u$ converges to $\pm\infty$.

9.0.2. *On the necessity of the non-degeneracy assumption.* It seems to be an interesting question as to whether or not the limit

$$\lim_{s \to +\infty} \mathrm{pr} \circ u(s,t)$$

exists without the non-degeneracy assumption.

It is illuminating to compare with the case where (W, L, J) is a compact symplectic manifold with Lagrangian L and compatible almost complex structure J. It can be shown that every finite energy holomorphic curve $w : [0,\infty) \times [0,1] \to (W, L)$ satisfies $\lim_{s\to\infty} w(s,-) = p \in L$.

In other words, every holomorphic strip has a well-defined asymptotic limit, even though the set of asymptotics is *not* discrete (i.e., the asymptotics are degenerate). Rather, the set of asymptotics forms the manifold L. In some sense, the set of asymptotics is degenerate in a controlled way, analogous to how the critical points of a Morse-Bott function are degenerate.[1]

In [**Bou02**], the author proves asymptotic convergence results in the case when c is a Reeb orbit in a *Morse-Bott* family of orbits, in the sense that there is a compact manifold Σ foliated by orbits of R, and the linearization of the Reeb flow operator is non-degenerate when restricted to variations lying in the normal bundle to Σ.

Indeed, a non-degenerate orbit γ can be considered as a simple case of a Morse Bott family of orbits. The analysis in §10 which proves that $u(s,t)$ converges to $\gamma(t)$, i.e., that there is a fixed limiting parametrization of the orbit, is fairly similar to the analysis in [**Bou02**], which proves that a holomorphic curve is asymptotic to a well-defined parametrized Reeb orbit in the Morse-Bott family.

9.0.3. *Outline of the proof.* Our proof Theorem 9.4 has four steps. The first step will reduce the proof of the C^k bound for every k to the case $k = 0$. The main technique in this step will be *elliptic bootstrapping*. The next step will be to show that if $|\mathrm{d}u|$ is unbounded, then a *non-constant* holomorphic plane or half-plane with boundary on $\mathbb{R} \times \Lambda$ with finite Hofer energy and *zero* $\mathrm{d}\alpha$-energy exists. The third step will be to show that there are *no* non-constant planes or half-planes with finite Hofer energy and zero $\mathrm{d}\alpha$-energy. The first three steps together prove the uniform C^k bounds. Finally, in the fourth step, we will investigate the asymptotic convergence of $u(s,t)$ as $s \to \infty$.

[1] Note that the Morse-Bott condition for f is not *just* that the critical points form a manifold L; it also requires that the Hessian $\nabla \mathrm{d}f$ is non-degenerate when restricted to the normal bundle of L.

9.1. Elliptic bootstrapping and bounding higher derivatives

In this section we will prove the following lemma:

Lemma 9.7. Let $u_n : \mathcal{D}(z_n, \frac{1}{2}) \to (\mathbb{R}\times Y, \mathbb{R}\times\Lambda)$ be a sequence of holomorphic curves whose first derivatives are uniformly bounded. Then $\sup_n \left|\nabla^k \mathrm{d}u_n(z_n)\right| < \infty$ for each k.

Here $\mathcal{D}(z, r)$ is the domain $D(z, r) \cap \mathbb{R}\times[0, 1]$, as shown in the figure below for various values of z.

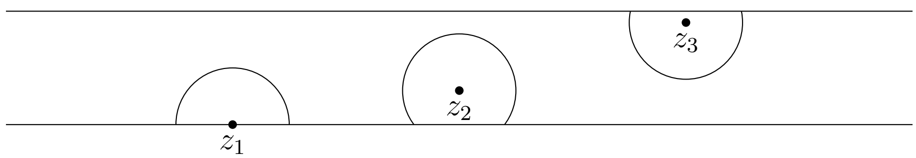


FIGURE 1. The domain $\mathcal{D}(z, \frac{1}{2})$ is a partial disk. Shown for three points z_1, z_2, z_3.

Remark 9.8. In the statement of the lemma we use metric g from (9.1) to measure sizes. We use the Levi-Civita connection ∇ associated to g to take the higher derivatives. Any translation invariant metric will suffice for this lemma.

In our proof of Lemma 9.7 we will require two analytical results: the *Sobolev embedding theorem* and the *elliptic estimates for the Laplacian*. We state these prerequisites here.

Lemma 9.9 (Sobolev embedding theorem)**.** For every bounded Lipshitz domain $\Omega \subset \mathbb{R}^2$ there exists constants $c_2(\Omega)$ $c_1(\Omega) > 0$ so that

$$\|f\|_{C^0(\Omega)} \leq c_1 \|f\|_{W^{1,4}(\Omega)} \leq c_2 \|f\|_{W^{2,2}(\Omega)}.$$

Proof. See [**MS12**, Theorem B.1.11] for a more general result. □

Lemma 9.10 (Elliptic estimates for the Laplacian)**.** For every pair of domains $\Omega_1, \Omega_2 \subset \mathbb{H}$ with $\bar{\Omega}_1 \subset \Omega_2$ there exists a constant $c(k, \Omega_1, \Omega_2)$ so that

$$\|u\|_{W^{k+2,2}(\Omega_1)} \leq c(\|\Delta u\|_{W^{k,2}(\Omega_2)} + \|u\|_{W^{k+1,2}(\Omega_2)}),$$

for all smooth $u : \Omega_2 \to \mathbb{R}^d$ satisfying the Dirichlet boundary conditions $u(\mathbb{R}\cap\Omega_2) = 0$ or the Neumann boundary conditions $\partial_t u(\mathbb{R}\cap\Omega_2) = 0$.

Proof. See [**RS01**, Lemma C.2] for a short proof. □

Note that we will only consider $\Omega = D(0, r)$ or $\Omega = D(0, r) \cap \mathbb{H}$, when applying Lemmas 9.9 and 9.10.

Proof (of Lemma 9.7)**.** This argument is inspired by the proof of [**RS01**, Lemma C.3].

In search of a contradiction, let us suppose that $\left|\nabla^k \mathrm{d}u_n(z_n)\right|$ is unbounded. Then, passing to a subsequence, we may assume that $\lim_{n\to\infty} \left|\nabla^k \mathrm{d}u_n(z_n)\right| = \infty$.

Let us redefine our curves by translating in the vertical direction $u_n := \mathrm{T}_n \circ u_n$. This does not change the sizes of the derivatives. We pick T_n so that $u_n(z_n)$ converges to some point $p \in \mathbb{R} \times Y$, after potentially taking a further subsequence.

Write $z_n = s_n + it_n$. By passing to a further subsequence, we may suppose that t_n converges to a point $t_\infty \in [0,1]$. By replacing $u_n(s,t) := u_n(-s,-t)$, we may suppose that $t_\infty \in [0, \frac{1}{2}]$. We consider two cases, either $t_\infty = 0$ or $t_\infty \in (0, \frac{1}{2}]$. We will prove the case $t_\infty = 0$, i.e., when the points z_n are converging to the boundary, and leave the other (simpler) case to the reader.

Consider the function $v_n(s+it) = u_n(s_n + s + it)$. Since $z_n = s_n + it_n$ and t_n converges to 0, eventually v_n is defined on the half-disk $\mathcal{D}(0, \frac{1}{3})$.

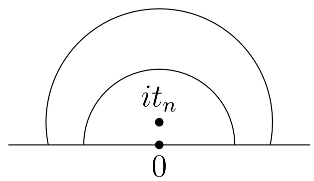


FIGURE 2. The half disk $\mathcal{D}(0, \frac{1}{3})$ is eventually contained in $\mathcal{D}(it_n, \frac{1}{2})$.

Since $v_n(it_n) = u_n(z_n)$, we conclude that $v_n(it_n)$ and $v_n(0)$ both converge to p. Therefore p lies on $\mathbb{R} \times \Lambda$. Choose now a coordinate chart $\varphi : \bar{U} \to \bar{B} \subset \mathbb{R}^{2n}$ centered at p which identifies $(\mathbb{R} \times \Lambda) \cap \bar{U}$ with $(\mathbb{R}^n \times \{0\}) \cap \bar{B}$ and so that the induced complex structure $\mathrm{d}\varphi \cdot J \cdot \mathrm{d}\varphi^{-1}$ is equal to J_0 along $\mathbb{R}^n \times \{0\}$. To see that such a coordinate chart exists one can, e.g., pick the first n coordinates $x_1, \cdots, x_n$ for $\mathbb{R} \times \Lambda$ and then define the remaining coordinates $y_1, \cdots, y_n$ by exponentiating the vector fields $J\partial_{x_i}$ (which are transverse to $\mathbb{R} \times \Lambda$ since J is compatible with ω).

By the assumed C^1 bound, we conclude that v_n eventually maps $\mathcal{D}(0, \delta)$ into U. Thus we may (eventually) define the $\mathbb{R}^{2n}$-valued function $w_n(z) = \varphi \circ v_n(z)$.

Then, abusing notation and letting $J := \mathrm{d}\varphi \cdot J \cdot \mathrm{d}\varphi^{-1}$, we conclude that w_n satisfies the boundary value problem:

$$\begin{cases} \partial_s w_n + J(w_n) \cdot \partial_t w_n = 0 \\ \qquad w_n(s,0) \in \mathbb{R}^n \times \{0\} \end{cases} \tag{9.3}$$

We decompose $w_n(s,t)$ into its real and imaginary parts:

$$w_n(s,t) = \begin{bmatrix} X_n(s,t) \\ Y_n(s,t) \end{bmatrix}.$$

We easily compute that $Y_n(s,0) = 0$ and $0 = \partial_s Y_n(s,0) = -\partial_t X_n(s,0)$. This means that X_n satisfies the Neumann boundary conditions and Y_n satisfies the Dirichlet boundary conditions. Therefore we conclude from Lemma 9.10 that, for $k \geq 2$, w_n satisfies the elliptic estimates:

$$\|w_n\|_{W^{k,2}(\mathcal{D}(\delta/k))} \leq c_k(\|\Delta w_n\|_{W^{k-2,2}(\mathcal{D}(\delta/(k-1)))} + \|w_n\|_{W^{k-1,2}(\mathcal{D}(\delta/(k-1)))}). \tag{9.4}$$

Here we abbreviate $\mathcal{D}(r) := \mathcal{D}(0,r)$.

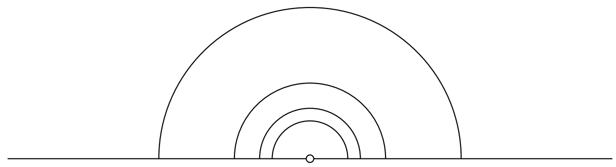


FIGURE 3. Nested disks $\mathcal{D}(\delta/k)$.

In order to use (9.4), we compute

$$\begin{aligned} &(\partial_s - J(w_n)\partial_t)(\partial_s w_n + J(w_n)\partial_t w_n) = 0 \\ \Longrightarrow\ &\Delta w_n = \partial_t[J(w_n)]\partial_s w_n - \partial_s[J(w_n)]\partial_t w_n. \end{aligned} \tag{9.5}$$

Our strategy will be to use (9.4) and (9.5) to bootstrap the initial C^1 bound to a $W^{k,2}$ bound on the disk $\mathcal{D}(\delta/k)$, for all k. To be more precise, we will prove:

$$\sup_n \|w_n\|_{W^{k,2}(\mathcal{D}(\delta/k))} < \infty \tag{9.6}$$

by induction on k. The base case $k = 1$ holds from the initial C^1 bound.

Since w_n is uniformly bounded in C^1, we conclude from (9.5) that Δw_n is uniformly bounded in $L^2(\mathcal{D}(\delta))$. Therefore (9.4) implies that (9.6) holds with $k = 2$.

It is well-known that:

$$\sup_n \|w_n\|_{W^{k,2}(\mathcal{D}(r))} < \infty \implies \sup_n \|J(w_n)\|_{W^{k,2}(\mathcal{D}(r))} < \infty, \tag{9.7}$$

for all $k \geq 0$, since J is smooth.

It is also easy to see that the following *quadratic estimate* holds:

$$\|fg\|_{W^{1,2}(\Omega)} \leq \|f\|_{W^{1,2}(\Omega)} \|g\|_{C^0(\Omega)} + \|f\|_{C^0(\Omega)} \|g\|_{W^{1,2}(\Omega)}. \tag{9.8}$$

In particular, applying (9.8) to (9.5) implies that

$$\sup_n \|\Delta w_n\|_{W^{1,2}(\mathcal{D}(\delta/2))} = \sup_n \|\partial_t[J(w_n)]\partial_s w_n - \partial_s[J(w_n)]\partial_t w_n\|_{W^{1,2}(\mathcal{D}(\delta/2))} < \infty,$$

since we know $J(w_n), w_n$ are uniformly bounded in $W^{2,2}(\mathcal{D}(\delta/2))$ and C^1.

Then we easily conclude from the elliptic estimates (9.4) that the desired result (9.6) holds with $k = 3$.

To continue the bootstrapping argument, we will require another quadratic estimate; for $W^{k,2}$ with $k \geq 2$ we can use the following estimate:

$$\|fg\|_{W^{k,2}(\Omega)} \leq C_k \|f\|_{W^{k,2}(\Omega)} \|g\|_{W^{k,2}(\Omega)} . \tag{9.9}$$

It is easy establish (9.9) for $k > 2$ by induction using

$$\|fg\|_{W^{k,2}(\Omega)} \leq \|fg\|_{W^{k-1,2}(\Omega)} + \|\nabla f \cdot g\|_{W^{k-1,2}(\Omega)} + \|f \cdot \nabla g\|_{W^{k-1,2}(\Omega)} .$$

The base case when $k = 2$ follows from a similar observation:

$$\|fg\|_{W^{2,2}(\Omega)} \leq \|fg\|_{W^{1,2}(\Omega)} + \|\nabla f \cdot \nabla g\|_{L^2} + \|\nabla\nabla f \cdot g\|_{L^2} + \|f \cdot \nabla\nabla g\|_{L^2} .$$

The first term above can be estimated using our first quadratic estimate (9.8), together with the Sobolev embedding for $C^0 \subset W^{2,2}$. The last two terms can be estimated using $\|ab\|_{L^2} \leq \|a\|_{L^2} \|b\|_{C^0}$ and the Sobolev embedding theorem. The hard term to estimate is $\|\nabla f \cdot \nabla g\|_{L^2}$. To do so, we will use the Sobolev embedding theorem for $W^{1,4} \subset W^{2,2}$, and the Hölder-type inequality

$$\|\nabla f \cdot \nabla g\|_{L^2} \leq \|\nabla f\|_{L^4} \|\nabla g\|_{L^4} .$$

Returning to our bootstrapping argument, we can now conclude from (9.7) and (9.9) that

$$\sup_n \|\Delta w_n\|_{W^{2,2}(\mathcal{D}(\delta/2))} = \sup_n \|\partial_t[J(w_n)]\partial_s w_n - \partial_s[J(w_n)]\partial_t w_n\|_{W^{2,2}(\mathcal{D}(\delta/2))} < \infty.$$

Then applying the elliptic estimates (9.4) proves (9.6) in the case $k = 4$. The argument repeats, without any further modification, to conclude (9.6) for all k.

We are almost finished with the proof. Recall that we assumed:

$$\lim_{n\to\infty} \left|\nabla^k \mathrm{d}u_n(z_n)\right| = \infty$$

in search of a contradiction. Since $\varphi \circ u_n(z) = w_n(z - s_n)$, we conclude that w_n is also unbounded in the C^{k+1} norm near $z_n - s_n = it_n$ (since φ is a diffeomorphism between compact domains, it distorts the C^{k+1} size by a bounded amount).

Since t_n converges to 0, it_n eventually enters the disk $\mathcal{D}(\delta/(k+3))$. On this domain the C^{k+1} norm is bounded by the $W^{k+3,2}$ norm, by the Sobolev embedding theorem. Then (9.6) with k replaced by $k+3$ contradicts the fact that the C^{k+1} size of w_n is unbounded. This contradiction completes the proof. □

9.2. The bubbling argument

In this section we will prove the following lemma:

Lemma 9.11. Suppose that $u_n : \mathcal{D}(z_n, \frac{1}{2}) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ is a sequence of holomorphic curves.

Then we have the following alternative: either $\sup_n |\mathrm{d}u_n(z_n)| < \infty$, or there exists a non-constant holomorphic plane $v_\infty : \mathbb{C} \to \mathbb{R} \times Y$ or half-plane $v_\infty : \mathbb{H} \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ with

$$(\text{Hofer energy of } v_\infty) \leq \limsup_{n\to\infty}(\text{Hofer energy of } u_n)$$

$$(\mathrm{d}\alpha\text{-energy of } v_\infty) \leq \limsup_{n\to\infty}(\mathrm{d}\alpha\text{-energy of } u_n).$$

Remark 9.12. The case that will be of interest to us is the following: suppose we have a single holomorphic curve $u : [0, \infty) \times [0, 1] \to \mathbb{R} \times Y$ with finite Hofer energy. Suppose that the derivative of u is unbounded. Then we will set

$$u_n = u|_{\mathcal{D}(z_n, \frac{1}{2})}$$

for a sequence of points with $\sup_n |\mathrm{d}u_n(z_n)| = \infty$. This forces $\lim_{n\to\infty} s(z_n) = \infty$, and so

$$\limsup_n(\mathrm{d}\alpha\text{-energy of } u_n) = 0.$$

Then Lemma 9.11 will imply the existence of a finite Hofer energy plane or half-plane with *zero* $\mathrm{d}\alpha$-energy. In §9.3 we will show that such planes/half-planes cannot exist. This argument shows u must have a bounded derivative.

One technical result needed in the proof of Lemma 9.11 is known as "Hofer's lemma,"

Lemma 9.13 (Hofer's Lemma)**.** Let $d : X \to [0, \infty)$ be a continuous function on a complete metric space; and let $\epsilon' > 0$ and $x' \in X$. One can find $0 < \epsilon \leq \epsilon'$ and $x \in X$ so that

(i) $\mathrm{dist}(x, x') < 2\epsilon'$,

(ii) $d(y) \leq 2d(x)$ for all $y \in D(x, \epsilon)$.

(iii) $\epsilon d(x) \geq \epsilon' d(x')$,

Hofer's lemma was introduced in [**HV92**, Lemma 3.3] (moreover, they show that the lemma gives a *characterization* of completeness). We will give the proof here for the reader's convenience.

Proof (of Lemma 9.13)**.** Let $\epsilon_n = 2^{-n}\epsilon'$, and define a (potentially terminating) sequence x_n as follows: let $x_0 = x'$, and choose $x_{n+1} \in D(x_n, \epsilon_n)$ so that $d(x_{n+1}) > 2d(x_n)$. If no such x_{n+1} exists (i.e., the sequence terminates at x_n), then we conclude that, for all $y \in D(x_n, \epsilon_n)$ we have $d(y) \leq 2d(x_n)$, so (ii) is satisfied with $x = x_n$, $\epsilon = \epsilon_n$. By construction, we have

$$\epsilon_n d(x_n) \geq 2\epsilon_n d(x_{n-1}) = \epsilon_{n-1} d(x_{n-1}) \geq \cdots \geq \epsilon_0 d(x_0) = \epsilon' d(x'),$$

so (iii) would also be satisfied. Since $\mathrm{dist}(x_0, x_n) \leq \epsilon_0 + \epsilon_1 + \cdots + \epsilon_n \leq 2\epsilon_0$, we conclude (i) also holds.

Thus the proof of the lemma is reduced to proving that the above recursion terminates. In search of a contradiction suppose it does not converge. Then the sequence x_n converges, however, $d(x_n)$ is unbounded since $d(x_n) > 2d(x_{n-1})$. This is impossible, and so we complete the proof. $\square$

Proof (of Lemma 9.11)**.** Let $u_n : \mathcal{D}(z_n, \frac{1}{2}) \to \mathbb{R} \times Y$ be a sequence of holomorphic curves. Without loss of generality, let us suppose that the derivative $\mathrm{d}u_n(z_n)$ is unbounded. By passing to a subsequence, we may suppose that $R'_n := |\mathrm{d}u_n(z_n)|$ satisfies $\lim_{n\to\infty} R'_n = +\infty$.

Now pick $0 < \epsilon'_n < 1/6$ so that $\lim_{n\to\infty} \epsilon'_n = 0$ but $\lim_{n\to\infty} \epsilon'_n R'_n = +\infty$.

Introduce the function $d_n(z) = |\mathrm{d}u_n(z)|$, and apply Hofer's lemma with $\epsilon' = \epsilon'_n$ and $x' = 0$ to conclude $\epsilon_n \leq \epsilon'_n$ and x_n so that

(i) $x_n \in \mathcal{D}(z_n, 2\epsilon'_n)$,

(ii) $|\mathrm{d}u_n(y)| \leq 2\,|\mathrm{d}u_n(x_n)|$ for $y \in \mathcal{D}(x_n, \epsilon_n)$,

(iii) $\epsilon_n\,|\mathrm{d}u_n(x_n)| \geq \epsilon'_n R'_n$.

The reader may complain that d_n is not defined on a complete metric space, but it is easy to see that every point and ball considered in the recursive proof of Hofer's lemma will remain entirely in $\mathcal{D}(z_n, 3\epsilon'_n)$. Since we chose $\epsilon'_n < 1/6$, we see that we can cut off d_n outside of $\mathcal{D}(z_n, 3\epsilon'_n)$ (and obtain a continuous function defined on all of $\mathbb{R} \times [0,1]$) without affecting our conclusions.

We abbreviate $R_n := |\mathrm{d}u_n(x_n)|$. Note that by item (iii) R_n is still diverging to ∞.

The idea now is to rescale the domains of u_n by the factor of R_n^{-1}; we introduce

$$v_n : D(0, R_n\epsilon_n) \cap R_n(\mathbb{R} \times [0,1] - x_n) \text{ given by } v_n(z) = u_n(x_n + R_n^{-1} z).$$

The domain of v_n seems a bit awkward, but we can simplify it by writing $x_n = s_n + it_n$ and observing that

$$R_n(\mathbb{R} \times [0,1] - x_n) = \mathbb{R} \times [-t_n R_n, (1-t_n)R_n].$$

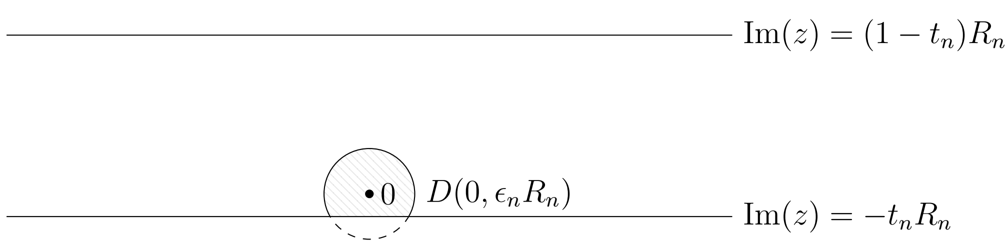


FIGURE 4. The domain of the rescaled map v_n is the shaded region. Depending on the limit of $t_n R_n$, the domain is either expanding to cover the entire complex plane $\mathbb{C}$, or an upper or lower half-plane.

We can pass to a further subsequence so that t_nR_n and $(1-t_n)R_n$ converge in $[0,\infty]$. There are then three cases to consider: if either t_nR_n converges to a finite number, then the domains of v_n converge to an upper half-plane. On the other hand, if $(1-t_n)R_n$ converges to a finite number, then the domains of v_n converge to a lower half-plane.

To be precise, by "converge to a half plane" we mean that there exists a half plane H with the property that any compact set in H is eventually contained in the domain of v_n.

If both t_nR_n and $(1-t_n)R_n$ diverge to ∞, the domains of v_n converge to the entire complex plane $\mathbb{C}$.

It is straightforward to conclude that

$$|\mathrm{d}v_n(0)| = 1 \text{ and } |\mathrm{d}v_n(z)| \leq 2 \tag{9.10}$$

for all z in the domain of v_n (using (ii)). As we did in the proof of Lemma 9.7, we now replace $v_n := \mathrm{T}_n \circ v_n$ where T_n is a sequence of vertical translations. This does not affect (9.10). We choose T_n so that $v_n(0)$ converges (taking a subsequence if necessary).

The Arzelà-Ascoli theorem implies that v_n converges in C^0_{loc} to a continuous function $v_\infty : \Omega \to \mathbb{R} \times Y$ with where Ω is either a half-plane or $\mathbb{C}$. If Ω is a half-plane, then the aforementioned C^0_{loc} convergence implies that v_∞ maps the boundary onto $\mathbb{R} \times \Lambda$.

By our elliptic bootstrapping lemma (Lemma 9.7), we conclude that the higher derivatives of v_n are uniformly bounded on compact sets. Here we use the fact that if z lives in a compact set of Ω, then eventually z has a neighborhood in the domain of v_n identical to one of the partial disks considered in Lemma 9.7.

The C^k_{loc} bounds allow us to upgrade the conclusion of the Arzelà-Ascoli theorem to conclude that (i) the limit map v_∞ is smooth and (ii) v_n actually converges in C^∞_{loc} to v_∞. In particular, v_∞ is holomorphic. Moreover, by the C^1_{loc} convergence, we conclude that $|\mathrm{d}v_\infty(0)| = 1$, and hence v_∞ is non-constant.

It remains only to prove the bounds on the energies of v_∞. This follows from a fairly standard argument, which we will briefly explain. If v_∞ has energy greater the E, then for $\epsilon > 0$ there is a compact domain $K \subset \Omega$ on which v_∞ has energy greater than $E - \epsilon$. Eventually v_n is defined on K, and since v_n converges to v_∞ in $C^1(K)$, we conclude that the energy of v_n on K is eventually greater than $E - 2\epsilon$. Therefore

$$E - 2\epsilon < \limsup_{n\to\infty}(\text{energy of } v_n).$$

Since E was an arbitrary number less than the energy of v_∞, and $\epsilon > 0$ was also chosen arbitrarily, we conclude that

$$\text{energy of } v_\infty < \limsup_{n\to\infty}(\text{energy of } v_n).$$

This argument works verbatim replacing "energy" with "dα-energy." This argument also applies if we set

$$\text{“energy” of } u = \int u^* \mathrm{d}(e^{f(\sigma)}\mathrm{d}\alpha).$$

Then we can take the supremum over all f as required by the definition of the Hofer energy. This completes the proof of the lemma. $\square$

9.3. Holomorphic curves with zero dα-energy

Our main goal in this section is to prove that there are no holomorphic planes or half-planes with finite Hofer energy and zero dα-energy. As a first step, we prove the following lemma:

Lemma 9.14. Let Σ be a connected Riemann surface, and let $u : \Sigma \to \mathbb{R} \times Y$ be a holomorphic map with zero dα-energy. Then there is a leaf $\mathcal{L} \to \mathbb{R} \times Y$ of the *Reeb foliation* (the foliation spanned by ∂_σ and R) so that u factors smoothly through $\mathcal{L}$.

Proof. Since the dα-energy is the integral of $u^*\mathrm{pr}^*\mathrm{d}\alpha$, and $\mathrm{pr}^*\mathrm{d}\alpha$ is a J-compatible symplectic form on the contact distribution $\mathrm{pr}^*\xi \subset T(\mathbb{R} \times Y)$, we conclude that

$$\mathrm{im}(\mathrm{d}u) \subset \ker \mathrm{pr}^*\mathrm{d}\alpha = \mathbb{R}\partial_\sigma \oplus \mathbb{R}R.$$

Pick a point $z \in \Sigma$ and let $u(z) = p$. Choose coordinates $x_1, x_2, y_1, \cdots, y_{2n}$ centered on p so that $\partial_{x_1} = \partial_\sigma$ and $\partial_{x_2} = R$ (this is possible since ∂_σ and R commute). On an open set around z we conclude that $\mathrm{d}(y_i \circ u) = 0$ for all i, since

$$\mathrm{im}(\mathrm{d}u) \subset \mathrm{span}\left\{\partial_{x_1}, \partial_{x_2}\right\}.$$

In particular u factors smoothly through the locus where $y_1 = \cdots = y_{2n-2} = 0$, which is evidently part of some leaf $\mathcal{L}$.

This argument shows that set of points $z \in \Sigma$ which land in $\mathcal{L}$ is an open set. However, since the complement of $\mathcal{L}$ is a union of *other leaves*, we conclude by the same argument that the set of points which *don't* land in $\mathcal{L}$ is also an open set. By the connectedness of Σ, we conclude that all points $u(\Sigma) \subset \mathcal{L}$. Our argument also shows the factorization of u through the inclusion $\mathcal{L} \to \mathbb{R} \times Y$ is smooth. This completes the proof. $\square$

It is easy to classify the leaves of the Reeb foliation: each leaf is of the form $\mathbb{R} \times \gamma$ where γ is a Reeb flow line. In particular, if $\mathcal{L}$ is a leaf, then

$$\Gamma : \sigma + i\tau \in \mathbb{C} \mapsto (\sigma, \gamma(\tau)) \in \mathcal{L} \tag{9.11}$$

is either a diffeomorphism or the universal cover (depending on whether γ is a closed orbit or not). Also note that the fact that J is assumed to be admissible implies that (9.11) is holomorphic.

We compute the following formula:

$$\Gamma^* \mathrm{d}(e^{f(\sigma)}\mathrm{pr}^*\alpha) = f'(\sigma)e^{f(\sigma)}\mathrm{d}\sigma \wedge \mathrm{d}\tau.$$

Combining Lemma 9.14 with (9.11) allows us to prove the following:

Corollary 9.15. Let Σ be a simply-connected Riemann surface. If $u : \Sigma \to \mathbb{R} \times Y$ is a holomorphic curve with zero $\mathrm{d}\alpha$-energy, then there exists a holomorphic map $w : \Sigma \to \mathbb{C}$ so that $\Gamma \circ w = u$. If u has finite Hofer energy, then

$$\sup_{f \in \mathcal{P}} \int_\Sigma w^*(f'(\sigma)e^{f(\sigma)}\mathrm{d}\sigma \wedge \mathrm{d}\tau) < \infty, \tag{9.12}$$

where $\mathcal{P}$ is the collection of functions from Definition 9.2.

With this results in place, we can now prove the following lemma:

Lemma 9.16. There are no non-constant holomorphic planes $u : \mathbb{C} \to \mathbb{R} \times Y$ or half-planes $u : \mathbb{H} \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ with finite Hofer energy and zero $\mathrm{d}\alpha$-energy.

Proof. We argue by contradiction. By Corollary 9.15, we conclude a either a map $w : \mathbb{C} \to \mathbb{C}$ or a map $w : \mathbb{H} \to \mathbb{C}$ so that the energy (9.12) is finite.

Consider the case when $u : \mathbb{H} \to \mathbb{R} \times Y$. Since $u(\mathbb{R}) \subset \mathbb{R} \times \Lambda$ and the inverse image $\Gamma^{-1}(\mathbb{R} \times \Lambda)$ is a union of horizontal[2] lines $\tau = \text{const}$, we conclude that the induced map w satisfies $w(\mathbb{R}) \subset L$ for some horizontal line L. Then w can be doubled by the *Schwarz reflection principle* to obtain a holomorphic plane $w : \mathbb{C} \to \mathbb{C}$.

The doubling process increases (9.12) by a factor of 2, since the energy only depends on the horizontal coordinate which is unchanged when we reflect.

Thus it suffices to prove the case $w : \mathbb{C} \to \mathbb{C}$. We observe that the integral of $f'(\sigma)e^{f(\sigma)}\mathrm{d}\sigma \wedge \mathrm{d}t$ over a region of the form $[a, b] \times \mathbb{R}$ is always infinite. In particular, if w surjects on $[a, b] \times R$, then the integral (9.12) would be infinite.

Picard's little theorem asserts that w must surject onto $\mathbb{C}$ or $\mathbb{C}$ minus a single point. In particular, we can find $a < b$ so that w surjects onto $[a, b] \times \mathbb{R}$. This completes the proof.

If the reader does not like using Picard's theorem, we can also argue as follows. Recall that under the stereographic projection $p : \mathbb{C} \to \mathbb{CP}^1$, the Fubini-Study form ω_{FS} is pulled back to

$$p^*\omega_{\mathrm{FS}} = \frac{\mathrm{d}\sigma \wedge \mathrm{d}\tau}{(1 + \sigma^2 + \tau^2)^2}.$$

(See [**MS17**, Exercise 4.3.4]). Now observe that

$$\int_{-\infty}^{+\infty} \frac{1}{(1 + \sigma^2)^2} = c < \infty.$$

[2]The σ coordinate is the real coordinate on $\mathbb{C}$, even though it corresponds to the "vertical" coordinate in the symplectization. This conflict between "horizontal" and "vertical" is a bit unfortunate.

If we define

$$f'(\sigma) = \frac{1}{c}\frac{1}{(1+\sigma^2)^2},$$

and $f(\sigma) = \int_{-\infty}^{\sigma} f'(\sigma')\mathrm{d}\sigma'$, then $f : \mathbb{R} \to (0,1)$ is an increasing diffeomorphism, and hence is in the family $\mathcal{P}$ of functions from Definition 9.2. Moreover, it is clear that

$$p^*\omega_{\mathrm{FS}} = \frac{\mathrm{d}\sigma \wedge \mathrm{d}\tau}{(1+\sigma^2+\tau^2)^2} \leq ce^{f(\sigma)}f'(\sigma)\mathrm{d}\sigma \wedge \mathrm{d}\tau. \tag{9.13}$$

In particular, we conclude that the composite $p \circ w : \mathbb{C} \to \mathbb{CP}^1$ has finite Fubini-Study area. By the removal of singularities theorem (see [**MS12**, Theorem 4.1.2]) we conclude that $p \circ w$ extends to a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$. In particular, $p \circ w$ is surjective, and hence $w : \mathbb{C} \to \mathbb{C}$ is surjective.

Thus w surjects onto $\mathbb{R} \times [a,b]$, contradicting (9.12) (as we already explained above). This completes the proof.[3] □

In Remark 9.12, we explained that if a holomorphic curve

$$u : [0,\infty) \times [0,1] \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$$

with finite Hofer energy had an unbounded first derivative, then the bubbling lemma (Lemma 9.11) would produce a finite energy plane or half-plane with zero dα-energy. This conclusion is obviously incompatible with Lemma 9.16 above. Thus we have concluded the first part of Theorem 9.4, namely that for every holomorphic curve $u : [0,\infty) \times [0,1] \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ with finite Hofer energy we have

$$\sup_{s,t} \left|\nabla^k \mathrm{d}u(s,t)\right| < \infty, \tag{9.14}$$

for all k.

Proof. Remark 9.12 together with Lemma 9.16 imply the C^1 bound ($k = 0$). The result for $k \geq 1$ follows from Lemma 9.7. □

Before we end this section, we wish to prove two more lemmas concerning holomorphic curves with zero dα-energy.

Lemma 9.17. Let $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ be a non-constant holomorphic curve with finite Hofer energy and zero dα-energy. Then there exists a Reeb orbit $c : \mathbb{R}/\mathbb{Z} \to Y$ of action T and a real number σ_0 so that either

$$\sigma \circ u(s,t) = \pm Ts + \sigma_0 \text{ and } \mathrm{pr} \circ u(s,t) = c(\pm t).$$

[3] After writing this, the author came up with another proof: by a bubbling argument, it suffices to exclude non-constant maps $w : \mathbb{C} \to \mathbb{C}$ satisfying (9.12) *with bounded derivative.* Non-constant maps with bounded derivative are affine, and hence are surjective.

Proof. First we prove that the map u must lie in $\mathbb{R}\times\gamma$ for a closed orbit γ. The argument is similar to the proof of Lemma 9.16; observe that otherwise the composed map $u : \mathbb{R}\times\mathbb{R}/\mathbb{Z} \to \mathbb{C} \to \mathbb{CP}^1$ would have finite Fubini study area, and would therefore extend to a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$. As argued previously, this would imply the map u has infinite Hofer energy (by considering regions $[a, b] \times \mathbb{R} \subset \mathbb{C}$).

Let $\Gamma(s, t) = (Ts, \gamma(t))$ where $\gamma : \mathbb{R}/\mathbb{Z} \to Y$ is the underlying simple Reeb orbit, parametrized to have constant speed equal to its action. Then Γ parametrizes the leaf which contains the image of u, so can write $u = \Gamma \circ w$, where $w : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ is a holomorphic map. We know that w has bounded derivatives, and hence w's lift $\mathbb{C} \to \mathbb{C}$ takes the form of an affine map, which must map $i\mathbb{Z}$ into $i\mathbb{Z}$, and hence has the form $z \mapsto kz + \sigma_0/T$ where $k \in \mathbb{Z}$ and $\sigma_0 \in \mathbb{R}$. It follows that

$$u(s, t) = (Tks + \sigma_0, c(kt)).$$

Redefining $T := Tk$ and $c(t) = c(kt)$, i.e., replacing the simple orbit by its multiple cover, we obtain the desired result. □

Lemma 9.18. Let $u : \mathbb{R} \times [0, 1] \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ be a holomorphic curve with finite Hofer energy and zero $d\alpha$-energy. Then there exists a Reeb chord $c : [0, 1] \to (Y, \Lambda)$ and a real number σ_0 so that either

$$\sigma \circ u(s, t) = Ts + \sigma_0 \text{ and } \mathrm{pr} \circ u(s, t) = c(t),$$

or

$$\sigma \circ u(s, t) = -Ts + \sigma_0 \text{ and } \mathrm{pr} \circ u(s, t) = c(1 - t).$$

Proof. The argument is similar to the proof of Lemma 9.17. If u is constant, then we can take $T = 0$ and c to be a constant map. Thus let us suppose that u is non-constant. Since the strip $\mathbb{R} \times [0, 1]$ is simply connected, we can apply Corollary 9.15 to conclude a holomorphic map

$$w : \mathbb{R} \times [0, 1] \to \mathbb{C} \text{ satisfying } \Gamma \circ w = u.$$

Here $\Gamma : \mathbb{C} \to \mathbb{R} \times Y$ is defined by $\sigma \circ \Gamma(\sigma, \tau) = \sigma$ and $\mathrm{pr} \circ \Gamma(\sigma, \tau) = c(\tau)$ for some Reeb flow line c (which is potentially non-embedded).

Since u has boundary on $\mathbb{R} \times \Lambda$, w must have boundary on $\Gamma^{-1}(\mathbb{R} \times \Lambda)$. As shown in Figure 5, $\Gamma^{-1}(\mathbb{R} \times \Lambda) \subset \mathbb{C}$ is a collection of horizontal lines $\tau = \text{const}$ corresponding to the places where $\gamma(\tau)$ intersects Λ.

Without loss of generality, suppose that $w(\mathbb{R} \times \{0\})$ lies in the line $\tau = 0$, and $w(\mathbb{R} \times \{1\})$ lies in the line $\tau = T$. If $T < 0$, then we replace $w(s, t) := w(-s, 1 - t)$, so now $T > 0$.

We apply the Schwarz Reflection principle repeatedly to reflect w across horizontal lines until we have extended w to a map $w : \mathbb{C} \to \mathbb{C}$, with the property that w still has bounded derivatives. To give some details of the construction, the first step is extend w to a map

$\mathbb{R} \times [-1, 1] \to \mathbb{C}$ by defining $w(s, -t) = \bar{w}(s, t)$. The subsequent steps are similar, and we leave them to the reader. See Figure 6 for an illustration of the construction.

Note that if $T = 0$, (i.e., both boundaries of w lie on the same line), then the extension $w : \mathbb{C} \to \mathbb{C}$ has a bounded imaginary part (noting that the original map w has a bounded real part because its derivative is bounded). However, it is well-known that there are no non-constant functions with bounded imaginary part.[4] Since we assume that w is non-constant, we conclude that $T > 0$.

Because the extension $w : \mathbb{C} \to \mathbb{C}$ has a bounded first derivative, w must be an affine function $w(z) = az + \sigma_0$. Since w sends the lines $t = 0$ to $\tau = 0$ and $t = 1$ to $\tau = T$, we must have $a = T$, and $\sigma_0 \in \mathbb{R}$. Thus we conclude that

$$w(s, t) = Ts + iTt + \sigma_0 \implies \sigma \circ u(s, t) = Ts + \sigma_0 \text{ and } \mathrm{pr} \circ u(s, t) = c(Tt),$$

or

$$\sigma \circ u(s, t) = -Ts + \sigma_0 \text{ and } \mathrm{pr} \circ u(s, t) = c(T(1 - t)),$$

depending on whether or not we replaced $w(s, t)$ by $w(-s, 1 - t)$. Finally, redefine $c(t) := c(Tt)$ to obtain the desired time 1 parametrization. This completes the proof. $\square$

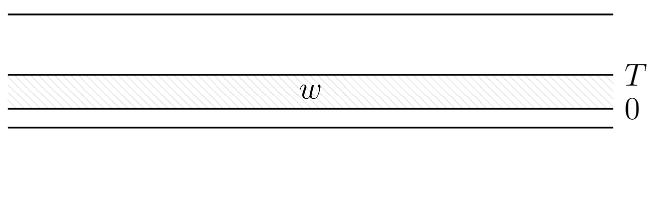


FIGURE 5. $\Gamma^{-1}(\mathbb{R} \times \Lambda)$ is a collection of horizontal lines. The holomorphic map $w : \mathbb{R} \times [0, 1] \to \mathbb{C}$ (shown as the shaded region) has boundary on $\Gamma^{-1}(\mathbb{R} \times \Lambda)$.

[4]proof: apply $z \mapsto e^{iz}$ to obtain a bounded holomorphic function.

$$
\begin{array}{rcl}
 & \vdots & \\
\hline & & T \\
t \in [0,1] & w(s,t) & \\
\hline & & 0 \\
t \in [-1,0] & \bar{w}(s,-t) & \\
\hline & & -T \\
t \in [-2,-1] & w(s,t+2) - 2iT & \\
\hline & & -2T \\
 & \vdots &
\end{array}
$$

FIGURE 6. Extending a function $w(s,t)$ with $w(s,0) \in \{0\} \times \mathbb{R}$ and $w(s,1) \in \{T\} \times \mathbb{R}$ to a holomorphic function $\mathbb{C} \to \mathbb{C}$.

9.4. Asymptotic convergence to Reeb orbits and chords

In this section we will analyze the convergence of the chords $\mathrm{pr} \circ u(s,-)$. Our goal is to prove the rest of Theorem 9.4. Let $u : [0,\infty) \times S \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ be a holomorphic map with finite Hofer energy. Pick a sequence s_n tending to $+\infty$. Consider the translated curves

$$v_n : [-\frac{1}{2}s_n, \infty) \times S \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda) \text{ given by } v_n(s,t) = u(s+s_n,t).$$

Note that $v_n(0,-) = u(s_n,-)$.

It is clear from (9.14) that $\sup_{s,t} \left|\nabla^k \mathrm{d}v_n(s,t)\right| < C_k$ for constants C_k independent of n. Moreover, the Hofer energy of v_n is bounded from above.

The $\mathrm{d}\alpha$-energy of v_n equals the $\mathrm{d}\alpha$-energy of u_n on the region $[\frac{1}{2}s_n, \infty) \times S$. As a consequence, the $\mathrm{d}\alpha$-energy of v_n is tending to zero.

As we have done before, replace $v_n := \mathrm{T}_n \circ v_n$, where T_n are vertical translations of $\mathbb{R} \times Y$, and T_n are chosen so that $v_n(0,0)$ converges.

The Arzelà-Ascoli theorem implies that v_n converges in C^∞_{loc} to a limiting holomorphic map $v_\infty : \mathbb{R} \times [0,1] \to \mathbb{R} \times Y$. By the same argument given in the proof of Lemma 9.11, the $\mathrm{d}\alpha$-energy of v_∞ is zero, and its Hofer energy is finite.

The Lemmas 9.17 and 9.18 apply, and we conclude that

$$v_\infty(s,t) = (\sigma_0 + Ts, c(t)) \text{ or } v_\infty(s,t) = (\sigma_0 - Ts, c(1-t))$$

for some Reeb chord c or orbit. Note that c may be constant (in which case $T = 0$).

The C^∞_{loc}-convergence of v_n to v_∞ implies that

$$\lim_{n\to\infty} \mathrm{dist}_{C^\infty(s_n)}(u_n(s,t); \sigma_0 + Ts, c(t)) = 0, \tag{9.15}$$

and a similar conclusion for negative punctures. Thus we have proved the second assertion of Theorem 9.4.

To complete the proof of Theorem 9.4, we need to upgrade the convergence of (9.15) by removing the dependence on the subsequence s_n. For this part we assume that the limit Reeb chord or orbit c is non-degenerate.

First we focus on the case of chords. For concreteness, let us suppose that in (9.15) $\mathrm{pr} \circ u_n(s_n, t)$ converges to $c(t)$; the case $c(1-t)$ is similar. We argue by contradiction: if $\lim_{s\to\infty} \mathrm{pr} \circ u_n(s,t)$ *does not* converge to $c(t)$, then we can find another subsequence $s'_n \to \infty$ so that $\lim_{n\to\infty} \mathrm{pr} \circ u_n(s'_n, t)$ converges to a different Reeb chord $c'(T't)$.

By further taking subsequences, we may suppose that $s_n < s'_n < s_{n+1}$. We consider $\mathrm{pr} \circ u_n(s, -)$ for $s \in [s_n, s'_n]$ as a path of chords joining $\mathrm{pr} \circ u_n(s_n, -)$ and $\mathrm{pr} \circ u_n(s'_n, -)$.

Since $c(t)$ is assumed to be a non-degenerate Reeb chord, there is a C^1 neighborhood U of $t \mapsto c(Tt)$ (in the space of chords of Λ) so that the only Reeb chord in $\bar{U}$ is c. The C^1 topology is metrizable, and hence we can find a smaller open set U' so that $c \in U'$, $\bar{U}' \subset U$.

Since $(\bar{U}')^c$ is open around c' and U' is open around c, eventually $\mathrm{pr} \circ u(s_n, -) \in U'$ and $\mathrm{pr} \circ u(s'_n, -) \in (\bar{U}')^c$. Then we conclude $s''_n \in (s_n, s'_n)$ so that $\mathrm{pr} \circ u(s''_n, -)$ lies in $U \setminus U'$.

By the same argument leading to (9.15), some subsequence of $\mathrm{pr} \circ u(s''_n, -)$ must converge to a map $t \mapsto c''(T''t)$ for some Reeb chord c''; moreover $\mathrm{pr} \circ u(s''_n, -)$ converges in $\bar{U} \setminus U'$. This contradicts the construction of $\bar{U} \setminus U'$.

Therefore $\lim_{s\to\infty} \mathrm{dist}_{C^\infty(s)}(u(s,t), c(t)) = 0$, as desired.

The argument for orbits is similar, except one can only conclude that

$$\lim_{s\to\infty} \inf_{t_0 \in \mathbb{R}/\mathbb{Z}} \mathrm{dist}_{C^\infty(s)}(u(s,t), c(t+t_0)) = 0,$$

as desired. This completes the proof of 9.4.

9.4.1. *Non-degeneracy and nearby Reeb chords/orbits.* To see why, non-degeneracy of a *chord* implies the chord is isolated in the C^1 topology, we can argue geometrically as follows. If $c_k(-)$ was a sequence of different Reeb chords converging to $c(-)$ in C^1 then (i) the periods T_k of c_k converge to T and (ii) the starting points $c_k(0)$ converge to $c(0)$. This implies that the points $(T_k, c_k(0)) \in \mathbb{R} \times \Lambda$ lie in the inverse image $\varphi^{-1}(\Lambda)$ for the Reeb flow φ. However, the transversality assumption implies that $(T, c) \in \varphi^{-1}(\Lambda)$ is an isolated point of $\varphi^{-1}(\Lambda)$. This contradicts the convergence of the sequence $(T_k, c_k(0))$.

A similar description is possible for orbits, but we can also argue using the definition of linearized operator from §3.3. Recall that if another Reeb orbit c_1 is sufficiently C^1 close to a fixed orbit c_0, then we can reparametrize $c_1 = \Phi_t(\eta(t))$, where η solves the non-linear equation $\eta'(t) + \Pi_t(\eta(t))\Phi'(\eta(t))$.

Then, by definition of linearization, we conclude that:

$$\eta'(t) - JS(t)\eta(t) = B(\eta) \cdot \eta(t) \cdot \eta(t),$$

where $A = -J\partial_t - S(t)$ is a *non-degenerate* asymptotic operator. Then, if we can find a sequence of solutions c_n which converge in C^1 to c_0, then we can rescale the resulting η_n to obtain a non-trivial element in the kernel of A, which we assume is impossible. The only possibility is if $\eta_n = 0$ holds identically, in which case c_n is a reparametrization of c_0.

Chapter 10

Exponential decay

Let σ, τ, x be the coordinates from Remark 4.2, centered on a *non-degenerate* Reeb orbit or chord c. We say that σ, τ form the *tangential* coordinates, while x forms the *transverse* coordinate.

Suppose that σ, τ, x are the coordinates of a holomorphic map $u : [s_0, s_1] \times S \to \mathbb{R} \times Y$, where $S = \mathbb{R}/\mathbb{Z}$ or $S = [0, 1]$. In the latter case, we suppose that u takes boundary on Λ, which is equivalent to $\tau = 0$ on $[s_0, s_1] \times \{0, 1\}$.

Let $\bar{f}$ denote the average $\int_0^1 f \mathrm{d}t$, and let $\sigma_0 = \bar{\sigma}(s_*)$ for $s_* = \frac{1}{2}(s_0 + s_1)$. In the $S = [0, 1]$ case, set $\tau_0 = 0$, while in the $S = \mathbb{R}/\mathbb{Z}$ case set $\tau_0 = \bar{\tau}(s_*)$.

We have the following exponential convergence result:

Theorem 10.1. For $\epsilon > 0$ sufficiently small, there exists $M_k = o(1)$ as $\epsilon \to 0$ so the following holds. Whenever σ, τ, x are the coordinates of a holomorphic curve u, and τ, x are C^{k+1} ϵ-small, the following estimate holds:

$$\sum_{k=0}^{\ell} \left|\nabla^\ell x\right| + \left|\nabla^\ell(\tau - \tau_0)\right| + \left|\nabla^\ell(\sigma - \sigma_0)\right| \leq M_k(e^{-\delta(s-s_0)} + e^{-\delta(s_1 - s)}),$$

for all $s \in [s_0, s_1]$. Here $\delta > 0$ depends only on the asymptotic operator A.

We give the proof at the end of §10.7.5.

10.1. The PDE near an orbit or chord

As explained in the proof of Lemma 4.7, we have the following equations

$$\begin{aligned} \partial_s \sigma - \partial_t \tau &= E_{11} \cdot \tau \cdot \partial_t x + E_{12} \cdot x \cdot x + E_{13} \cdot x \cdot \partial_t x + E_{14} \cdot x \cdot \tau \\ \partial_t \sigma + \partial_s \tau &= E_{21} \cdot \tau \cdot \partial_t x + E_{22} \cdot x \cdot x + E_{23} \cdot x \cdot \partial_t x + E_{24} \cdot x \cdot \tau \\ \partial_s x + J \partial_t x + S(t)x &= E_{31} \cdot \tau \cdot \partial_t x + E_{32} \cdot x \cdot x + E_{33} \cdot x \cdot \partial_t x + E_{34} \cdot x \cdot \tau. \end{aligned} \tag{10.1}$$

where E_{ij} are all smooth tensor valued functions of τ, x. For future use, we let:

$$R_j = E_{j1} \cdot \tau \cdot \partial_t x + E_{j2} \cdot x \cdot x + E_{j3} \cdot x \cdot \partial_t x + E_{j4} \cdot x \cdot \tau.$$

10.2. Differential inequality for the transverse coordinate

In this section we prove a differential inequality for the x coordinate.

Lemma 10.2. Suppose that σ, τ, x solve the above holomorphic curve equation (10.1). Let $\gamma(s) = \frac{1}{2}\|x\|^2$. There exists $\epsilon > 0$ with the following property. Suppose that τ and x are C^2 ϵ-small on $[s_0, s_1] \times S$, where $S = \mathbb{R}/\mathbb{Z}$ or $S = [0,1]$, and that the asymptotic operator $A = -J\partial_t x - S(t)x$ is *non-degenerate*. Then

$$\gamma''(s) - d^2\gamma(s) \geq \frac{1}{3}(\|\partial_s x\|^2 + \|A(x)\|^2),$$

where $d = d(A) > 0$ is the largest constant so that $d^2\|\xi\|^2 \leq \frac{1}{3}\|A(\xi)\|^2$.

Proof. A straightforward computation establishes that

$$\gamma''(s) = \|\partial_s x\|^2 + \langle x, \partial_s\partial_s x\rangle.$$

We replace $\partial_s x = Ax + R_3$, to obtain

$$\langle x, \partial_s\partial_s x\rangle = \langle x, A\partial_s x + \partial_s R_3\rangle = \langle x, A\partial_s x\rangle + \langle x, \partial_s R_3\rangle,$$

where we have used the fact that A is s-independent. Since x and $\partial_s x$ both take boundary values in $\mathbb{R}^n$, (or we are on $\mathbb{R}/\mathbb{Z}$), we can use the self-adjointness of A to conclude

$$\langle x, A\partial_s x\rangle = \langle Ax, \partial_s x\rangle = \|Ax\|^2 + \langle Ax, R_3\rangle.$$

Applying $d^2\|x\|^2 \leq \frac{1}{3}\|Ax\|^2$, we conclude:

$$\gamma'' - d^2\gamma \geq \frac{1}{3}(\|\partial_s x\|^2 + \|Ax\|^2) + (\frac{2}{3}\|\partial_s x\|^2 + \frac{1}{3}\|Ax\|^2 + \langle Ax, R_3\rangle + \langle x, \partial_s R_3\rangle).$$

Thus, in order to prove the lemma, it is sufficient to prove that

$$|\langle Ax, R_3\rangle + \langle x, \partial_s R_3\rangle| \leq \frac{1}{3}(\|Ax\|^2 + 2\|\partial_s x\|^2), \tag{10.2}$$

provided the C^2 sizes of x and τ are less than ϵ. We will now proceed to do this. It is fairly easy to estimate:

$$|\langle Ax, R_3\rangle| \leq \frac{1}{9}\|Ax\|^2,$$

provided ϵ is sufficiently small (using $\|x\|_{W^{1,2}} \leq C_A\|Ax\|$ since $A : W^{1,2} \to L^2$ is an isomorphism).

It is similarly easy to estimate:

$$|\langle x, \partial_s(E_{31}\cdot\tau)\cdot\partial_t x + \partial_s(E_{32}x\cdot x + E_{33}x\cdot\partial_t x + E_{34}\tau\cdot x)\rangle| \leq \frac{1}{9}\|Ax\|^2 + \frac{1}{9}\|\partial_s x\|^2,$$

one uses Cauchy-Schwarz, $2\|a\|\|b\| \leq \|a\|^2 + \|b\|^2$, and $\|x\|_{W^{1,2}} \leq C_A\|Ax\|_{L^2}$.

The remaining term involves second derivatives of x and needs to be integrated by parts. We have

$$\langle x, E_{31} \cdot \tau \cdot \partial_t \partial_s x\rangle = -\langle \partial_t x, E_{31} \cdot \tau \cdot \partial_s x\rangle - \langle x, \partial_t(E_{31} \cdot \tau) \cdot \partial_s x\rangle.$$

In the $\mathbb{R}/\mathbb{Z}$ case we can always integrate by parts. In the $[0,1]$ case we use the fact that τ vanishes at both endpoints. Then it is clear that both terms can be bounded above by $\frac{1}{9}(\|Ax\|^2 + \|\partial_s x\|^2)$. Combining everything gives (10.2), and this completes the proof. □

10.3. The relationship between $\gamma'' - \delta^2\gamma$ and exponential estimates.

There are many results in the theory of holomorphic curves which involve quantities decaying exponentially. Many of these results begin by establishing an estimate involving the combination $\gamma'' - \delta^2\gamma$, and usually γ is the L^2 size (squared) of some quantity. We give four examples from the literature, arranged chronologically:

(i) In [**Flo89b**, Lemma 5.2], Floer establishes an estimate of the form

$$\gamma''(s) - \delta^2\gamma(s) \geq 0 \tag{10.3}$$

for a certain quantity $\gamma : \mathbb{R} \to [0, \infty)$. Since γ is non-negative and defined on all $\mathbb{R}$, γ must be identically zero. This is part of the argument Floer uses to show that the Floer homology of L with L_f (the graph of $\mathrm{d}f$ in T^*L) can be computed in terms of the Morse complex of f.

(ii) In [**Sal97**, Lemma 2.11], Salamon considers the quantity

$$\gamma(s) := \frac{1}{2}\int_0^1 |\xi(s,t)|^2 \, dt$$

where $\xi(s,t)$ solves an equation of the form $\partial_s \xi + J_0 \partial_t \xi + S(t)\xi = 0$. He then shows that γ satisfies (10.3) for s sufficiently large, using the assumption that $\xi \mapsto J_0\partial_t\xi + S(t)\xi$ is an isomorphism. Salamon then shows that $\gamma(s)$ decays exponentially with rate δ, for s sufficiently large.

(iii) In [**RS01**, Lemma 3.1], Robbin-Salamon consider a function $\gamma : [0, \infty) \to \mathbb{R}^+$ satisfying inequality of the form

$$\gamma''(s) - \delta^2\gamma(s) \geq -c_0 e^{-\epsilon s}.$$

The authors use to show that $\gamma(s)$ decays exponentially with rate δ, provided $\epsilon > \delta$.

(iv) In [**HWZ02**, Lemma 3.6], Hofer-Wysocki-Zehnder consider the L^2 size

$$\gamma(s) = \int_{\mathbb{R}/\mathbb{Z}} |z(s,t)|^2 \, \mathrm{d}t,$$

where z is a $\mathbb{R}^{2n-2}$-valued coordinate near a Reeb orbit measuring the directions transverse to ∂_σ and R. They show that $\gamma : [-r, r] \to \mathbb{R}^+$ satisfies the estimate (10.3). The authors then use (10.3) to show that $\gamma(s) \leq Ae^{-\delta(r+s)} + Be^{-\delta(r-s)}$ for appropriately chosen constants A, B.

We will prove the following lemma:

Lemma 10.3. Let $\gamma : [-r, r] \to \mathbb{R}^+$, $\alpha : [-r, r] \to \mathbb{R}^+$ and $\kappa : [-r, r] \to \mathbb{R}^+$ be smooth functions satisfying

$$\gamma'' - \delta^2\gamma \geq \alpha - \kappa, \tag{10.4}$$

for some constant $\delta > 0$. Suppose that $\kappa \leq K_1 e^{-D(r+s)} + K_2 e^{-D(r-s)}$ for $K_1, K_2 \geq 0$ and $D > \delta$. Then

$$\gamma \leq C_1 e^{-\delta(r+s)} + C_2 e^{-\delta(r-s)} \text{ and } \int_{s-0.5}^{s+0.5} \alpha(s)\,\mathrm{d}s \leq A_1 e^{-\delta(r+s)} + A_2 e^{-\delta(r-s)},$$

where the inequality involving α holds only for $s \in [-r+1, r-1]$, while the inequality involving γ holds for all $s \in [-r, r]$. The constants are given by

$$\begin{aligned} C_1 &= \gamma(-r) + \frac{K_1 + K_2 e^{-2Dr}}{D^2 - \delta^2} \\ C_2 &= \frac{\gamma'(r) - \delta\gamma(r)}{2\delta} + \frac{K_2}{2\delta(D-\delta)}. \\ A_i &= e^{\delta}(40 + 2\delta^2)C_i + 2e^{D}K_i \end{aligned}$$

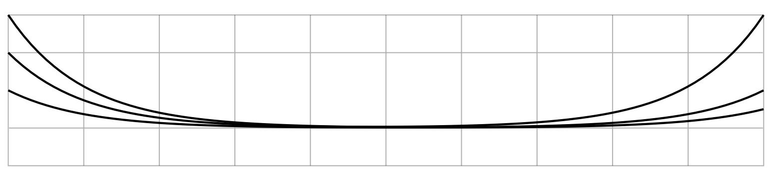


FIGURE 1. Plots of $C_1 e^{-\delta(r+s)} + C_2 e^{-\delta(r-s)}$ for various values of C_1 and C_2. The values C_1, C_2 are the values taken by the function at the left and right endpoints, respectively.

Proof. We begin by introducing the function

$$\beta := \gamma - Be^{-\delta(r-s)} + \frac{K_1}{D^2 - \delta^2} e^{-D(r+s)} + \frac{K_2}{D^2 - \delta^2} e^{-D(r-s)},$$

for a constant B to be determined at a later stage. It is straightforward to observe that

$$\beta'' - \delta^2\beta = \gamma'' - \delta^2\gamma + \left[K_1 e^{-D(r+s)} + K_2 e^{-D(r-s)}\right] \geq \alpha \geq 0.$$

The trick now is to observe that

$$\frac{\mathrm{d}}{\mathrm{d}s}(e^{-\delta s}(\beta' + \delta\beta)) \geq 0 \implies e^{-\delta s}(\beta' + \delta\beta) \text{ is increasing.} \tag{10.5}$$

Now we will pick the constant $B > 0$ so that $\beta' + \delta\beta$ is non-positive at the right endpoint $s = r$. We compute

$$\beta'(r) + \delta\beta(r) = \gamma'(r) + \delta\gamma(r) - 2\delta B + \frac{(D+\delta)K_2 - (D-\delta)K_1 e^{-2Dr}}{D^2 - \delta^2}.$$

Therefore

$$\beta'(r) + \delta\beta(r) \leq \gamma'(r) + \delta\gamma(r) + \frac{K_2}{D-\delta} - 2\delta B.$$

This leads us to make the choice

$$B = \frac{\gamma'(r) + \delta\gamma(r)}{2\delta} + \frac{K_2}{2\delta(D-\delta)}.$$

With this choice of B we can conclude from (10.5) that $\beta'(s) + \delta\beta(s) \leq 0$ holds *for all* s.

Then we can integrate $\beta'(s) + \delta\beta(s) \leq 0$ to conclude

$$e^{\delta s}\beta(s) \leq e^{-\delta r}\beta(-r) \text{ for } s \in [-r, r].$$

Thus $\beta(s) \leq e^{-\delta(r+s)}\beta(-r)$, and hence

$$\gamma(s) \leq \beta(-r)e^{-\delta(r+s)} + Be^{-\delta(r-s)}.$$

We estimate $\beta(-r)$ as follows

$$\beta(-r) \leq \gamma(-r) + \frac{K_1 + K_2 e^{-2Dr}}{D^2 - \delta^2} =: C_1.$$

We set $C_2 = B$. This completes the first part of the proof.

To estimate the integral $\int_{s-0.5}^{s+0.5} \alpha(s)\,\mathrm{d}s$, we will use a convolution trick. We introduce a smooth symmetric bump function ρ which is supported in $(-1, 1)$ and which equals 1 on $[-0.5, 0.5]$, as shown in Figure 2.

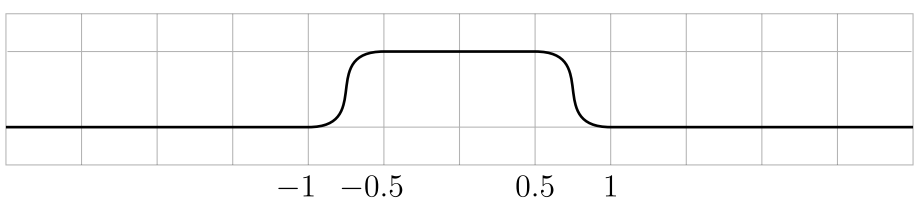


FIGURE 2. The bump function ρ. We have $\|\rho\|_{L^1} \leq 2$. Moreover, this can be achieved with $\|\rho''\|_{L^2} \leq 4\pi^2 + \epsilon \leq 40$, because one can take a smooth ϵ-approximation of the C^2 function $0.5 - 0.5\cos(2\pi x)$ which interpolates from 0 to 1 over an interval of size 0.5 (we need two copies of this function).

We convolve both sides of (10.4) with ρ, yielding

$$(\rho'') * \gamma - \delta^2(\rho * \gamma) \geq \rho * \alpha - \rho * \kappa.$$

This holds only on the restricted domain $s \in [-r+1, r-1]$. For functions f supported in $[-1,1]$ it is simple to estimate

$$|f * \gamma|(s) \le \max_{s' \in [s-1,s+1]} |\gamma(s)| \, \|f\|_{L^1} \le e^{\delta} \|f\|_{L^1} (C_1 e^{-\delta(r+s)} + C_2 e^{-\delta(r-s)}).$$

A similar conclusion holds with γ replaced by κ. We conclude

$$\begin{aligned} \rho * \alpha(s) \le e^{\delta}(\|\rho''\|_{L^1} + \delta^2 \|\rho\|_{L^1})(C_1 e^{-\delta(r+s)} + C_2 e^{-\delta(r-s)}) \\ + e^{D} \|\rho\|_{L^1} (K_1 e^{-D(r+s)} + K_2 e^{-D(r-s)}). \end{aligned}$$

Using the fact that $D > \delta$, and $\|\rho\|_{L^1} \le 2$ and $\|\rho''\|_{L^1} \le 40$, we obtain

$$\rho * \alpha(s) \le \left[e^{\delta}(40 + 2\delta^2)C_1 + 2e^{D}K_1\right] e^{-\delta(r+s)} + \left[e^{\delta}(40 + 2\delta^2)C_2 + 2e^{D}K_2\right] e^{-\delta(r-s)}.$$

Setting $A_i := e^{\delta}(40 + 2\delta^2)C_i + 2e^{D}K_i$, we conclude

$$\int_{s-0.5}^{s+0.5} \alpha(s) \, \mathrm{d}s \le \rho * \alpha(s) \le A_1 e^{-\delta(r+s)} + A_2 e^{-\delta(r-s)}.$$

This completes the proof. $\square$

Remark 10.4. A simpler proof in the case when $\kappa = \alpha = 0$ (with different constants C_1, C_2) is possible using a slightly different argument: observe that $\beta = \gamma - c\cosh(\delta s)$ satisfies $\beta'' - \delta^2 \beta \ge 0$, and hence cannot attain a positive interior maximum. Thus if c is chosen so that β is non-positive at both endpoints, then β must be everywhere non-positive, hence $\gamma \le c\cosh(\delta s)$, as desired. This is the argument used in [**HWZ02**, Lemma 3.6].

10.4. Exponential estimates on the $W^{1,2}$ norm of the transverse coordinates

Throughout this section let $u : [s_0, s_1] \times S \to \mathbb{R} \times Y$ be a holomorphic curve so that $\mathrm{pr} \circ u(s,t)$ remains sufficiently close to the non-degenerate Reeb chord c so that the differential inequality from §10.2 can be applied.

We have the following result:

Lemma 10.5. Let ϵ, d be the constants from Lemma 10.2, and suppose that x, τ are C^2 ϵ-small on $[s_0, s_1]$. Then there is a constant C, depending only on the asymptotic operator A, so that:

$$\int_{s-0.5}^{s+0.5} \int_0^1 |x|^2 + |\partial_s x|^2 + |\partial_t x|^2 \, \mathrm{d}s\mathrm{d}t \le C \|x\|_{C^1}^2 (e^{-d(s-s_0)} + e^{-d(s_1-s)}), \tag{10.6}$$

for $s \in [s_0 + 0.5, s_1 - 0.5]$.

Proof. Note that the estimate holds for $s \in [s_0 + 0.5, s_0 + 1] \cup [s_1 - 1, s_1 - 0.5]$, simply by making C larger than $3/e^{-d}$.

For the other regions, we will apply Lemma 10.3 to the differential inequality from §10.2. In this case we have $\kappa = 0$, and we conclude that for $s \in [s_0 + 1, s_1 - 1]$ that

$$\int_{s-0.5}^{s+0.5} \|\partial_s x\|^2 + \|A(x)\|^2 \leq c_1(x)[e^{-d(s-s_0)} + e^{-d(s_1-s)}],$$

where

$$c_1(x) = 3e^d(40 + 2d^2)\gamma(-s_0 - 1) + 3(40 + 2d^2)e^d \frac{\gamma'(s_0 + 1) - d\gamma(s_0 + 1)}{2d}.$$

It is clear that $c_1(x) \leq C_1 \|x\|_{C^1}^2$, where C_1 is independent of ϵ. Since C_1 depends on d, it implicitly depends on the asymptotic operator A.

Next we use $\|x\|^2 + \|\partial_t x\|^2 \leq C_A \|A(x)\|^2$, and conclude that

$$\int_{s-0.5}^{s+0.5} \|x\|^2 + \|\partial_s x\|^2 + \|\partial_t x\|^2 \,\mathrm{d}s \leq C \|x\|_{C^1}^2 [e^{-d(s-s_0)} + e^{-d(s_1-s)}],$$

for a modified constant. This completes the proof. □

Corollary 10.6. Assume that σ, τ, x solve (10.1) and x, τ are C^2 ϵ-small on $[s_0, s_1] \times S$. Write:

$$\|x\|_{W^{1,2,1}(s_0,s_1)} := \int_{s_0}^{s_1} \|x\| + \|\partial_s x\| + \|\partial_t x\| \,\mathrm{d}s$$

Then $\|x\|_{W^{1,2,1}(s_0,s_1)} < C_1 \|x\|_{C^1}$ where C_1 is independent of s_0, s_1. Keep in mind that $\|-\|$ denotes the L^2 norm over S. Moreover, if we write:

$$\|x\|_{W^{1,1}(s_0,s_1)} := \int_{s_0}^{s_1} \int_0^1 |x| + |\partial_s x| + |\partial_t x| \,\mathrm{d}t\mathrm{d}s$$

Then $\|x\|_{W^{1,1}(s_0,s_1)} < C_2 \|x\|_{C^1}$ where C_2 is independent of s_0, s_1.

Proof. We begin by proving the estimate on the mixed norm. We have:

$$\int_{s_0+k-0.5}^{s_0+k+0.5} \|x\| + \|\partial_s x\| + \|\partial_t x\| \,\mathrm{d}s \leq c[\int_{s_0+k-0.5}^{s_0+k+0.5} \|x\|^2 + \|\partial_s x\|^2 + \|\partial_t x\|^2 \,\mathrm{d}s]^{1/2}.$$

This can be proved by Cauchy-Schwarz and estimating $(\|x\| + \|\partial_s x\| + \|\partial_t x\|)^2$. The exponential estimates give:

$$\int_{s_0+k-0.5}^{s_0+k+0.5} \|x\|^2 + \|\partial_s x\|^2 + \|\partial_t x\|^2 \,\mathrm{d}s \leq C \|x\|_{C^1}^2 (e^{-dk} + e^{-d(s_1-s_0+k)}).$$

Taking square roots and using the above estimate, we have

$$\int_{s_0+k-0.5}^{s_0+k+0.5} \|x\| + \|\partial_s x\| + \|\partial_t x\| \,\mathrm{d}s \leq C' \|x\|_{C^1} (e^{-dk/2} + e^{-d(s_1-s_0+k)/2}).$$

It is easy to see that this sum is finite, and bounded independently of s_0, s_1. This completes the proof of the first part.

For the second part, we simply use the fact that $\int_0^1 |f| \,\mathrm{d}t \leq \|f\|$. □

10.5. Uniform convergence for the tangential coordinates

Suppose throughout this section that σ, τ, x solves (10.1) on $[0, \infty) \times S$.

Proposition 10.7. There exist two constants σ_0, τ_0 so that

$$\lim_{s\to\infty} \mathrm{dist}_{C^\infty(s)}(\sigma, \sigma_0) + \lim_{s\to\infty} \mathrm{dist}_{C^\infty(s)}(\tau, \tau_0) = 0, \tag{10.7}$$

where $\mathrm{dist}_{C^\infty(s)}$ is the C^∞ distance for the restriction to $[s-1, s+1] \times S$.

Proof. This uses a trick I learned from [**Bou02**, §3].

We know that from the bubbling analysis that, for any sequence $s_k \to \infty$, there is a further subsequence and constants σ_0, τ_0 (depending on the subsequence), so that

$$\lim_{\ell\to\infty} \mathrm{dist}_{C^\infty(s_k)}(\sigma, \sigma_0) + \mathrm{dist}_{C^\infty(s_k)}(\tau, \tau_0) = 0.$$

Let us abbreviate the above by saying that $\sigma(s_k, t), \tau(s_k, t)$ converges to σ_0, τ_0.

If we can show that there are σ_0, τ_0 so that *every* sequence has a convergent subsequence with the same limits σ_0, τ_0, then (10.7) follows. This argument is called the *subsequence trick*, and we assume the reader is familiar with it.

Rotate the coordinate system, if necessary, so that τ has 0 as limit point (i.e., $\tau_0 = 0$ is a limit for some subsequence).

The trick is to integrate the first two equations of (10.1) over $[s_0, s_1] \times S$ to conclude:

$$\int_0^1 \sigma(s_1, t) - \sigma(s_0, t)\mathrm{d}t \leq C \left\| x \right\|_{W^{1,1}(s_0,s_1)}, \tag{10.8}$$

and

$$\int_0^1 \tau(s_1, t) - \tau(s_0, t)\mathrm{d}t + \int_{s_0}^{s_1} \sigma(s, 1) - \sigma(s, 0)\mathrm{d}s \leq C \left\| x \right\|_{W^{1,1}(s_0,s_1)}. \tag{10.9}$$

Corollary 10.6 implies $\left\| x \right\|_{W^{1,1}(s_0,s_1)} \leq C \left\| x \right\|_{C^1}$, where C does not depend on the size of the domain.

Thus the left hand sides of (10.8) and (10.9) converge to zero whenever s_0^n, s_1^n both converge to ∞, and the C^2 size of x, τ remains ϵ-small on $[s_0^n, s_1^n]$.

We can find a sequence s_0^n, s_1^n where $\sigma(s_0^n, t), \tau(s_0^n, t)$ converges to $\sigma_0, 0$ and s_1^n is the largest number so that the C^2 size of x, τ remains ϵ small on $[s_0^n, s_1^n + 1]$. In particular, there must be some point so that the C^2 size of x, τ is at least $\epsilon/2$.

A subsequence of $\sigma(s_1^n, t)$ must converge to σ_0'. But clearly we must have $\sigma_0' = \sigma_0$, by (10.8). Similarly, a subsequence of $\tau(s_1^n, -)$ must converge to 0 in $\mathrm{dist}_{C^\infty(s_1^n)}$, by (10.9), but this contradicts the fact that the C^2 size was at least $\epsilon/2$. Thus we conclude that we can take s_1^n

to be any number larger than s_0^n. It follows by the subsequence argument that the $s \to \infty$ limit of σ, τ is unique, and hence (10.7) holds, as desired. $\square$

The above argument also shows that the *average* of σ converges exponentially to σ_0, and a similar result for τ.

Corollary 10.8. Let $\delta := d/2$. Suppose that x, τ are ϵ C^2-small on $[s_0, s_1] \times S$. Then

$$|\bar{\sigma}(s) - \bar{\sigma}(s_*)\mathrm{d}t| \leq C \|x\|_{C^1} (e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

and, if $S = \mathbb{R}/\mathbb{Z}$,

$$|\bar{\tau}(s) - \bar{\tau}(s_*)\mathrm{d}t| \leq C \|x\|_{C^1} (e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

where $s_* = \frac{1}{2}(s_0 + s_1)$ and $\bar{f}$ is the average of f.

Proof. Without loss, suppose $s < s_*$. The above argument implies that

$$|\bar{\sigma}(s) - \bar{\sigma}(s_*)\mathrm{d}t| \leq C \|x\|_{W^{1,1}(s,s_*)}.$$

Cover $[s, s_*] \times S$ by finitely many intervals I of the form $[s' - 0.5, s' + 0.5]$ and for each apply the exponential estimate to conclude:

$$\|x\|_{W^{1,1}(I)} \leq C_1 \|x\|_{W^{1,2}(I)} \leq C_2 \|x\|_{C^1} (e^{-\delta(s'-s_0)} + e^{-\delta(s_1-s')}).$$

To get the full $W^{1,1}$ norm over $[s, s_*]$, we need to sum the above for all s' in "lattice" of step length 1 contained in $[s, s_*]$. We can estimate this from above as

$$\sum_{k=0}^{\infty} e^{-\delta(s-s_0+k)} + \sum_{k=0}^{\infty} e^{-\delta(s_1-s_*-k)} \leq C_3[e^{-\delta(s-s_0)} + e^{-\delta(s_1-s_*)}] \leq 2C_3 e^{-\delta(s-s_0)}.$$

A similar argument works when $s_* < s$, but with $e^{-\delta(s_1-s)}$ instead. Adding the two estimates concludes the desired result. The same argument works for τ in the $\mathbb{R}/\mathbb{Z}$ case, using that $\sigma(s, 0) = \sigma(s, 1)$. $\square$

Remark 10.9. Henceforth, we will typically denote $\bar{\sigma}(s_*) = \sigma_0$ and $\bar{\tau}(s_*) = \tau_0$.

10.6. C^k exponential estimates on the transverse coordinate.

The goal in this section is to establish C^k exponential estimates on x. The key idea is to bootstrap the $W^{1,2}$ exponential estimates in Lemma 10.5 to establish C^k exponential estimates on x.

10.6.1. *Elliptic bootstrapping.* We will need to use the following linear elliptic estimates for $\bar{\partial} = \partial_s + J_0 \partial_t$ and $\Delta = \partial_s^2 + \partial_t^2$.

Lemma 10.10. Consider a sequence $0.5 = \rho_0 > \rho_1 > \rho_2 > \cdots > 0.1$. Abbreviate the domains $\Omega_k = [-\rho_k, \rho_k] \times S$. There exist constants L_k so that for $k \geq 1$

$$\|x\|_{W^{k,2}(\Omega_k)} \leq L_k(\|\bar{\partial}x\|_{W^{k-1,2}(\Omega_{k-1})} + \|x\|_{W^{k-1,2}(\Omega_{k-1})}) \tag{10.10}$$

for every smooth function $x : \Omega_0 \to \mathbb{R}^{2n}$, with $x(s,0), x(s,1) \in \mathbb{R}^n$ when $S = [0,1]$. Similarly there are constants c_k so that for $k \geq 2$

$$\|\tau\|_{W^{k,2}(\Omega_k)} \leq L_k(\|\Delta\tau\|_{W^{k-2,2}(\Omega_{k-1})} + \|\tau\|_{W^{k-1}(\Omega_{k-1})}) \tag{10.11}$$

for every smooth function $\tau : \Omega_0 \to \mathbb{R}$, with $\tau(s,0) = \tau(s,1) = 0$ when $S = [0,1]$.

Proof. We prove the case when $S = [0,1]$, leaving the easier $S = \mathbb{R}/\mathbb{Z}$ case to the reader.

First we prove the elliptic estimate for $\bar{\partial}$, following [**RS01**, Lemma C.1]. If x has compact support with $\mathbb{R}^n$ boundary conditions, we compute

$$\int_{\Omega_{k-1}} |\bar{\partial}x|^2 = \int_{\Omega_{k-1}} |\partial_s x|^2 + |\partial_t x|^2 \,\mathrm{d}s\mathrm{d}t.$$

Now let β_k be a bump function which is 1 on Ω_k and supported in Ω_{k-1}, and compute

$$\begin{aligned}
\|x\|_{W^{1,2}(\Omega_k)} &\leq \|\beta x\|_{W^{1,2}(\Omega_{k-1})} \leq \|\beta x\|_{L^2(\Omega^{k-1})} + \|\mathrm{d}(\beta x)\|_{L^2(\Omega_{k-1})} \\
&\leq \|\beta x\|_{L^2(\Omega_{k-1})} + \|(\bar{\partial}\beta)x\|_{L^2(\Omega_{k-1})} + \|\beta\bar{\partial}(x)\|_{L^2(\Omega_{k-1})} \\
&\leq c_k(\|\bar{\partial}x\|_{L^2(\Omega_{k-1})} + \|x\|_{L^2(\Omega_{k-1})}).
\end{aligned}$$

A similar estimate holds with x replaced by $\nabla^\ell x$ (since $\bar{\partial}$ commutes with derivatives). Summing over $\ell = 0, \cdots, k-1$ we conclude

$$\|x\|_{W^{k,2}(\Omega_k)} \leq L_k(\|\bar{\partial}x\|_{W^{k-1,2}(\Omega_{k-1})} + \|x\|_{W^{k-1,2}(\Omega_{k-1})}),$$

as desired.

To establish the elliptic estimate for $\Delta\tau$, we insert an intermediate domain $\Omega_k \subset \Omega' \subset \Omega_{k-1}$ and compute

$$\|i\tau\|_{W^{k,2}(\Omega_k)} \leq c'(\|\bar{\partial}i\tau\|_{W^{k-1,2}(\Omega')} + \|i\tau\|_{W^{k-1,2}(\Omega')}).$$

Observe that $\partial = \partial_s - J_0\partial_t$ is conjugate to $\bar{\partial}$, and hence satisfies the same elliptic estimates. Since $\bar{\partial}(i\tau)$ is real along the boundary, we can apply these estimates to $\bar{\partial}(i\tau)$. Thus

$$\|\bar{\partial}\tau\|_{W^{k-1,2}(\Omega')} \leq c''(\|\partial\bar{\partial}\tau\|_{W^{k-2,2}(\Omega_{k-1})} + \|\bar{\partial}\tau\|_{W^{k-2,2}(\Omega_{k-1})}).$$

It is easy to see that $\partial\bar{\partial} = \Delta$, and hence the combination of our two estimates yields the desired result (10.11). ☐

10.6.2. *Bootstrapping the estimate for the transverse coordinate.* The main result of this section is the following exponential estimate on the x coordinate:

Lemma 10.11. There is $C_k > 0$ so that for $\epsilon > 0$ sufficiently small we have the following: Suppose that x, σ, τ solve (10.1) and x, τ are C^{k+1} ϵ-small on $[s_0, s_1]$. Then:

$$\sum_{\ell=1}^{k} \left|\nabla^\ell x(s,t)\right| \le C_k \left\|x\right\|_{C^1} (e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

for $s \in [s_0 + 0.5, s_1 - 0.5]$, where $\delta = \frac{1}{2}d$.

We can easily modify the statement to get an estimate which holds on all of $[s_0, s_1]$, although this modification obscures the fact that the first derivatives of x control the higher derivatives on the interior.

Corollary 10.12. For $\epsilon > 0$ sufficiently small there is $D_k = o(1)$ as $\epsilon \to 0$ satisfying the following. Suppose that x, σ, τ solve (10.1) and x, τ are C^{k+1} ϵ-small on $[s_0, s_1]$. Then:

$$\sum_{\ell=1}^{k} \left|\nabla^\ell x(s,t)\right| \le D_k (e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

for $s \in [s_0, s_1]$. $\square$

Proof. The estimate is trivial to establish on the ends $[s_0, s_0 + 1] \cup [s_1 - 1, s_1]$ (i.e., we can pick $D_k = \epsilon$), and hence it suffices to establish the estimate on the interior interval $[s_0 + 1, s_1 - 1]$.

Fix some $s \in [s_0 + 0.5, s_1 - 0.5]$, and let $\Omega_k = [s - \rho_k, s + \rho_k] \times S$, as in §10.6.1. The holomorphic curve equation for x implies that:

$$\bar{\partial} x = -S(t)x + R_3.$$

It is straightforward to estimate $\left\|R_3\right\|_{W^{1,2}(\Omega_1)} \le o(1) \left\|x\right\|_{W^{2,2}(\Omega_1)}$ as $\epsilon \to 0$. One inspects (10.1) and uses estimates of the form $\left\|a \cdot b \cdot c\right\|_{W^{k,2}} \le \left\|a\right\|_{C^k} \left\|b\right\|_{C^k} \left\|c\right\|_{W^{k,2}}$, etc.

Hence, for ϵ sufficiently small we can estimate:

$$\left\|\bar{\partial} x\right\|_{W^{1,2}(\Omega_1)} \le C \left\|x\right\|_{W^{1,2}(\Omega_1)} + o(1) \left\|x\right\|_{W^{2,2}(\Omega_1)},$$

where C depends on $S(t)$. Then we apply the elliptic estimates to conclude:

$$\left\|x\right\|_{W^{2,2}(\Omega_2)} \le 2L_2 C \left\|x\right\|_{W^{1,2}(\Omega_1)}.$$

Continuing in this fashion, we conclude, up to $\left\|R_3\right\|_{W^{k+1,2}(\Omega_1)} \le o(1) \left\|x\right\|_{W^{k+2,2}(\Omega_1)}$ as $\epsilon \to 0$, which yields:

$$\left\|x\right\|_{W^{k+2,2}(\Omega_{k+2})} \le 2^{k+1} L_{k+2} \cdots L_2 C^{k+1} \left\|x\right\|_{W^{1,2}(\Omega_1)},$$

provided ϵ is sufficiently small.

Lemma 10.5 then yields:

$$\left\|x\right\|_{W^{1,2}(\Omega_1)} \le C \left\|x\right\|_{C^1} (e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

Finally, the Sobolev embedding theorem yields

$$\|x\|_{C^k(\Omega_{k+2})} \leq D_k \|x\|_{C^1} (e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

where D_k depends only on $S(t)$, the constants $L_1, \ldots, L_{k+2}$, and the constant from the Sobolev embedding theorem. Since (s,t) lies in Ω_{k+2}, we conclude the desired result. $\square$

10.7. Exponential convergence for the tangential coordinates

In this section we prove exponential decay estimates for the τ and σ coordinate. The argument splits into two cases, depending on whether $S = [0,1]$ or $S = \mathbb{R}/\mathbb{Z}$.

10.7.1. *Differential inequality for τ when $S = [0,1]$.* The idea is to prove that τ satisfies a differential inequality for which we can apply Lemma 10.3.

Lemma 10.13. Introduce the quantity:

$$\Gamma(s) = \frac{1}{2}\int_0^1 |\tau(s,t)|^2 \, dt.$$

There is $c > 0$ (depending only on the constant from the Poincaré lemma for $[0,1]$) and $\epsilon > 0$ with the following property: if the C^2 sizes of x, τ is less than ϵ, then

$$\Gamma''(s) - c^2\Gamma(s) \geq \frac{1}{3}(\|\partial_s\tau\|^2 + \|\partial_t\tau\|^2 - \|A(x)\|^2). \tag{10.12}$$

Proof. The proof uses the Poincaré inequality for functions on $[0,1]$ which vanish on both endpoints. In particular, there is a constant c so that:

$$\frac{c^2}{2}\|\tau\|^2 \leq \frac{1}{3}\|\partial_t\tau\|^2.$$

Using the holomorphic curve equations (10.1), we compute

$$\begin{aligned}\Gamma''(s) &= \|\partial_s\tau\|^2 + \langle\tau, \partial_s\partial_s\tau\rangle = \|\partial_s\tau\|^2 - \langle\tau, \partial_t\partial_s\sigma\rangle + \langle\tau, \partial_s R_2\rangle \\ &= \|\partial_s\tau\|^2 + \langle\partial_t\tau, \partial_s\sigma\rangle + \langle\tau, \partial_s R_2\rangle = \|\partial_s\tau\|^2 + \|\partial_t\tau\|^2 + \langle\partial_t\tau, R_1\rangle + \langle\tau, \partial_s R_2\rangle,\end{aligned}$$

We have used the fact that τ vanishes on both endpoints in order to do the integration by parts.

We easily estimate (by inspecting (10.1)):

$$|\langle\partial_t\tau, R_1\rangle| \leq \frac{1}{6}(\|\partial_t\tau\|^2 + \|A(x)\|^2),$$

provided the C^0 sizes of x, τ are sufficiently small.

Similarly, we estimate:

$$|\langle\tau, \partial_s R_2\rangle| \leq \frac{1}{6}(\|\partial_t\tau\|^2 + \|\partial_s\tau\|^2 + \|A(x)\|^2),$$

provided the C^2 sizes of τ and x are sufficiently small, using the Poincaré inequality when we need to estimate terms involving $\|\tau\|^2$.

Then we conclude:

$$\Gamma''(s) - c^2\Gamma(s) \geq \frac{1}{3}(\|\partial_s\tau\|^2 + \|\partial_t\tau\|^2 - \|A(x)\|^2),$$

as desired. □

10.7.2. *Differential inequality for τ when $S = \mathbb{R}/\mathbb{Z}$.* The analogous differential inequality in the case $S = \mathbb{R}/\mathbb{Z}$ is a bit harder to establish, since we cannot apply the Poincaré inequality to τ. However, we know from Corollary 10.8 that the mean of τ converges exponentially to 0. Therefore, it suffices to establish a differential inequality for

$$f(s,t) = \tau(s,t) - \bar{\tau}(s)$$

There is a Poincaré inequality for functions with mean 0, and hence there is hope that we can argue as we did in the previous section. We compute:

$$\partial_s f = -\partial_t\sigma + R_2 - \int_0^1 R_2 =: -\partial_t\sigma + R_2' \text{ and } \partial_t f = \partial_s\sigma + R_1.$$

Then we conclude:

Lemma 10.14. Introduce the quantity: $\Gamma(s) = \frac{1}{2}\int_0^1 |f(s,t)|^2\,dt$. There is $c > 0$ (depending only on the constant from the Poincaré lemma for $\mathbb{R}/\mathbb{Z}$) and $\epsilon > 0$ with the following property: if the C^2 sizes of x, τ are less than ϵ, then

$$\Gamma''(s) - c^2\Gamma(s) \geq \frac{1}{3}(\|\partial_s f\|^2 + \|\partial_t f\|^2 - \|A(x)\|^2 - \|\partial_s x\|^2). \tag{10.13}$$

Proof. The argument is similar to the one in the previous section. We have

$$\begin{aligned}\Gamma''(s) &= \|\partial_s f\|^2 + \langle f, \partial_s\partial_s f\rangle = \|\partial_s f\|^2 - \langle f, \partial_t\partial_s\sigma\rangle + \langle f, \partial_s(R_2')\rangle \\ &= \|\partial_s f\|^2 + \langle \partial_t f, \partial_s\sigma\rangle + \langle f, \partial_s(R_2')\rangle \\ &= \|\partial_s f\|^2 + \|\partial_t f\|^2 + \langle \partial_t f, R_1\rangle + \langle f, \partial_s(R_2')\rangle,\end{aligned}$$

Once again, it is easy to estimate:

$$|\langle \partial_t f, R_1\rangle| \leq \frac{1}{6}(\|\partial_t f\|^2 + \|A(x)\|^2),$$

by inspection of the terms appearing in R_1 from (10.1), provided that ϵ is sufficiently small. The harder term to estimate is the one involving $\partial_s(R_2')$. We have

$$|\langle f, \partial_s(R_2')\rangle| \leq \frac{1}{6}(\|\partial_s f\|^2 + \|\partial_t f\|^2 + \|A(x)\|^2 + \|\partial_s x\|^2). \tag{10.14}$$

Most of the terms are straightforward to estimate, using the Poincaré inequality whenever we need to estimate $\|f\|^2 \leq \|\partial_t f\|^2$. For instance, one of the terms appearing involves $\partial_s\int_0^1 R_2$, which can be estimated as follows:

$$\langle f, 1\rangle \int_0^1 \partial_s(E_{34}\cdot x\cdot\tau) \leq C_1 \|f\| (\|x\| + \|\partial_s x\|) \leq C_2(\|\partial_t f\|^2 + \|A(x)\|^2 + \|\partial_s x\|^2),$$

where $C_1, C_2 = o(1)$ as $\epsilon \to 0$. Arguably the hardest term to estimate is:

$$\langle f, E_{31}\cdot\tau\cdot\partial_s\partial_t x - \int_0^1 E_{31}\cdot\tau\cdot\partial_s\partial_t x\rangle.$$

However, we can integrate this term by parts to get the second derivatives off of x and then bound the result by (10.14), as desired. Then, as in the previous section, we conclude:

$$\Gamma''(s) - c^2\Gamma(s) \geq \frac{1}{3}(\|\partial_s f\|^2 + \|\partial_t f\|^2 - \|A(x)\|^2 - \|\partial_s x\|^2),$$

as desired. □

10.7.3. $W^{1,2}$ *exponential estimates for the tangential coordinate.* Suppose that x, τ are C^2 ϵ-small on $[s_0, s_1]\times S$, so that the previous results apply. Then we have

Lemma 10.15. There exists a constant T which is $o(1)$ as $\epsilon \to 0$ so that the following holds. If $S = \mathbb{R}/\mathbb{Z}$, and $\tau_0 = \bar\tau(s_*)$, or $S = [0,1]$ and $\tau_0 = 0$, then

$$\int_{s-0.5}^{s+0.5}\int_0^1 |\tau - \tau_0|^2 + |\partial_t\tau|^2 + |\partial_s\tau|^2 \leq T(e^{-c(s-s_0)} + e^{-c(s_1-s)}).$$

Proof. Appealing to either the differential inequality (10.12), in the case when $S = [0,1]$, or (10.13), in the case when $S = \mathbb{R}/\mathbb{Z}$, we conclude that:

$$\Gamma''(s) - c^2\Gamma(s) \geq \frac{1}{3}(\|\partial_s f\|^2 + \|\partial_t f\|^2) - \kappa,$$

where $f = \tau$ or $f = \tau - \bar\tau$ depending on whether $S = [0,1]$ or $S = \mathbb{R}/\mathbb{Z}$, and

$$\kappa \leq D_1^2 \|x\|_{C^1}^2 (e^{-2\delta(s-s_0)} + e^{-2\delta(s_1-s_0)}),$$

by appealing to the C^1 exponential estimates in the previous section (note that the C^1 estimates hold on all of $[s_0, s_1]$). In particular, assuming that $c < 2\delta$, shrinking it if necessary, then we can apply Lemma 10.3 to conclude:

$$\int_{s-0.5}^{s+0.5} \|f\|^2 + \|\partial_s f\|^2 + \|\partial_t f\|^2 \,\mathrm{d}s \leq T(e^{-c(s-s_0)} + e^{-c(s_1-s)}),$$

where $T = o(1)$ as $\epsilon \to 0$. However, $\partial_t f = \partial_t\tau$, $f = \tau - \int_0^1 \tau$, and, $\partial_s f = \partial_s\tau - \int_0^1 R_2$, which implies that:

$$\int_{s-0.5}^{s+0.5} \|\tau - \tau_0\|^2 + \|\partial_s\tau\|^2 + \|\partial_t\tau\|^2 \,\mathrm{d}s \leq T(e^{-c(s-s_0)} + e^{-c(s_1-s)}),$$

Here we have used:

$$\|f\| \leq \|\tau - \tau_0\| + |\bar\tau - \tau_0|,$$

and Corollary 10.8, which asserts that:

$$|\bar{\tau} - \tau_0| \leq C \|x\|_{C^1} (e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

We conclude the desired result. □

Remark 10.16. Henceforth, let $2\delta < c$ and $2\delta < d$, so that our results apply. Thus we see that there are two conditions placed on δ, one comes from the spectral properties of A (namely, d), while the other condition (namely, $2\delta < c$) comes from the Poincaré inequality for $\mathbb{R}$ valued functions on $[0,1]$ with vanishing endpoints and functions on $\mathbb{R}/\mathbb{Z}$ with zero mean.

10.7.4. *Bootstrapping the estimate for the tangential coordinate.* We bootstrap the previous exponential estimate on the τ coordinate.

Lemma 10.17. For each $\epsilon > 0$ sufficiently small there is $T_k = o(1)$ as $\epsilon \to 0$ with the following property. Suppose that x, σ, τ solve (10.1) and x, τ are C^{k+1} ϵ-small on $[s_0, s_1]$. Then:

$$\sum_{\ell=1}^{k} \left|\nabla^\ell (\tau - \tau_0)(s,t)\right| \leq T_k (e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

for $s \in [s_0, s_1]$. Moreover, T_k can be chosen $o(1)$ as $\epsilon \to 0$.

Proof. As we argued previously, it suffices to establish the estimate on the interior interval $[s_0+1, s_1-1]$.

Fix some $s \in [s_0+1, s_1-1]$, and let $\Omega_k = [s-\rho_k, s+\rho_k] \times S$, as in §10.6.1. The holomorphic curve equation for τ implies that:

$$\Delta(\tau - \tau_0) = \partial_s R_2 - \partial_t R_1.$$

It is straightforward to estimate $\|\partial_s R_2 - \partial_t R_1\|_{W^{k,2}(\Omega_k)} \leq o(1) \|x\|_{W^{k+2,2}(\Omega_k)}$ as $\epsilon \to 0$. This only relies on having C^{k+1} bounds on τ, x.

Hence,we can estimate:

$$\|\Delta(\tau - \tau_0)\|_{W^{k,2}(\Omega_k)} \leq o(1) \|x\|_{W^{k+2,2}(\Omega_k)}.$$

Then we apply the elliptic estimates for Δ to conclude:

$$\|\tau - \tau_0\|_{W^{k+2,2}(\Omega_{k+2})} \leq o(1) \|x\|_{W^{k+2,2}(\Omega_{k+1})} + L_{k+2} \|\tau - \tau_0\|_{W^{k+1,2}(\Omega_{k+1})}.$$

Continuing in this fashion,

$$\|\tau - \tau_0\|_{W^{k+2,2}(\Omega_{k+2})} \leq o(1) \|x\|_{W^{k+2,2}(\Omega_1)} + L_{k+2} \cdots L_2 \|\tau - \tau_0\|_{W^{1,2}(\Omega_1)}.$$

Applying the $W^{1,2}$ elliptic estimate for τ from §10.7.3, the $W^{k+2,2}(\Omega_1)$ estimate on x from the proof of Lemma 10.10, and the Sobolev embedding theorem, we conclude that

$$\|\tau - \tau_0\|_{C^k(\Omega_{k+2})} \leq T_k(e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

where $T_k = o(1)$ as $\epsilon \to 0$, as desired. □

10.7.5. *Exponential estimates on the σ coordinate.* The goal in this section is to use the equations (10.1), and the estimates on x, τ, to derive exponential estimates for σ. We have the following result:

Lemma 10.18. For sufficiently small $\epsilon > 0$, there exist constants $S_k = o(1)$ as $\epsilon \to 0$ so that the following holds. If x, τ are C^{k+1} ϵ-small, then, for all $s, t \in [s_0, s_1] \times S$ we have:

$$\sum_{\ell=0}^{k} \left|\nabla^\ell(\sigma(s,t) - \sigma_0)\right| \leq S_k(e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

where σ_0 is the average value of σ along the central circle $s_* = \frac{1}{2}(s_0 + s_1)$.

Proof. It is trivial to use (10.1) to estimate:

$$\sum_{\ell=1}^{k} \left|\nabla^\ell(\sigma(s,t) - \sigma_0)\right| = \sum_{\ell=1}^{k} \left|\nabla^\ell(\sigma(s,t))\right| \leq S'_k(e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}),$$

for some $S'_k = o(1)$ as $\epsilon \to 0$. This is because the derivatives of σ can be expressed entirely in terms of the derivatives of τ, x. The tricky part is estimating the $\ell = 0$ term. We have:

$$|\sigma(s,t) - \sigma_0| = |\sigma(s,t) - \bar{\sigma}(s)| + |\bar{\sigma}(s) - \bar{\sigma}(s_*)|\,.$$

The first term can be bounded in terms of $\partial_t \sigma$, and hence the desired estimate holds for this term. For the second term we appeal to Corollary 10.8 to conclude:

$$|\bar{\sigma}(s) - \bar{\sigma}(s_*)| \leq C\,\|x\|_{C^0}\,(e^{-\delta(s-s_0)} + e^{-\delta(s_1-s)}).$$

This completes the proof. □

Proof (of Theorem 10.1)**.** One simply combines Lemmas 10.11, 10.17, and 10.18. □

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